Differential Calculus. Part 2

1 The Differential for Branch Functions

This section contains examples of computing the differential function attached to **real** functions of **vector variable**, so the codomain is $\mathbb{R}$. Unlike the exercises proposed for study in Seminar 6, the functions considered now are defined on branches. On the interior of each branch, the study goes smooth, but at the meeting point of the branches, the differentiability might suffer. In examples it is easy not notice all possible cases.

This is actually the place where we have to apply the entire algorithm for analyzing the differentiability of a function at point. Let us once more recall the Algorithm:

Algorithm fornReal Functions of Vector Variables

$$\begin{cases} A \subseteq \mathbb{R}^n \\ a \in intA \\ f : A \to \mathbb{R} \end{cases}$$

Step 1: Study if f has partial derivatives at a with respect to all variables.

$$\begin{cases} YES & \to \text{ go to Step 2} \\ NO & \to \text{STOP the function does not have a differential at a} \end{cases}$$

In fact, we have to determine

$$\frac{\partial f}{\partial x_j}(a) \quad \forall j \in \{1, ..., n\}.$$

Step 2: Study the limit

$$l = \lim_{h \to 0_n} \frac{1}{\|h\|} \Big\{ f(a+h) - f(a) - \langle h, \nabla f(a) \rangle \Big\}. \tag{1}$$

Recall that $a = (a_1, ..., a_n)$ and $h = (h_1, ..., h_n) \in \mathbb{R}^n$

$$a + h = (a_1 + h_1, ..., a_n + h_n) \quad \text{and} \quad \|h\| = \sqrt{h_1^2 + ... + h_n^2}.$$

Moreover, the gradient of a function at a point is the vector composed of all of its partial derivatives at that point. Thus

$$\langle h, \nabla f(a) \rangle = h_1 \cdot \frac{\partial f}{\partial x_1}(a) + ... + h_n \cdot \frac{\partial f}{\partial x_n}(a) = \sum_{j=1}^{n} h_j \cdot \frac{\partial f}{\partial x_j}(a).$$

This means that the limit (1) is actually

$$l = \lim_{h \to 0_n} \frac{f(a_1 + h_1, ..., a_2 + h_2) - f(a_1, ..., a_n) - \sum_{j=1}^{n} h_j \cdot \frac{\partial f}{\partial x_j}(a)}{\sqrt{h_1^2 + ... + h_n^2}}. \tag{2}$$

We encounter the following cases

$$\begin{cases} l = 0 & \to \text{ go to Step 3} \\ \textit{otherwise} & \to \text{STOP the function does not have a differential at a} \end{cases}$$

Step 3: We are in the case when $l = 0$. This means that the function f is differentiable at the point a. Its differential is the linear function

$$df(a) : \mathbb{R}^n \to \mathbb{R}$$

defined by

$$df(a)(h) = \langle h, \nabla f(a) \rangle = \sum_{j=1}^{n} h_j \cdot \frac{\partial f}{\partial x_j}(a), \quad \forall h \in \mathbb{R}^n .$$

Example 1.1: Study the differentiability of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Solution:

We notice that the obvious problem appears at the point $(0, 0)$. Let us consider the set

$$A := \mathbb{R}^2 \setminus \{(0, 0)\}$$

This set is open. We may compute the partial derivatives of f with respect to both x and y at each random point. Let $(x, y) \in A$ be randomly chosen. Then

$$\frac{\partial f}{\partial x}(x, y) = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2}$$

and

$$\frac{\partial f}{\partial y}(x, y) = -\frac{y^4 + 3x^2y^2 + 2x^3y}{(x^2 + y^2)^2}.$$

These functions are continuous, therefore, **f is differentiable on A.**

The following step is to study the existence of the partial derivatives at $(0, 0)$, starting with respect to x:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \to 0} \frac{\frac{x^3 - 0}{x^2 + 0}}{x - 0} = 1 \in \mathbb{R} .$$

This means that f has a partial derivative at $(0, 0)$ with respect to x.

Then we study the existence of the partial derivatives at $(0,0)$ with respect to y:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{x\to 0} \frac{f(0,y)-f(0,0)}{x-0} = \lim_{y\to 0} \frac{\frac{0-y^3}{0+y^2}}{y-0} = \frac{-y^3}{y^3} = -1 \in \mathbb{R}\,.$$

This means that f has a partial derivative at $(0,0)$ with respect to y.

We follow now the steps of the Algorithm from Section 4 of Seminar 6. We are currently at **Step 2**. Recall that **if the function were is differentiable at $(0,0)$, the differential would be the function**

$$df(0,0) : \mathbb{R}^2 \to \mathbb{R},$$

defined by

$$df(0,0)(h_1,h_2) = \langle (h_1,h_2,), \nabla f(0,0)\rangle = h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0), \quad \forall (h_1,h_2)\in\mathbb{R}^2\,.$$

WE DO NOT KNOW IF f IS DIFFERENTIABLE AT $(0,0)$, SO

In order to study the differentiability of f at $(0,0)$ we have to study the limit

$$l = \lim_{(h_1,h_2)\to(0,0)} \frac{f\Big((0,0)+(h_1,h_2)\Big) - f(0,0) - \Big(h_1\frac{\partial f}{\partial x}(0,0) + h_2\frac{\partial f}{\partial y}(0,0)\Big)}{\|(h_1,h_2)-(0,0)\|}.$$

In order for the things to be simpler to analyse, we denote by

$$\omega(h_1,h_2)$$

the value of a function at a point $(h_1,h_2)\in\mathbb{R}^2$ generated by the limit that we have to compute. So

$$\begin{aligned}\omega(h_1,h_2) &= \frac{f(h_1,h_2) - f(0,0) - h_1\frac{\partial f}{\partial x}(0,0) - h_2\frac{\partial f}{\partial y}(0,0)}{\sqrt{h_1^2+h_2^2}}\\ &= \frac{\dfrac{h_1^3-h_2^3}{h_1^2+h_2^2} - h_1 + h_2}{\sqrt{h_1^2+h_2^2}} = \frac{h_1^2h_2 - h_1h_2^2}{(h_1^2+h_2^2)^{3/2}}\,.\end{aligned}$$

We show that

$$\nexists \lim_{(h_1,h_2)\to(0,0)} \omega(h_1,h_2),$$

with the help of sequences. So we consider the sequences, having as general terms, for all $k\in\mathbb{N}$

$$a_k = \left(\frac{1}{k},\frac{1}{k}\right) \quad \text{and} \quad b_k = \left(\frac{2}{k},\frac{1}{k}\right).$$

We have that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = (0,0),$$

but

$$\lim_{k\to\infty} \omega\left(\frac{1}{k}, \frac{1}{k}\right) = 0 \quad \text{și} \quad \lim_{k\to\infty} \omega\left(\frac{2}{k}, \frac{1}{k}\right) = \frac{2}{5\sqrt{5}} \neq 0.$$

so

$$\lim_{k\to\infty} \omega(a_k) \neq \lim_{k\to\infty} \omega(b_k).$$

Therefore

$$\nexists \lim_{(h_1,h_2)\to(0,0)} \omega(h_1, h_2)$$

so the limit that we are looking for does not exist. This means that **the function f is not differentiable at $(0,0)$.**

In conclusion the function f is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Example 1.2: **Study the differentiability of the function**

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} x^{\frac{4}{3}} \sin \frac{y}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Solution:

To begin with, we see that the problem is this time represented by a large set..., not just the point $(0,0)$, like it was the case in the other exercise. Set now

$$B := \{(x,y) \in \mathbb{R}^2 : x \neq 0\}.$$

Is is an open set, on which f has partial derivatives with respect to both x and y at each given point (x,y) randomly chosen.

$$\frac{\partial f}{\partial x}(x,y) = \frac{4}{3} x^{1/3} \sin \frac{y}{x} - x^{-2/3} \cos \frac{y}{x}$$

and

$$\frac{\partial f}{\partial y}(x,y) = x^{1/3} \cos \frac{y}{x}$$

These two partial derivatives are continuous functions on B, this means that the function f is differentiable on B. (We are not specifically asked to write the expression of the differential function).

Part 2 We are left to study the differentiablity of f on $\mathbb{R} \setminus B$. Consider a random point in this set,

$$(0,a) \text{ with } a \in \mathbb{R}.$$

First we analyze the partial derivatives. For $\frac{\partial f}{\partial x}(0,a)$ we have to analyze

$$\lim_{x\to 0}\frac{f(x,a)-f(0,a)}{x-0}=\lim_{x\to 0}\frac{x^{4/3}\sin\dfrac{a}{x}}{x}=\lim_{x\to 0}x^{1/3}\sin\frac{a}{x}=0$$

while for $\frac{\partial f}{\partial y}(0,a)$ we have to analyze

$$\lim_{y\to a}\frac{f(0,y)-f(0,a)}{y-a}=\lim_{y\to a}\frac{0}{y-a}=0,$$

We arrive at the conlusion that both partial derivatives at $(0,a)$ exist, and

$$\frac{\partial f}{\partial x}(0,a)=\frac{\partial f}{\partial y}(0,a)=0.$$

We have arrived at the **Step 2** of the Algorithm, in which we analyze

$$\ell \;=\; \lim_{(h_1,h_2)\to(0,0)}\frac{f(0+h_1,a+h_2)-f(0,a)-h_1\frac{\partial f}{\partial x}(0,a)-h_2\frac{\partial f}{\partial y}(0,a)}{\sqrt{h_1^2+h_2^2}}$$

Once again we make use of the notation ω the function whose limit we have to compute now at $(0,a)$, thus we are interested in

$$\lim_{(h_1,h_2)\to(0,0)}\omega(h_1,h_2)$$

with

$$\omega(h_1,h_2)=\frac{f(h_1,a+h_2)}{\sqrt{h_1^2+h_2^2}}.$$

In this case **we will prove that f is indeed differentiable at $(0,a)$**, so the limit has to be 0. We make use of the sandwich theorem. We distinguish two cases:

Case 1: $h_1\neq 0$, when

$$\begin{aligned}|\omega(h_1,h_2)| &= \frac{|h_1|^{4/3}\left|\sin\dfrac{a+h_2}{h_1}\right|}{\sqrt{h_1^2+h_2^2}}\le\frac{|h_1|^{4/3}}{\sqrt{h_1^2+h_2^2}}=\frac{|h_1|}{\sqrt{h_1^2+h_2^2}}|h_1|^{1/3}\\ &\le |h_1|^{1/3}.\end{aligned}$$

Case 2: $h_1 = 0$, when $f(0, a + h_2) = 0$.

From the Cases 1 and 2, we conclude that

$$\|\omega(h_1, h_2)\| \leq \|h_1\|^{\frac{1}{3}}, \forall (h_1, h_2) \in \mathbb{R}^2$$

Therefore,

$$\lim_{(h_1,h_2)\to(0,0)} \|\omega(h_1, h_2)\| \leq \lim_{(h_1,h_2)\to(0,0)} \|h_1\|^{\frac{1}{3}} = 0$$

So, the limit l of the **Step 2** of the Algorithm exists, and is equal to 0. This means that **the function f is differentiable at $(0, a)$ and the differential is the fucntion**

$$df(0, a) : \mathbb{R}^2 \to \mathbb{R}$$

defined by

$$df(0, a)(h_1, h_2) = \langle (h_1, h_2), \nabla f(0, a) \rangle = h_1 \cdot 0 + h_2 \cdot 0 = 0, \quad \forall (h_1, h_2) \in \mathbb{R}^2 .$$

In conclusion, f is differentiable on $\mathbb{R}^2$.

Example 1.3: Study the differentiability of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Solution:

We notice that the obvious problem appears at the point $(0, 0)$. Let us consider the set

$$A := \mathbb{R}^2 \setminus \{(0, 0)\}$$

This set is open. We may compute the partial derivatives of f with respect to both x and y at each random point. Let $(x, y) \in A$ be randomly chosen. Then

$$\frac{\partial f}{\partial x}(x, y) = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

and

$$\frac{\partial f}{\partial y}(x, y) = \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2}.$$

These functions are continuous, therefore, **f is differentiable on A**.

The following step is to study the existence of the partial derivatives at $(0, 0)$, starting with respect to x:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x\to 0} \frac{\frac{x \cdot 0(x-0)-0}{x^2+0}}{x - 0} = \lim_{x\to 0} \frac{0}{x^3} = \lim_{x\to 0} 0 = 0 \in \mathbb{R} .$$

This means that f has a partial derivative at $(0, 0)$ with respect to x.

Then we study the existence of the partial derivatives at $(0,0)$ with respect to y:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{x\to 0}\frac{f(0,y)-f(0,0)}{x-0} = \lim_{y\to 0}\frac{\frac{0\cdot y(0-y^3)}{0+y^2}}{y-0} = \lim_{y\to 0}\frac{0}{y^3} = \lim_{y\to 0} 0 = 0 \in \mathbb{R}\,.$$

This means that f has a partial derivative at $(0,0)$ with respect to y.

We follow now the steps of the Algorithm from Section 4 of Seminar 6. We are currently at **Step 2**. Recall that **if the function were is differentiable at** $(0,0)$**, the differential would be the function**

$$df(0,0) : \mathbb{R}^2 \to \mathbb{R},$$

defined by

$$df(0,0)(h_1,h_2) = \langle (h_1,h_2,), \nabla f(0,0)\rangle = h_1\frac{\partial f}{\partial x}(0,0)+h_2\frac{\partial f}{\partial y}(0,0) = h_1\cdot 0+h_2\cdot 0, \quad \forall (h_1,h_2)\in\mathbb{R}^2\,.$$

WE DO NOT KNOW IF f IS DIFFERENTIABLE AT $(0,0)$, SO

In order to study the differentiability of f at $(0,0)$ we have to study the limit

$$l = \lim_{(h_1,h_2)\to(0,0)} \frac{f\Big((0,0)+(h_1,h_2)\Big) - f(0,0) - \Big(h_1\frac{\partial f}{\partial x}(0,0)+h_2\frac{\partial f}{\partial y}(0,0)\Big)}{\|(h_1,h_2)-(0,0)\|}.$$

In order for the things to be simpler to analyse, we denote by

$$\omega(h_1,h_2)$$

the value of a function at a point $(h_1,h_2)\in\mathbb{R}^2$ generated by the limit that we have to compute. So

$$\omega(h_1,h_2) = \frac{\frac{h_1\cdot h_2(h_1^2-h_2^2)}{h_1^2+h_2^2} - 0 - 0}{\sqrt{h_1^2+h_2^2}} = \frac{h_1\cdot h_2(h_1^2-h_2^2)}{\Big(\sqrt{h_1^2+h_2^2}\Big)^{\frac{3}{2}}}.$$

We will make use once again of the sandwich theorem. First of all let us recall that by the classical mean theorem

$$\sqrt{h_1^2\cdot h_2^2} \le \frac{h_1^2+h_2^2}{2} \iff \frac{1}{h_1^2+h_2^2} \le \frac{1}{2}|h_1h_2|.$$

Moreover, it is clear that

$$\frac{|h_1|}{\sqrt{h_1^2+h_2^2}} \le 1 \quad \text{and} \quad \frac{|h_2|}{\sqrt{h_1^2+h_2^2}} \le 1.$$

This is why

$$|\omega(h_1,h_2)| = \frac{|h_1|}{\sqrt{h_1^2+h_2^2}}\cdot\frac{|h_2|}{\sqrt{h_1^2+h_2^2}}\cdot\frac{h_1^2-h_2^2}{\sqrt{h_1^2+h_2^2}} \le \frac{h_1^2-h_2^2}{\sqrt{h_1^2+h_2^2}}$$

$$\leq \frac{h_1^2}{\sqrt{h_1^2+h_2^2}} \leq h_1^2 \cdot \sqrt{\frac{1}{2}|h_1 h_2|}.$$

Due to the fact that

$$\lim_{(h_1,h_2)\to 0} h_1^2 \cdot \sqrt{\frac{1}{2}|h_1 h_2|} = 0,$$

from the sandwich theorem we have that

$$\lim_{(h_1,h_2)\to 0} \omega(h_1,h_2) = 0.$$

So, the limit exists and is 0 in Step 2 of the algorithm. This means that the function f is differentiable at $(0,0)$

In conclusion the function f is differentiable on $\mathbb{R}^2$.

2 Partial Derivatives of Composed Functions.

Even tough the general theory is proved for vector spaces of random dimension, we will just analyze particular cases so, let us begin with the following example

Eample 2.1:

$$\begin{cases} f : \mathbb{R}^2 \to \mathbb{R}^3 \\ g : \mathbb{R}^3 \to \mathbb{R}^2 \quad g(x,y,z) = \left(x^2 - y + 2yz^2, z^3 e^{xy}\right), \quad \forall (x,y,z) \in \mathbb{R}^3 \\ F : \mathbb{R}^3 \to \mathbb{R}, \quad F(x,y,z) = (f \circ g)(x,y,z), \quad \forall (x,y,z) \in \mathbb{R}^3. \end{cases}$$

We will determine the partial derivatives of F with respect to x, y and z, respectively, in terms of the partial derivatives of f. Note that the actual expression of f is unknown, but it is assumed that it has partial derivatives with respect to all variables at all the points in its domain.

Let $(x,y,z) \in \mathbb{R}^3$ be a random point.

Step 1: The partial derivative with respect to x, i.e

$$\frac{\partial F}{\partial x}(x,y,z) = \frac{\partial (f \circ g)}{\partial x}(x,y,z).$$

Be aware of the fact that g is a vector function, of vector variable, so from its expression, since its codomain is R^2 it may be written as a pair of two real functions of vector variable, namely

$$g = (g_1, g_2), \quad \text{where} \quad g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$$

and

$$g_1(x,y,z) = x^2 - y + 2yz^2 \quad \text{and} \quad g_2(x,y,z) = z^3 e^{xy}.$$

Then

$$\frac{\partial (f\circ g)}{\partial x}(x,y,z) = \frac{\partial f}{\partial g_1}\big(g(x,y,z)\big)\cdot\frac{\partial g_1}{\partial x}(x,y,z) + \frac{\partial f}{\partial g_2}\big(g(x,y,z)\big)\cdot\frac{\partial g_2}{\partial x}(x,y,z).$$

In exercises, in order not to get too many thing to write, we might skip some details, and the formula above may be written as

$$\frac{\partial F}{\partial x} = \frac{\partial (f\circ g)}{\partial x}(x,y,z) = \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial x}.$$

With a similar reasoning, we deduce the partial derivatives with respect to y and z respectively,

$$\frac{\partial F}{\partial y} = \frac{\partial (f\circ g)}{\partial y}(x,y,z) = \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial y} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial y}$$

and

$$\frac{\partial F}{\partial z} = \frac{\partial (f\circ g)}{\partial x}(x,y,z) = \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial z} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial z}.$$

We see that we actually need the partial derivatives of g, with respect to all variables.

$$\frac{\partial g}{\partial x}(x,y,z) = \left(\frac{\partial g_1}{\partial x}(x,y,z), \frac{\partial g_2}{\partial x}(x,y,z)\right) = \left(2x, z^3e^{xy}\cdot y\right) = \left(2x, yz^3e^{xy}\right)$$

$$\frac{\partial g}{\partial y}(x,y,z) = \left(\frac{\partial g_1}{\partial y}(x,y,z), \frac{\partial g_2}{\partial y}(x,y,z)\right) = \left(-1+2z^2, z^3e^{xy}\cdot x\right) = \left(-1+2z^2, xz^3e^{xy}\right)$$

$$\frac{\partial g}{\partial z}(x,y,z) = \left(\frac{\partial g_1}{\partial z}(x,y,z), \frac{\partial g_2}{\partial z}(x,y,z)\right) = \left(4yz, 3z^2e^{xy}\right).$$

Coming back to F we conclude that

$$\frac{\partial F}{\partial x}(x,y,z) = \frac{\partial f}{\partial g_1}(g(x,y,z))\cdot 2x + \frac{\partial f}{\partial g_2}(g(x,y,z)\cdot yz^3e^{xy}$$

$$= 2x\frac{\partial f}{\partial g_1}\left(x^2-y+2yz^2, z^3e^{xy}\right) + yz^3e^{xy}\frac{\partial f}{\partial g_2}\left(x^2-y+2yz^2, z^3e^{xy}\right)$$

$$\frac{\partial F}{\partial y}(x,y,z) = \frac{\partial f}{\partial g_1}(g(x,y,z)) \cdot (-1+2z^2) + \frac{\partial f}{\partial g_2}(g(x,y,z) \cdot xz^3 e^{xy}$$

$$= (-1+2z^2)\frac{\partial f}{\partial g_1}\left(x^2-y+2yz^2, z^3e^{xy}\right) + xz^3e^{xy}\frac{\partial f}{\partial g_2}\left(x^2-y+2yz^2, z^3e^{xy}\right)$$

$$\frac{\partial F}{\partial z}(x,y,z) = \frac{\partial f}{\partial g_1}(g(x,y,z)) \cdot 4yz + \frac{\partial f}{\partial g_2}(g(x,y,z) \cdot 3z^2 e^{xy}$$

$$= 4yz\frac{\partial f}{\partial g_1}\left(x^2-y+2yz^2, z^3e^{xy}\right) + 3z^2e^{xy}\frac{\partial f}{\partial g_2}\left(x^2-y+2yz^2, z^3e^{xy}\right)$$

Here $\frac{\partial f}{\partial g_1}$ means the partial derivative of f with respect to the first variable, while $\frac{\partial f}{\partial g_2}$ means the partial derivative of f with respect to the second variable.

Sometimes, f is written as $f(u,v)$. Pay attention, it is not $f=(u,v)$, and instead of $\frac{\partial f}{\partial g_1}$ we may write

$$\frac{\partial f}{\partial u}$$

, and, instead of $\frac{\partial f}{\partial g_2}$ we may write

$$\frac{\partial f}{\partial v}$$

.

Example 2.2:

$$\begin{cases} f:\mathbb{R}^3 \to \mathbb{R} \\ \\ g:\mathbb{R}^2 \to \mathbb{R}^3 \quad g(x,y) = \left(-3x+2y, x^2+y^2, 2x^3-y^3\right), \quad \forall (x,y) \in \mathbb{R}^2 \\ \\ F:\mathbb{R}^2 \to \mathbb{R}, \quad F(x,y) = (f \circ g)(x,y), \quad \forall (x,y) \in \mathbb{R}^2 . \end{cases}$$

We will determine the partial derivatives of F with respect to x and y, respectively, in terms of the partial derivatives of f.
Note that the actual expression of f is unknown, but it is assumed that it has partial derivatives with respect to all variables at all the points in its domain.

Let $(x,y) \in \mathbb{R}^2$ be a random point.

Step 1: The partial derivative with respect to x, i.e

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial (f\circ g)}{\partial x}(x,y).$$

Be aware of the fact that g is a vector function, of vector variable, so from its expression, since its codomain is R^2 it may be written as a pair of two real functions of vector variable, namely

$$g = (g_1, g_2, g_3), \quad \text{where} \quad g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}$$

and

$$g_1(x,y) = -3x + 2y, \quad g_2(x,y) = x^2 + y^2 \quad \text{and} \quad g_3(x,y) = 2x^3 - y^3.$$

Then

$$\frac{\partial (f\circ g)}{\partial x}(x,y) = \frac{\partial f}{\partial g_1}\big(g(x,y,z)\big)\cdot\frac{\partial g_1}{\partial x}(x,y) + \frac{\partial f}{\partial g_2}\big(g(x,y)\big)\cdot\frac{\partial g_2}{\partial x}(x,y) + \frac{\partial f}{\partial g_3}\big(g(x,y)\big)\cdot\frac{\partial g_3}{\partial x}(x,y).$$

In exercises, in order not to get too many thing to write, we might skip some details, and the formula above may be written as

$$\frac{\partial F}{\partial x} = \frac{\partial (f\circ g)}{\partial x} = \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial x} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial x} + \frac{\partial f}{\partial g_3}\cdot\frac{\partial g_3}{\partial x}.$$

With a similar reasoning, we deduce the partial derivatives with respect to y and z respectively,

$$\frac{\partial F}{\partial y} = \frac{\partial (f\circ g)}{\partial y} = \frac{\partial f}{\partial g_1}\cdot\frac{\partial g_1}{\partial y} + \frac{\partial f}{\partial g_2}\cdot\frac{\partial g_2}{\partial y} + \frac{\partial f}{\partial g_3}\cdot\frac{\partial g_3}{\partial y}$$

We see that we actually need the partial derivatives of g, with respect to all variables.

$$\frac{\partial g}{\partial x}(x,y) = \left(\frac{\partial g_1}{\partial x}(x,y), \frac{\partial g_2}{\partial x}(x,y), \frac{\partial g_3}{\partial x}(x,y)\right) = \left(-3, 2x, 6x^2\right)$$

$$\frac{\partial g}{\partial y}(x,y) = \left(\frac{\partial g_1}{\partial y}(x,y), \frac{\partial g_2}{\partial y}(x,y), \frac{\partial g_3}{\partial y}(x,y)\right) = \left(2, 2y, -3y^2\right)$$

Coming back to F we conclude that

$$\frac{\partial F}{\partial x}(x,y) = \frac{\partial f}{\partial g_1}(g(x,y))\cdot(-3) + \frac{\partial f}{\partial g_2}(g(x,y)\cdot 2x + \frac{\partial f}{\partial g_3}(g(x,y)\cdot 6x^2$$

$$= -3\frac{\partial f}{\partial g_1}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right) + 2x\frac{\partial f}{\partial g_2}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right)$$

$$+6x^2\frac{\partial f}{\partial g_3}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right).$$

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial f}{\partial g_1}(g(x,y)) \cdot (2) + \frac{\partial f}{\partial g_2}(g(x,y) \cdot 2y + \frac{\partial f}{\partial g_3}(g(x,y) \cdot (-3y^2)$$
$$= 2\frac{\partial f}{\partial g_1}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right) + 2y\frac{\partial f}{\partial g_2}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right)$$
$$-3y^2\frac{\partial f}{\partial g_3}\left(-3x+2y, x^2+y^2, 2x^3-y^3\right).$$

Here $\frac{\partial f}{\partial g_1}$ means the partial derivative of f with respect to the first variable, while $\frac{\partial f}{\partial g_2}$ means the partial derivative of f with respect to the second variable, and $\frac{\partial f}{\partial g_3}$ means the partial derivative of f with respect to the second variable.

Sometimes, f is written as $f(u,v,w)$. Pay attention, it is not $f = (u,v,w)$, and instead of $\frac{\partial f}{\partial g_1}$ we may write
$$\frac{\partial f}{\partial u}$$
instead of $\frac{\partial f}{\partial g_2}$ we may write
$$\frac{\partial f}{\partial v}$$
and instead of $\frac{\partial f}{\partial g_3}$ we may write
$$\frac{\partial f}{\partial w}$$
.

Example 2.3: Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a differentiable function on $\mathbb{R}^3$, and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$F(x,y) = f(\cos x + \sin x, \sin x + \cos x, e^{x-y})$$
Knowing that $J(f)(1,1,1) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}$ determine $dF(\frac{\pi}{2}, \frac{\pi}{2})$.

Solution:

This exercise once again deals with the differentials of composed functions. First of all we have to know good all the entry data. Thus:
$$\begin{cases} & f : \mathbb{R}^3 \to \mathbb{R}^2 \\ & F : \mathbb{R}^2 \to \mathbb{R}^2 \\ g : \mathbb{R}^2 \to \mathbb{R}^3 & g(x,y) = (\cos x + \sin x, \sin x + \cos x, e^{x-y}) \\ & F = (f \circ g) \end{cases}$$

In order to solve the exercise we must acknowledge a result from the lecture, which states that the Jacobi matrix of composed differentiable functions satisfies the following equality

$$J(f\circ g)(a) = J(f)(g(a))\cdot J(g)(a), \tag{3}$$

where a is in the domain of g.

We have to do a detective work now :), but fortunately for us, we have all the date we need. Thus, the conclusion asks as to determine $J(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right)$, so our point is

$$a = \left(\frac{\pi}{2},\frac{\pi}{2}\right).$$

We notice that $g\left(\frac{\pi}{2},\frac{\pi}{2}\right) = (1,1,1)$. In order to fill in the expressions in (3),

$$J(f\circ g)\left(\frac{\pi}{2},\frac{\pi}{2}\right) = J(f)(1,1,1)\cdot J(f\circ g)\left(\frac{\pi}{2},\frac{\pi}{2}\right) = J(f)(1,1,)\cdot J(g)(\left(\frac{\pi}{2},\frac{\pi}{2}\right).$$

Since $J(f)(1,1,1)$ is already given, we just need $J(g)\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

Lucky for us, we have already determined this Martrix in Seminar 6, the last exercise. (for details, take a look at Seminar6 , just note that the function g was there denoted by f.)

The searched for Jacobi matrix is

$$J(g)\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Finally we are able to conclude that

$$J(f\circ g)\left(\frac{\pi}{2},\frac{\pi}{2}\right)\cdot J(f\circ g)\left(\frac{\pi}{2},\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}.$$

Having this Jacobi matrix, we are able to write the expression of the differential function associated to F at the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$, namely

$$d(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right):\mathbb{R}^2\to\mathbb{R}^2$$

$$d(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right)(h_1,h_2) = J(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right)\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}\cdot\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 3h_1-7h_2 \\ h_1-2h_2 \end{pmatrix},$$

for all $(h_1,h_2)\in\mathbb{R}^2$, differential written in its matrix form. In the vector form we have that

$$d(F)\left(\frac{\pi}{2},\frac{\pi}{2}\right)(h_1,h_2) = (3h_1-7h_2, h_1-2h_2)\quad \forall (h_1,h_2)\in\mathbb{R}^2.$$

Example 2.4: Let us consider the following:

$$\begin{cases} f:(0,\infty)\times(0,\infty)\to\mathbb{R} \ \textbf{ a differentiable function} \\ g:(0,\infty)\times\left(0,\frac{\pi}{2}\right)\to(0,\infty)\times(0,\infty), \\ g(\rho,\theta)=(\rho\cos\theta,\rho\sin\theta) \\ F:(0,\infty)\times\left(0,\frac{\pi}{2}\right)\to\mathbb{R} \qquad F=(f\circ g) \end{cases}$$

Determine the partial derivatives of F in terms of the partial derivatives of f. - Homework