Calculus on $\mathbb{R}^n$
Seminar 5 **2020**

Partial derivatives

We start by considering partial derivatives of real functions, of vector variable. The general context is the following:

$$\begin{cases} A \subseteq \mathbb{R}^n \\ a \in intA \\ f : A \to \mathbb{R} \end{cases}$$

Let $j \in \{1, ..., n\}$. The function f is said to be **partially differentiable** with respect to the variable x_j at the point a, if the following limit exists:

$$\exists \quad \lim_{x_j \to a_j} \frac{f(a_1, ..., a_{j-1}, \mathbf{x_j}, a_{j+1}, ..., a_n) - f(a_1, ..., a_n)}{x_j - a_j} \in \mathbb{R}\,.$$

If this limit exists, it is usually denoted by:

$$\frac{\partial f}{\partial x_j}(a),$$

and is called **the first-order partial derivative of f, with respect to the variable x_j, at a.**

When solving simple examples, we consider at most $n = 3$, case in which we have to analyze three different partial derivatives, usually denoted by

$$\frac{\partial f}{\partial x}(a), \quad \frac{\partial f}{\partial y}(a), \quad \frac{\partial f}{\partial z}(a).$$

(And in most cases instead of a we have (x, y, z).)

When there is no obvious problem with the interior of the set A, we compute very simply the partial derivative with respect (let's say) x by considering the other variable as constants. Thus, we may sometimes even use the notation

$$\frac{\partial f}{\partial x}(a) = (f)'_x(a).$$

The same happens for y and z, respectively.

The procedure may continue inductively, so the second order partial derivatives, mean actually computing the partial derivatives of the first-order partial derivatives (of course, if these functions are differentiable).

Starting for example with the first order partial derivative with respect to y, we obtain three new functions, namely

$$\frac{\partial^2 f}{\partial x \partial y}(a) = ((f)'_y))'_x(a), \quad \frac{\partial^2 f}{\partial y \partial y}(a) = ((f)'_y)'_y(a) = \frac{\partial^2 f}{\partial y^2}(a)$$

$$\text{and} \quad \frac{\partial^2 f}{\partial z \partial y}(a) = ((f)'_y))'_z(a).$$

There are cases when the first order partial derivatives may be computed directly, but also, there are cases when we have to go back to the limit in order ot get a correct conclusion, (so, for let's say z), we have to see that

$$\frac{\partial f}{\partial z}(a_1, a_2, a_3) = \lim_{z \to a_3} \frac{f(a_1, a_2, z) - f(a_1, a_2, a_3)}{z - a_3}.$$

When we have a real function of vector variable $f : A \to \mathbb{R}$, and all of the first order partial derivatives exist on the entire $intA$, one may consider the **Gradient** of the function,

$$\nabla f(a) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(z), \frac{\partial f}{\partial z}(a) \right).$$

Excercise 1:

Let $f : \mathbb{R}^* \times R^2 \to R$ be a function defined by

$$f(x, y, z) = \frac{z^2 e^y}{x}.$$

Determine:

a) all first-order partial derivatives of f;

b) all second-order partial derivatives of f;

c) Consider the vectors $u = \nabla f(1, 0, 2)$ and $v = \nabla f(2, 1, 1)$. Determine $< u, v >$.

Solution:

a) Let $(x, y, z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y,z) = -\frac{z^2e^y}{x^2},\ \frac{\partial f}{\partial y}(x,y,z) = \frac{z^2e^y}{x} \text{ and } \frac{\partial f}{\partial z}(x,y,z) = \frac{2ze^y}{x}.$$

b) Let $(x,y,z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2\frac{z^2e^y}{x^3},\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = \frac{z^2e^y}{x} \text{ and } \frac{\partial^2 f}{\partial z^2}(x,y,z) = \frac{2e^y}{x}.$$

The mixed second-order partial derivatives are

$$\frac{\partial^2 f}{\partial y\partial x}(x,y,z) = -\frac{z^2e^y}{x^2} = \frac{\partial^2 f}{\partial x\partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial y\partial z}(x,y,z) = \frac{2ze^y}{x} = \frac{\partial^2 f}{\partial z\partial y}(x,y,z),$$

$$\frac{\partial^2 f}{\partial z\partial x}(x,y,z) = -\frac{2ze^y}{x^2} = \frac{\partial^2 f}{\partial x\partial z}(x,y,z).$$

c) Determine the vectors $u := \nabla f(1,0,2)$, $v := \nabla f(2,1,1)$ and their scalar product $\langle u,v\rangle$. Let us first recall the formulation of the Gradient of a vector function. Let $(x,y,z) \in \mathbb{R}^* \times \mathbb{R}^2$ be arbitrarily chosen. Then

$$\nabla f = \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right).$$

For the given vectors we obtain

$$u = \nabla f(1,0,2) = \left(\frac{\partial f}{\partial x}(1,0,2), \frac{\partial f}{\partial y}(1,0,2), \frac{\partial f}{\partial z}(1,0,2)\right) = \left(-\frac{2^2e^0}{1^2}, \frac{2^2e^0}{1}, \frac{2ze^0}{1}\right)$$

$$= (-4,4,4)$$

and

$$u = \nabla f(2,1,1) = \left(\frac{\partial f}{\partial x}(2,1,1), \frac{\partial f}{\partial y}(2,1,1), \frac{\partial f}{\partial z}(2,1,1)\right) = \left(-\frac{1^2e^1}{2^2}, \frac{1^2e^1}{2}, \frac{2e^1}{2}\right)$$

$$= (-\frac{1}{4}e, \frac{1}{2}e, e).$$

The required scalar product is

$$\langle u,v\rangle = -4\cdot\left(-\frac{1}{4}e\right) + 4\cdot\frac{1}{2}e + 4\cdot e = 7e.$$

Exercise 2: For the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x,y) = \begin{cases} \frac{x^4-y^4}{2(x^4+y^4)} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

study

a) the partial derivatives of f with respect to both variables x and y at $0_2 = (0,0)$.

b) the continuity at 0_2.

Solution:

a) First we analyze the partial derivative of f with respect to x at 0_2:

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{\frac{x^4-0}{2(x^4)+0} - 0}{x} = \lim_{x\to 0} \frac{1}{2x}.$$

Since $\lim\limits_{\substack{x\to 0\\x<0}} \frac{1}{2x} = -\infty$ and $\lim\limits_{\substack{x\to 0\\x>0}} \frac{1}{2x} = +\infty$, we conclude that $\lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0}$ does not exist. Hence f is not partially differentiable with respect to x at 0_2.

Then we analyze the partial derivative of f with respect to y at 0_2:

$$\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{\frac{0-y^4}{2(0+y^4)} - 0}{y} = \lim_{y\to 0} -\frac{1}{2y}.$$

Since $\lim\limits_{\substack{y\to 0\\y<0}} -\frac{1}{2y} = +\infty$ and $\lim\limits_{\substack{y\to 0\\y>0}} -\frac{1}{2y} = -\infty$ we conclude that $\lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{x-0}$ does not exist. Hence f is not partially differentiable with respect to y at 0_2.

b) We will prove that f is not continuous at 0_2. Assume by contradiction that f is continuous at 0_2. for every sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^2$, with $\lim\limits_{k\to\infty} x^k = 0_2$, one should have that $\lim\limits_{k\to\infty} f(x^k) = f(0_2)$.

If we consider the sequence with the general term $a^k = (\frac{1}{k}, 0)$, then $\lim\limits_{k\to\infty} a^k = 0_2$ and $\lim\limits_{k\to\infty} f(a^k) = \lim\limits_{k\to\infty} \frac{\left(\frac{1}{k}\right)^4 - 0}{2\left(\left(\frac{1}{k}\right)^4 + 0\right)} = \lim\limits_{k\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0 =$

$f(0_2)$. Hence we have obtained a contradiction. Thus f is not continuous at 0_2.

Exercise 3: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq 0_2 \\ 0, & (x,y) = 0_2 \end{cases}$$

a) Show that f has partial derivatives (with respect to both variables) on $\mathbb{R}^2$, and determine both first-order partial derivatives of f.

b) Show that f is not continuous at 0_2 (although f is partially differentiable at 0_2).

Solution:

a) We first study the partial derivative f at 0_2 with respect to x:

$$\lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{\frac{x\cdot 0}{x^2+0^2}}{x} = \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0.$$

Therefore

$$\frac{\partial f}{\partial x}(0,0) = 0,$$

thus f has a partial derivative 0_2 with respect to x.

Then we analyze the partial derivative of f with respect to y at 0_2:

$$\lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{\frac{0\cdot y}{0^2+y^2}}{y} = \lim_{y\to 0} 0 = 0.$$

Therefore

$$\frac{\partial f}{\partial y}(0,0) = 0,$$

thus f has a partial derivative with respect to y at 0_2.

If $(x,y) \in \mathbb{R}^2 \setminus \{0_2\}$ then the partial derivatives of f at $(x,y)$with respect to x and y, respectively, are

$$\frac{\partial f}{\partial x}(x,y) = \frac{y\left(x^2+y^2\right) - 2x^2 y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{x\left(x^2+y^2\right) - 2xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

Therefore

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{y(y^2-x^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2 \end{cases} \text{ and } \frac{\partial f}{\partial y}(x,y) = \begin{cases} \frac{x(x^2-y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq 0_2 \\ 0 & \text{if } (x,y) = 0_2 \end{cases}.$$

b)The existence of both partial derivatives (sometimes called Partial differentiability at a point) **does not imply** continuity at that point (this fact being in contrast to what we know from real functions of one variable). In this particular example the function f has partial derivatives with respect to both x and y (is partially differentiable) at 0_2 but is not continuous at 0_2. To serve our purpose we use the characterization of the continuity with sequences.

Consider the sequence $(x_k, y_k)_{k\in\mathbb{N}^*}$ with the general term defined by

$$(x_k, y_k) = \left(\frac{1}{k}, \frac{1}{k}\right), k \in \mathbb{N}^*.$$

Then

$$\lim_{k\to\infty}(x_k,y_k) = 0_2 \text{ while } \lim_{k\to\infty} f(x_k,y_k) = \lim_{k\to\infty} \frac{\frac{1}{k}\cdot\frac{1}{k}}{\frac{1}{k^2}+\frac{1}{k^2}} = \frac{1}{2} \neq f(0_2).$$

So, f is not continuous at 0_2.

Exercise 4: Determine all first-order and all second-order partial derivatives of the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = z^2\cos(x-y).$$

Solution: Let $(x,y,z) \in \mathbb{R}^3$ be arbitrarily chosen. Then

$$\frac{\partial f}{\partial x}(x,y,z) = -z^2\sin(x-y),\ \frac{\partial f}{\partial y}(x,y,z) = -z^2\sin(x-y)(x-y)'_y = z^2\sin(x-y) \text{ and}$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z\cos(x-y).$$

Moreover, we have

$$\frac{\partial^2 f}{\partial x^2}(x,y,z) = -z^2\cos(x-y),\ \frac{\partial^2 f}{\partial y^2}(x,y,z) = -z^2\cos(x-y)$$

and

$$\frac{\partial^2 f}{\partial z^2}(x,y,z) = 2\cos(x-y).$$

The mixed second-order partial derivatives are

$$\frac{\partial^2 f}{\partial y\partial x}(x,y,z) = z^2\cos(x-y) = \frac{\partial^2 f}{\partial x\partial y}(x,y,z),$$
$$\frac{\partial^2 f}{\partial y\partial z}(x,y,z) = 2z\sin(x-y) = \frac{\partial^2 f}{\partial z\partial y}(x,y,z),$$
$$\frac{\partial^2 f}{\partial z\partial x}(x,y,z) = -2z\sin(x-y) = \frac{\partial^2 f}{\partial x\partial z}(x,y,z).$$

Exercise 5: Determine the gradient of the function f at the point a in the following cases:

a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=e^{-x}\sin(x+2y)$ **and** $a=\left(0,\frac{\pi}{2}\right)$;

b) $f:\mathbb{R}^3\to\mathbb{R}$, $f(x,y,z)=(x-y)\cos(\pi z)$ **and** $a=\left(1,0,\frac{1}{2}\right)$.

Solution:

a) Let $(x,y)\in\mathbb{R}^2$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y) = -e^{-x}\sin(x+2y)+e^{-x}\cos(x+2y)$ and $\frac{\partial f}{\partial y}(x,y) = 2e^{-x}\cos(x+2y)$. Thus

$$\nabla f\left(0,\frac{\pi}{4}\right) = \left(\frac{\partial f}{\partial x}\left(0,\frac{\pi}{4}\right), \frac{\partial f}{\partial y}\left(0,\frac{\pi}{4}\right)\right)$$
$$= \left(-e^{-0}\sin\left(0+\frac{\pi}{2}\right)+e^{-0}\cos\left(0+\frac{\pi}{2}\right), 2e^{-0}\cos\left(0+\frac{\pi}{2}\right)\right) = (-1,0).$$

b) Let $(x,y,z)\in\mathbb{R}^3$ be arbitrarily chosen. Then $\frac{\partial f}{\partial x}(x,y,z) = \cos\pi z$, $\frac{\partial f}{\partial x}(x,y,z) = -\cos\pi z$ and $\frac{\partial f}{\partial x}(x,y,z) = -(x-y)\pi\sin\pi z$. Thus

$$\nabla f\left(1,0,\frac{1}{2}\right) = \left(\cos\frac{\pi}{2}, -\cos\frac{\pi}{2}, -(1-0)\pi\sin\frac{\pi}{2}\right) = (0,0,-\pi).$$

For further details, take a look at the solved exercises on Seminar5-ttrif

From there, **without df** (which is the differential will be dealt with in our Seminar 6), please try to **solve individually all the exercises from 1 to 10.**