Seminar 8

1. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$F(x, y) := (1 - x^2) \cos y - e^x \sin y \ln(1 + x^2 + y^2).$$

Prove that there exists an open neighborhood $U \subseteq \mathbb{R}$ of 1 and there exists a function $f : U \to \mathbb{R}$, which is of class C^1 on U and satisfies

$$f(1) = 0 \quad \text{and} \quad F(x, f(x)) = 0 \quad \text{for all } x \in U.$$

Determine $f'(1)$.

2. Let $a > 0$ be a real number, and let C be the set defined by

$$C := \left\{ (x, y) \;\middle|\; x^2 + y^2 - \frac{a}{2}\left(x + \sqrt{x^2 + y^2}\right) = 0 \right\}.$$

The points of C lie on a plane curve, called *cardioid* (see figure 1). Determine all the points of the cardioid around which one can express the variable y as a function of x.

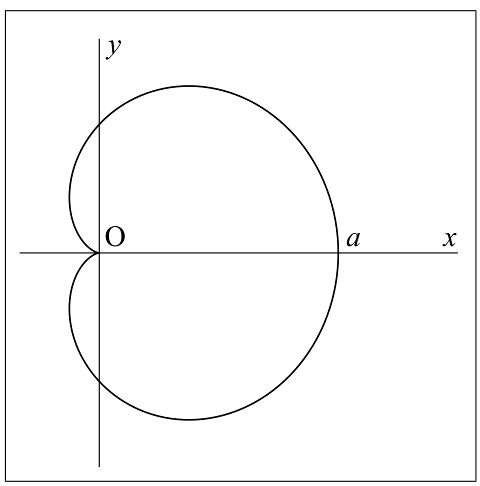

Figure 1: The cardioid

3. **(Homework)** Prove that the equation $x^2+xy+2y^2+3z^4-z=9$ defines implicitly around the point $(1,-2)$ a function $z=f(x,y)$, which is of class C^1 and satisfies $f(1,-2)=1$. Determine the first order partial derivatives and the differential of f at $(1,-2)$.

4. **(Homework)** Let $a>0$ be a real number, and let C be the set defined by

$$C := \{\, (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=a^2,\ x^2+y^2=ax \,\}.$$

The points of C lie on a space curve, called *Viviani's curve or Viviani's window* (see figure 2). Determine all the points of Viviani's curve around which it can be parameterized by using y as parameter.

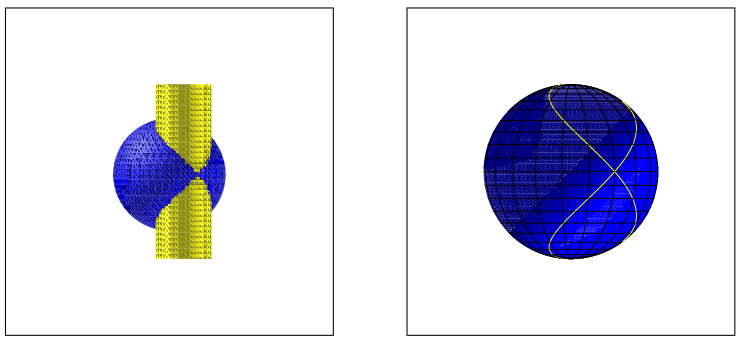

Figure 2: Curba lui Viviani

5. Prove that the system

$$\begin{cases} x^2+uy+e^v=0 \\ 2x+u^2-uv=5 \end{cases}$$

defines implicitly in a neighborhood of $(2,5)$ a function

$$f=(f_1(x,y),\ f_2(x,y)),$$

which is of class C^1 and satisfies $f(2,5)=(-1,0)$. Determine $\mathrm{d}f(2,5)$.

6. **(Homework)** Prove that the system

$$\begin{cases} x^2-y\cos(uv)+z^2=0 \\ x^2+y^2-\sin(uv)+2z^2=2 \\ xy-\cos u\cos v+z=1 \end{cases}$$

defines implicitly in a neighborhood of the point $\left(\frac{\pi}{2}, 0\right)$ a function

$$f = (f_1(u,v),\ f_2(u,v),\ f_3(u,v)),$$

which is of class C^1 and satisfies $f\left(\frac{\pi}{2}, 0\right) = (1,1,0)$. Determine the differential $\mathrm{d}f\left(\frac{\pi}{2}, 0\right)$.

7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x,y,z) := x + y + z$, and let C be the set defined by

$$C := \{\, (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ 2x + y + 2z = 1 \,\}.$$

Determine $\min f(C)$ and $\max f(C)$.

8. **(Homework)** Let f be the function defined by $f(x,y,z) := x^2+y^2-z^2$, and let C be the set defined by

$$C := \{\, (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9,\ x + y + z = 5 \,\}.$$

Determine $\min f(C)$ and $\max f(C)$.

9. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x,y,z) := x^2 + y^2 + z^2 - 2x + 2\sqrt{2}y + 2z$$

and the set $B := \{\, (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1 \,\}$. Determine $\min f(B)$ and $\max f(B)$.

10. Determine the stationary points of the following functions and investigate their nature:

a) $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) := 2x^2 - xy + 2xz - y + y^3 + z^2;$

b) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) := 3x - 3y - 2x^3 - xy^2 + 2x^2y + y^3;$

c) $f : \mathbb{R}^3 \to \mathbb{R},\ f(x,y,z) := x^2 + y^2 + z^2 - 2xyz;$

d) $f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) := x^4 + y^4 - 2x^2.$

Solutions

$\boxed{\textbf{1.}}$ We have to prove that the equation

$$F(x,y)=0 \quad \Leftrightarrow \quad (1-x^2)\cos y - e^x \sin y \ln(1+x^2+y^2)=0$$

defines implicitly the variable y as a function of the variable x around the point $(a,b)=(1,0)$. We check the hypotheses of the implicit function theorem (Theorem 2.15.3 in the lecture notes). Obviously, F is of class C^1 on $\mathbb{R}^2$ and it satisfies $F(a,b)=F(1,0)=0$. The other condition is

$$\det J_y(F)(1,0)\neq 0 \quad \Leftrightarrow \quad \frac{\partial F}{\partial y}(1,0)\neq 0.$$

Since

$$\frac{\partial F}{\partial y}(x,y) = -(1-x^2)\sin y - e^x\cos y \ln(1+x^2+y^2) - e^x \sin y \cdot \frac{2y}{1+x^2+y^2}\,,$$

it follows that $\dfrac{\partial F}{\partial y}(1,0)=-e\ln 2\neq 0$, hence the implicit function theorem can be applied. According to it, there exists an open neighborhood $U\subseteq\mathbb{R}$ of 1 and there exists a function $f:U\to\mathbb{R}$, which is of class C^1 on U, such that $f(1)=0$ and

$$F(x,f(x)) = (1-x^2)\cos f(x) - e^x \sin f(x)\ln(1+x^2+f^2(x)) = 0 \quad \forall\, x\in U.$$

Differentiating both sides of this equality, we deduce that for all $x\in U$ it holds

$$\begin{aligned}
&-2x\cos f(x) - (1-x^2)\sin f(x)\cdot f'(x) - e^x\sin f(x)\ln(1+x^2+f^2(x))\\
&\quad -e^x\cos f(x)\cdot f'(x)\ln(1+x^2+f^2(x))\\
&\quad\quad -e^x\sin f(x)\,\frac{2x+2f(x)f'(x)}{1+x^2+f^2(x)} = 0.
\end{aligned}$$

Letting $x=1$ and tacking into consideration that $f(1)=0$, we get

$$-2-e\ln 2\cdot f'(1)=0 \quad\Rightarrow\quad f'(1)=-\frac{2}{e\ln 2}\,.$$

$\boxed{\textbf{2.}}$ Let $F:\mathbb{R}^2\to\mathbb{R}$ be the function defined by

$$F(x,y) := x^2+y^2-\frac{a}{2}\left(x+\sqrt{x^2+y^2}\right),$$

and let $(x_0, y_0) \in C$. The condition that y can be expressed as a function of x around the point (x_0, y_0) is actually the condition that the equation $F(x, y) = 0$ defines implicitly y as a function of x around (x_0, y_0). This is

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \quad \Leftrightarrow \quad 2y_0 - \frac{a y_0}{2\sqrt{x_0^2 + y_0^2}} \neq 0.$$

Solving the system

$$\begin{cases} 2y - \dfrac{ay}{2\sqrt{x^2+y^2}} = 0 \\ \\ x^2 + y^2 - \dfrac{a}{2}\left(x + \sqrt{x^2+y^2}\right) = 0, \end{cases}$$

we get $(x, y) \in \left\{(0,0),\, (a,0),\, \left(-\frac{a}{8}, \frac{a\sqrt{3}}{8}\right), \left(-\frac{a}{8}, -\frac{a\sqrt{3}}{8}\right)\right\}$. Therefore, the points of the cardioid around which one can express the variable y as a function of x are $(x, y) \in C \setminus \left\{(0,0),\, (a,0),\, \left(-\frac{a}{8}, \frac{a\sqrt{3}}{8}\right), \left(-\frac{a}{8}, -\frac{a\sqrt{3}}{8}\right)\right\}$ (see figure 3).

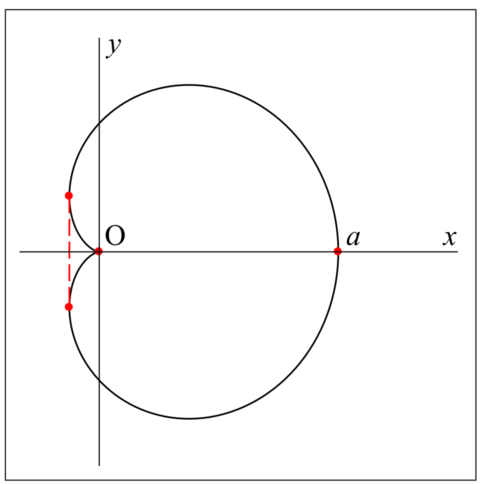

Figure 3:

$\boxed{\mathbf{3.}}$ Consider the function $F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, defined by

$$F(x, y, z) := x^2 + xy + 2y^2 + 3z^4 - z - 9$$

and the points $a := (1, -2)$, $b := 1$. We have to prove that the equation $F(x, y, z) = 0$ defines implicitly the variable z as a function of x and y around the point $(a, b) = (1, -2, 1)$. We check the hypotheses of the implicit function theorem. Obviously, F is of class C^1 on $\mathbb{R}^3$ and $F(a, b) = F(1, -2, 1) = 0$. The other condition is

$$\det J_z(F)(1, -2, 1) \neq 0 \quad \Leftrightarrow \quad \frac{\partial F}{\partial z}(1, -2, 1) \neq 0.$$

Since

$$\frac{\partial F}{\partial z}(x, y, z) = 12z^3 - 1,$$

it follows that $\dfrac{\partial F}{\partial z}(1, -2, 1) = 11 \neq 0$, hence the implicit function theorem can be applied. According to it, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ of a and there exists a function $f : U \to \mathbb{R}$, of class C^1 on U, such that $f(1, -2) = 1$ and

$$x^2 + xy + 2y^2 + 3f^4(x, y) - f(x, y) - 9 = 0 \quad \forall\ (x, y) \in U.$$

Differentiating both sides of this equality, first with respect to x, then with respect to y, we obtain

$$2x + y + 12f^3(x, y)\frac{\partial f}{\partial x}(x, y) - \frac{\partial f}{\partial x}(x, y) = 0 \quad \forall\ (x, y) \in U$$

and

$$x + 4y + 12f^3(x, y)\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(x, y) = 0 \quad \forall\ (x, y) \in U.$$

Letting $(x, y) = (1, -2)$ and tacking into account that $f(1, -2) = 1$, we get

$$\frac{\partial f}{\partial x}(1, -2) = 0 \quad \text{și} \quad \frac{\partial f}{\partial y}(1, -2) = \frac{7}{11}\,.$$

Consequently, we have $\mathrm{d}f(1, -2)(h_1, h_2) = \frac{7}{11}\, h_2$ for all $(h_1, h_2) \in \mathbb{R}^2$.

$\boxed{\textbf{4.}}$ Consider the function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$F(x, y, z) := \big(x^2 + y^2 + z^2 - a^2,\ x^2 + y^2 - ax\big).$$

We have to find all the points $(x_0, y_0, z_0) \in C$ with the property that the equation $F(x, y, z) = (0, 0)$ defines implicitly around (x_0, y_0, z_0) the variables

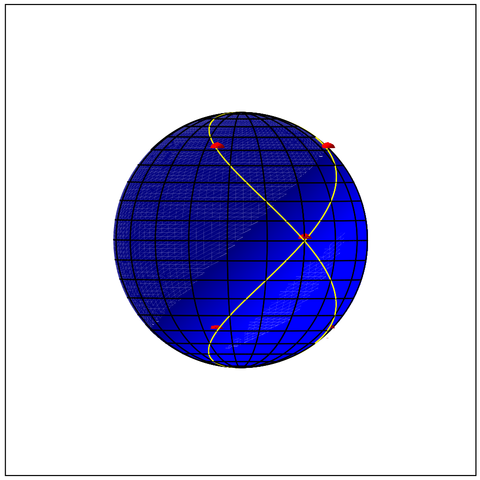

Figure 4:

x and z as functions of y. Proceeding as in the solution of problem $\boxed{\mathbf{2}}$, we obtain (see figure 4)

$$(x_0, y_0, z_0) \in C \setminus \left\{ (a,0,0), \left(\frac{a}{2}, -\frac{a}{2}, -\frac{a\sqrt{2}}{2}\right), \left(\frac{a}{2}, -\frac{a}{2}, \frac{a\sqrt{2}}{2}\right), \left(\frac{a}{2}, \frac{a}{2}, -\frac{a\sqrt{2}}{2}\right), \left(\frac{a}{2}, \frac{a}{2}, \frac{a\sqrt{2}}{2}\right) \right\}.$$

$\boxed{\mathbf{5.}}$ Consider the function $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$F(x, y, u, v) := \big(\underbrace{x^2 + uy + e^v}_{F_1(x,y,u,v)}, \underbrace{2x + u^2 - uv - 5}_{F_2(x,y,u,v)}\big)$$

and the points $a := (2, 5)$, $b := (-1, 0)$. We have to prove that the vector equation (or that the system of scalar equations) $F(x, y, u, v) = (0, 0)$ defines implicitly the variables u and v as functions of x and y around the point $(a, b) = (2, 5, -1, 0)$. We check the hypotheses of the implicit function theorem. Obviously, F is of class C^1 on $\mathbb{R}^4$ and $F(2, 5, -1, 0) = (0, 0)$. The other condition is

$$\det J_{(u,v)}(F)(2, 5, -1, 0) \neq 0.$$

We have

$$
\begin{aligned}
J_{(u,v)}(F)(x,y,u,v) &= \begin{pmatrix} \dfrac{\partial F_1}{\partial u}(x,y,u,v) & \dfrac{\partial F_1}{\partial v}(x,y,u,v) \\[2ex] \dfrac{\partial F_2}{\partial u}(x,y,u,v) & \dfrac{\partial F_2}{\partial v}(x,y,u,v) \end{pmatrix} \\
&= \begin{pmatrix} y & e^v \\ 2u-v & -u \end{pmatrix}.
\end{aligned}
$$

Therefore,

$$
\det J_{(u,v)}(F)(2,5,-1,0) = \begin{vmatrix} 5 & 1 \\ -2 & 1 \end{vmatrix} = 7 \neq 0,
$$

hence the implicit function theorem can be applied. According to it, there exists an open neighborhood $U \subseteq \mathbb{R}^2$ of the point a as well as a function $f = (f_1, f_2) : U \to \mathbb{R}^2$, which is of class C^1 pe U, such that $f(2,5) = (-1,0)$ and

$$
(1) \qquad \forall\ (x,y) \in U\ : \quad \begin{cases} x^2 + f_1(x,y)y + e^{f_2(x,y)} = 0 \\ 2x + f_1^2(x,y) - f_1(x,y)f_2(x,y) - 5 = 0. \end{cases}
$$

Differentiating with respect to x both equalities in (1), we obtain

$$
\begin{cases} 2x + \dfrac{\partial f_1}{\partial x}(x,y)\, y + e^{f_2(x,y)}\, \dfrac{\partial f_2}{\partial x}(x,y) = 0 \\[2ex] 2 + 2f_1(x,y)\, \dfrac{\partial f_1}{\partial x}(x,y) - \dfrac{\partial f_1}{\partial x}(x,y)\, f_2(x,y) - f_1(x,y)\, \dfrac{\partial f_2}{\partial x}(x,y) = 0 \end{cases}
$$

for all $(x,y) \in U$. Then letting $(x,y) = (2,5)$ and tacking into consideration that $f_1(2,5) = -1$ and $f_2(2,5) = 0$, we get

$$
\begin{cases} 5\, \dfrac{\partial f_1}{\partial x}(2,5) + \dfrac{\partial f_2}{\partial x}(2,5) = -4 \\[2ex] -2\, \dfrac{\partial f_1}{\partial x}(2,5) + \dfrac{\partial f_2}{\partial x}(2,5) = -2, \end{cases}
$$

whence $\dfrac{\partial f_1}{\partial x}(2,5) = -\dfrac{2}{7}$, $\dfrac{\partial f_2}{\partial x}(2,5) = -\dfrac{18}{7}$.

Differentiating now with respect to y both equalities in (1), and proceeding as above, we obtain $\dfrac{\partial f_1}{\partial y}(2,5) = \dfrac{1}{7}$, $\dfrac{\partial f_2}{\partial y}(2,5) = \dfrac{2}{7}$. Consequently, we have

$$J(f)(2,5) = \begin{pmatrix} -\dfrac{2}{7} & \dfrac{1}{7} \\ -\dfrac{18}{7} & \dfrac{2}{7} \end{pmatrix},$$

whence

$$\mathrm{d}f(2,5)(h_1,h_2) = \begin{pmatrix} -\dfrac{2}{7} & \dfrac{1}{7} \\ -\dfrac{18}{7} & \dfrac{2}{7} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \dfrac{-2h_1 + h_2}{7} \\ \dfrac{-18h_1 + 2h_2}{7} \end{pmatrix}.$$

$\boxed{\textbf{7.}}$ C is the set of points lying on the circle obtained by intersecting the sphere $(S)\ :\ x^2 + y^2 + z^2 = 1$ with the plane $(\pi)\ :\ 2x + y + 2z = 1$ (see figure 5).

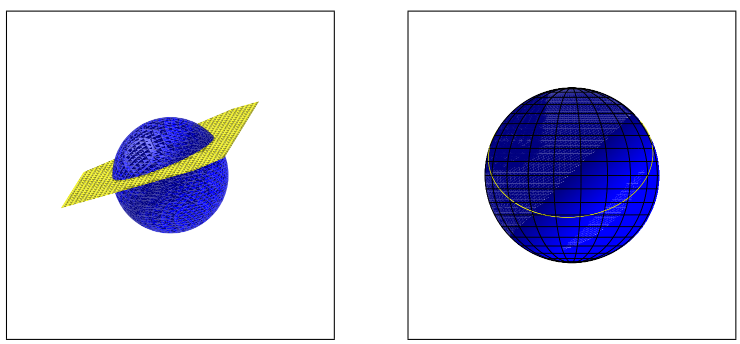

Figure 5:

Therefore, C is a compact set. Since the function f is continuous, in virtue of the Weierstrass theorem it follows that it is bounded and attains its bounds on C. Consequently, there exist two points $(a,b,c) \in C$ and $(a',b',c') \in C$ such that

$$f(a,b,c) = \min f(C) \quad \text{and} \quad f(a',b',c') = \max f(C).$$

According to the method of Lagrange multipliers (Theorem 2.17.1 in the lecture notes), for each of the two extrema there exists a pair of multipliers

$(\lambda_0, \mu_0), (\lambda'_0, \mu'_0) \in \mathbb{R}^2$ such that

$$(a, b, c, \lambda_0, \mu_0) \quad \text{and} \quad (a', b', c', \lambda'_0, \mu'_0)$$

are stationary points for the Lagrange function. Let

$$F_1(x, y, z) := x^2 + y^2 + z^2 - 1 \quad \text{and} \quad F_2(x, y, z) := 2x + y + 2z - 1$$

be the functions expressing the constraints on the variables in the definition of the set C, and let

$$\begin{aligned} L(x, y, z, \lambda, \mu) &:= f(x, y, z) + \lambda F_1(x, y, z) + \mu F_2(x, y, z) \\ &= x + y + z + \lambda(x^2 + y^2 + z^2 - 1) + \mu(2x + y + 2z - 1) \end{aligned}$$

be the Lagrange function. All that remains to be done is to determine the stationary points of L. They are solutions to the system

$$\begin{aligned} L'_x(x, y, z, \lambda, \mu) &= 1 + 2\lambda x + 2\mu = 0 \\ L'_y(x, y, z, \lambda, \mu) &= 1 + 2\lambda y + \mu = 0 \\ L'_z(x, y, z, \lambda, \mu) &= 1 + 2\lambda z + 2\mu = 0 \\ L'_\lambda(x, y, z, \lambda, \mu) &= x^2 + y^2 + z^2 - 1 = 0 \\ L'_\mu(x, y, z, \lambda, \mu) &= 2x + y + 2z - 1 = 0. \end{aligned}$$

Subtracting side by side the first and the third equation, we get $2\lambda(x - z) = 0$, whence $x = z$ (we cannot have $\lambda = 0$ because, otherwise, it would result that $1 + 2\mu = 0 = 1 + \mu$, which is absurd). Therefore, we have

$$\begin{cases} 2x^2 + y^2 - 1 = 0 \\ 4x + y - 1 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x = 0, \ y = 1 \\ \quad\text{or} \\ x = \frac{4}{9}, \ y = -\frac{7}{9}. \end{cases}$$

Finally, the stationary points of L are

$$(0, 1, 0, \dots, \dots) \quad \text{and} \quad \left(\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}, \dots, \dots\right).$$

We have not yet determined the values of the multipliers because they have no relevance to the problem. Since $f(0, 1, 0) = 1$ and $f\left(\frac{4}{9}, -\frac{7}{9}, \frac{4}{9}\right) = \frac{1}{9}$, it follows that $\min f(C) = \frac{1}{9}$ și $\max f(C) = 1$.

$\boxed{\textbf{9.}}$ The set B is compact, while the function f is continuous. According to the Weierstrass theorem, there exist two points (a, b, c), $(a', b', c') \in B$ such that

$$f(a, b, c) = \min f(B) \quad \text{and} \quad f(a', b', c') = \max f(B).$$

Note that

$$\nabla f(x, y, z) = (0, 0, 0) \quad \Leftrightarrow \quad \begin{cases} 2x - 2 = 0 \\ 2y + 2\sqrt{2} = 0 \\ 2z + 2 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = 1 \\ y = -\sqrt{2} \\ z = -1, \end{cases}$$

and $(1, -\sqrt{2}, -1) \notin \operatorname{int} B$. Therefore, (a, b, c), $(a', b', c') \in \operatorname{bd} B$. We have

$$\operatorname{bd} B = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

Consider the function $F(x, y, z) := x^2 + y^2 + z^2 - 1$. According to the method of Lagrange multipliers, for each of the two extrema there exists a multiplier $\lambda_0, \lambda_0' \in \mathbb{R}$ such that (a, b, c, λ_0) and (a', b', c', λ_0') are stationary points for the Lagrange function

$$\begin{aligned} L(x, y, z, \lambda) &:= f(x, y, z) + \lambda F(x, y, z) \\ &= x^2 + y^2 + z^2 - 2x + 2\sqrt{2}y + 2z + \lambda(x^2 + y^2 + z^2 - 1). \end{aligned}$$

A simple calculation shows that the only stationary points of L are

$$\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}, 1\right) \quad \text{and} \quad \left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, -3\right).$$

Tacking into account that $f\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right) = -3$ and $f\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = 5$, we conclude that $\min f(B) = -3$ and $\max f(B) = 5$.

Geometric interpretation. The set B is the closed unit ball in $\mathbb{R}^3$. We have $f(x, y, z) = (x - 1)^2 + (y + \sqrt{2}\,)^2 + (z + 1)^2 - 4$, hence

$$f(x, y, z) = PM^2 - 4, \quad \text{where } P(x, y, z),\ M(1, -\sqrt{2}, -1).$$

Therefore, the problem is to determine the smallest and largest distance PM, when the point P lies in B. The points P' and P'' for which the distance PM is minimum/maximum can be obtained by intersecting the line OM with the sphere $(S) : x^2 + y^2 + z^2 = 1$ (see figure 6). The parametric

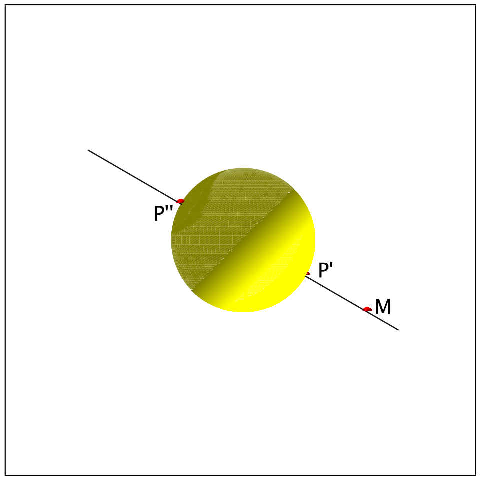

Figure 6:

equations of the line OM are $OM\ :\ x=t,\ y=-\sqrt{2}t,\ z=-t$. Assuming that $(t,-\sqrt{2}t,-t)\in S$, we deduce that $t=\pm\frac{1}{2}$, whence $P'\left(\frac{1}{2}\,,\,-\frac{\sqrt{2}}{2}\,,\,-\frac{1}{2}\right)$ and $P''\left(-\frac{1}{2}\,,\,\frac{\sqrt{2}}{2}\,,\,\frac{1}{2}\right)$.

$\boxed{\textbf{10.}}$ a) **I.** We first determine the stationary points of f. They are solutions to the system

$$\begin{aligned} f'_x(x,y,z) &= 4x-y+2z=0\\ f'_y(x,y,z) &= -x-1+3y^2=0\\ f'_z(x,y,z) &= 2x+2z=0. \end{aligned}$$

Solving the system, we find the stationary points $\left(\frac{1}{3}\,,\,\frac{2}{3}\,,\,-\frac{1}{3}\right)$, $\left(-\frac{1}{4}\,,\,-\frac{1}{2}\,,\,\frac{1}{4}\right)$.

II. We determine the second order partial derivatives of f and the Hessian matrix. We have

$$\begin{aligned} f''_{xx}(x,y,z) &= 4, & f''_{xy}(x,y,z) &= f''_{yx}(x,y,z)=-1,\\ f''_{yy}(x,y,z) &= 6y, & f''_{yz}(x,y,z) &= f''_{zy}(x,y,z)=0,\\ f''_{zz}(x,y,z) &= 2, & f''_{zx}(x,y,z) &= f''_{xz}(x,y,z)=2. \end{aligned}$$

Therefore,

$$H(f)(x,y,z)=\begin{pmatrix} 4 & -1 & 2\\ -1 & 6y & 0\\ 2 & 0 & 2 \end{pmatrix}.$$

III.1. The Hessian matrix in the first stationary point is

$$H(f)\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix},$$

while the diagonal principal minors in the Sylvester theorem are

$$\Delta_1 = 4, \quad \Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 15, \quad \Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 14.$$

Since $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, it follows that $\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$ is a local minimum point for f.

III.2. For the second stationary point we have

$$H(f)\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

and

$$\Delta_1 = 4, \quad \Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & -3 \end{vmatrix} = -13, \quad \Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{vmatrix} = -14.$$

Therefore, $\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$ is a saddle point.

Remark. Although f has only one extremum point (a minimum), this point is not a global minimum for f. Indeed, we have

$$f(0, y, 0) = y^3 - y \longrightarrow -\infty \quad \text{as } y \to -\infty.$$

b) **I.** The stationary points of f are solutions to the system

$$\begin{aligned} f'_x(x, y) &= 3 - 6x^2 - y^2 + 4xy = 0 \\ f'_y(x, y) &= -3 - 2xy + 2x^2 + 3y^2 = 0. \end{aligned}$$

Solving the system, we find the stationary points $(1, 1)$, $(-1, -1)$, $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ and $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$.

II. We determine the second order partial derivatives of f and the Hessian matrix. We have

$$f''_{xx}(x,y) = -12x + 4y, \qquad f''_{xy}(x,y) = f''_{yx}(x,y) = -2y + 4x,$$
$$f''_{yy}(x,y) = -2x + 6y.$$

Therefore,

$$H(f)(x,y) = \begin{pmatrix} 4y - 12x & 4x - 2y \\ 4x - 2y & 6y - 2x \end{pmatrix}.$$

III.1. Since

$$H(f)(1,1) = \begin{pmatrix} -8 & 2 \\ 2 & 4 \end{pmatrix},$$

it follows that

$$\Delta_1 = -8, \quad \Delta_2 = \begin{vmatrix} -8 & 2 \\ 2 & 4 \end{vmatrix} = -36,$$

hence $(1,1)$ is a saddle point.

III.2. Since

$$H(f)\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) = \begin{pmatrix} -\frac{20}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} & -\frac{14}{\sqrt{6}} \end{pmatrix},$$

it follows that

$$\Delta_1 = -\frac{20}{\sqrt{6}}, \quad \Delta_2 = \begin{vmatrix} -\frac{20}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} & -\frac{14}{\sqrt{6}} \end{vmatrix} = 36,$$

hence $\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ is a local maximum for f.

Proceeding as above, it is found that **(Homework)** $(-1,-1)$ is a saddle point, while $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ is a local minimum.

c) **I.** The stationary points of f are solutions to the system

$$\begin{aligned} f'_x(x,y,z) &= 2x - 2yz = 0 \\ f'_y(x,y,z) &= 2y - 2zx = 0 \\ f'_z(x,y,z) &= 2z - 2xy = 0. \end{aligned}$$

Solving the system, we find that the the stationary points of f are $(0,0,0)$, $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ and $(-1,1,-1)$.

II. We determine the second order partial derivatives of f and the Hessian matrix. We have

$$\begin{aligned}
f''_{xx}(x,y,z) &= 2, & f''_{xy}(x,y,z) &= f''_{yx}(x,y,z) = -2z,\\
f''_{yy}(x,y,z) &= 2, & f''_{yz}(x,y,z) &= f''_{zy}(x,y,z) = -2x,\\
f''_{zz}(x,y,z) &= 2, & f''_{zx}(x,y,z) &= f''_{xz}(x,y,z) = -2y.
\end{aligned}$$

Therefore,

$$H(f)(x,y,z) = \begin{pmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{pmatrix}.$$

III.1. Since

$$H(f)(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

we have $\Delta_1 = 2$, $\Delta_2 = 4$, $\Delta_3 = 8$, hence $(0,0,0)$ is a local minimum.

III.2. Since

$$H(f)(1,1,1) = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix},$$

we have $\Delta_1 = 2$, $\Delta_2 = 0$, hence the Sylvester theorem cannot be applied. We determine the second order differential of f at the stationary point $(1,1,1)$. This is the quadratic form

$$\mathrm{d}^2 f(1,1,1)(h_1,h_2,h_3) = 2h_1^2 + 2h_2^2 + 2h_3^2 - 4h_1h_2 - 4h_2h_3 - 4h_3h_1.$$

Since $\mathrm{d}^2 f(1,1,1)(1,0,0) = 2 > 0$ and $\mathrm{d}^2 f(1,1,1)(1,1,1) = -6 < 0$, it follows that $\mathrm{d}^2 f(1,1,1)$ is an indefinite quadratic form. Consequently, $(1,1,1)$ is a saddle point.

Proceeding analogously, it is found that **(Homework)** the three remaining stationary points of f are saddle points, too.

d) The stationary points of f are **(Homework)** $(0,0)$, $(1,0)$ and $(-1,0)$, while the Hessian matrix is

$$H(f)(x,y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 \end{pmatrix}.$$

Since $H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$, we have $\Delta_1 = -4$, $\Delta_2 = 0$, hence the Sylvester theorem cannot be applied. The second order differential of f at $(0,0)$, $\mathrm{d}^2 f(0,0)(h_1, h_2) = -4h_1^2$ is a negative **semi**-definite quadratic form, hence it is useless in establishing the nature of $(0,0)$. Note that

$$f(0,y) = y^4 > 0 = f(0,0) \quad \text{for all } y \in \mathbb{R} \setminus \{0\},$$

hence $(0,0)$ cannot be a local maximum for f. On the other hand, since

$$f(x,0) = x^2(x^2 - 2) < 0 = f(0,0) \quad \text{for all } x \in (-\sqrt{2}, \sqrt{2}\,) \setminus \{0\},$$

it follows that $(0,0)$ cannot be a local minimum for f. In conclusion, $(0,0)$ is a saddle point.

Since $H(f)(1,0) = H(f)(-1,0) = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$, we have $\Delta_1 = 8$, $\Delta_2 = 0$, hence the Sylvester theorem cannot be applied even now. The second order differential of f at $(1,0)$ and at $(-1,0)$,

$$\mathrm{d}^2 f(1,0)(h_1, h_2) = \mathrm{d}^2 f(-1,0)(h_1, h_2) = 8h_1^2$$

is a positive **semi**-definite quadratic form, hence it is useless in establishing the nature of of the points $(1,0)$ and $(-1,0)$. Note that

$$f(x,y) = (x^2 - 1)^2 + y^4 - 1 \geq -1 = f(1,0) = f(-1,0)$$

for all $(x,y) \in \mathbb{R}^2$, hence $(1,0)$ and $(-1,0)$ are global minimum points for f.