Calculus on $\mathbb{R}^n$
Seminar 4 **2020**

Limits and continuity

Definition: Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a function. Let $a \in A'$ and $b \in \mathbb{R}^m$. The point b is said to be the limit of the function f at a, if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A \backslash \{a\}, \text{ with } \|x - a\| < \delta, \text{ it holds } \|f(x) - f(a)\| < \varepsilon.$$

The usual notation for this is

$$\lim_{x \to a} f(x) = b.$$

During the first semester, we used a sequential definition, and we gave the ε as a characterization theorem. This term is the other way arround. Now the definition is with ε, and in practice, especially when we try to prove that the limit does not exist, we use the following sequential characterization of limits:

$$\lim_{x \to b} f(x) = b \Longleftrightarrow \forall (x_k) \subseteq A \backslash \{a\}, \text{ with } \lim_{k \to \infty} x_k = a \quad \text{ it holds } \lim_{k \to \infty} f(x_k) = b.$$

Obviously, in order for us to prove that the limit at a certain $a \in A$ does not exist, we may emphasize two sequences (a_k) and (b_k) from $A \backslash \{a\}$ such that

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = a \text{ but } \lim_{k \to \infty} f(a_k) \neq \lim_{k \to \infty} f(b_k).$$

Recall that during the first semester, we used for the common limit $n \to \infty$. This term we usually have $k \to \infty$. This is due to the fact that we work in $\mathbb{R}^n$, so n is now just a fixed number.

Exercise 1: Compute

$$\lim_{(x,y) \to (0,2)} \frac{\sin(xy)}{x}.$$

Solution: We use for the solution, the sequential characterization.

Let $(x_k, y_k) \subseteq \mathbb{R}^2 \setminus \{0, 2)\}$ be a sequence such that

$$\lim_{k\to\infty}(x_k, y_k) = (0, 2).$$

This is equivalent to the fact that $\lim_{k\to\infty} x_k = 0$ and $\lim_{k\to\infty} y_k = 2$. Then

$$\lim_{k\to\infty}\frac{\sin(x_k y_k)}{x_k} = \lim_{k\to\infty}\frac{\sin(x_k y_k)}{x_k y_k}\cdot y_k = 1\cdot\lim_{k\to\infty} y_k = 1\cdot 2 = 2.$$

This is due to the fact that $\lim\limits_{k\to\infty} x_k y_k = 0\cdot 2 = 0$, and obviously $\lim\limits_{k\to\infty}\frac{\sin(x_k y_k)}{x_k y_k} = 1$. Since the sequence (x_k, y_k) was chosen randomly, the proof is complete.

Excercise 2: Compute the following limits

$$a)\ \lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{1+x^2+y^2}-1}$$

$$b)\ \lim_{(x,y)\to(0,0)}\frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4}$$

Solution

We treat both $a)$ and $b)$ with $(x, y) \to (0, 0)$ directly.

$$a)\ \lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{1+x^2+y^2}-1} = \lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)(\sqrt{1+x^2+y^2}+1)}{x^2+y^2} =$$

$$\lim_{(x,y)\to(0,0)}\sqrt{1+x^2+y^2}+1 = 2.$$

$b)$ Let us recall the classical mean inequality. Given $a_1, ..., a_n > 0$ then

$$\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 ... a_n} \leq \frac{a_1+...+a_n}{n} \leq \sqrt{\frac{a_1^2+...+a_n^2}{n}}.$$

We apply the last for x^2 and y^2. Thus

$$\frac{x^2+y^2}{2} \leq \sqrt{\frac{x^4+y^4}{2}}.$$

We put this to the second power and exchange factors. Thus we get

$$\frac{1}{x^4+y^4} \leq \frac{2}{(x^2+y^2)^2}.$$

If we multiply this by $e^{-\frac{1}{x^2+y^2}}$, and get

$$\frac{e^{-\frac{1}{x^2+y^2}}}{x^4+y^4} \leq \frac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2}.$$

Due to the fact that $(x, y) \to (0, 0)$ we may substitute

$$\frac{1}{x^2+y^2} \text{ to } t, \text{ and } t \to \infty.$$

So

$$\frac{2e^{-\frac{1}{x^2+y^2}}}{(x^2+y^2)^2} = \lim_{t\to\infty} \frac{2e^{-t}}{\frac{1}{t^2}} = \lim_{t\to\infty} \frac{2t^2}{e^t} = 0.$$

Excercise 3: Study

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Solution:

This is actually an exercise where we show that the limit does not exist. The idea behind solving such exercising is passing the problem stated in two variables (x, y) to a problem in just one variable. Since $(x, y) \to (0, 0)$ we try to consider some particular instances.

The solution relies on emphasising a sequence $(x_k, y_k) \to (0, 0)$ for which $\lim_{k\to\infty} f(x_k, y_k)$ does not exists.

Consider a random sequence $(x_k) \subset \mathbb{R}$ such that $\lim_{k\to\infty} x_k = 0.$. Moreover, take $m \in \mathbb{R} \setminus \{0\}$, and consider

$$y_k = m \cdot x_k, \quad \forall k \in \mathbb{N}. \text{ Thus } \lim_{k\to\infty} y_k = 0,$$

and therefore

$$\lim_{k\to\infty} (x_k, y_k) = (0, 0).$$

We get

$$\lim_{k\to\infty} f(x_k, y_k) = \lim_{k\to\infty} \frac{x_k \cdot m x_k}{x_k^2 + m^2 x_k^2} = \frac{m}{1+m^2}.$$

We may assign to m different, values, and get for $m = 1$ and for $m = 2$

$$\lim_{k\to\infty} f(x_k, 1 \cdot x_k) = \frac{1}{2} \neq \frac{2}{5} = \lim_{k\to\infty} f(x_k, 2 \cdot x_k).$$

Thus the limit does not exists.

Excercise 4: Study

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

Solution:

Let $(x_k, y_k) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be s.t.

$$\lim_{k\to\infty} (x_k, y_k) = (0,0).$$

We notice that

$$0 \le \left|\frac{x_k^3 + y_k^3}{x_k^2 + y_k^2}\right| \le \frac{|x_k^3|}{x_k^2 + y_k^2} + \frac{|y_k^3|}{x_k^2 + y_k^2} \le\le \left|\frac{x_k^3 + y_k^3}{x_k^2 + y_k^2}\right| \le |x_k|\frac{x_k^2}{x_k^2 + y_k^2} + |y_k|\frac{y_k^2}{x_k^2 + y_k^2} \le |x_k| + |y_k|.$$

We have that

$$\lim_{k\to\infty} (x_k, y_k) = (0,0) \implies \lim_{k\to\infty} |x_k| + |y_k| = 0.$$

Then, according to the sandwich theorem

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} \le \lim_{k\to\infty} |x_k| + |y_k| = 0.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0.$$

Excercise 5: Study

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{xy}.$$

Solution:

Obviously, the function for which we study the limit is

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \quad f(x,y) = \frac{x^3 + y^3}{xy}.$$

Once again we make use of the sequential characterization of limits. We consider, to begin with, a random sequence $(x_k) \subseteq \mathbb{R} \setminus \{0\}$ such that $\lim_{k\to\infty} x_k = 0$.

Just like in the case of Exercise 3, we make use of this sequence, in order to generate $(y_k) \subseteq \mathbb{R} \setminus \{0\}$, with the help of a $m \in \mathbb{R}^*$. Thus

$$y_k = m \cdot x_k, \quad \forall k \in \mathbb{N}.$$

Then

$$\frac{x_k^3 + y_k^3}{x_k y_k} = \frac{x_k^3 + m^3 x_k^3}{x_k \cdot m \cdot x_k} = \frac{1 + m^3}{m} x_k.$$

This means that

$$\lim_{k\to\infty} f(x_k, y_k) = \lim_{k\to 0} \frac{1 + m^3}{m} x_k = \frac{1 + m^3}{m} \cdot 0 = 0. \qquad (1)$$

But this does not mean that the limit of the function exists in general, because... in order to be so, it has to be 0 for all the sequences that tend to $(0,0)$.

Consider now another sequence, with the general term

$$z_k = m \cdot x_k^2, \quad \forall k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} (x_k, z_k) = (0,0)$$

and

$$\lim_{k\to\infty} f(x_k, z_k) = \lim_{k\to\infty} \frac{x_k^3 + m^3 \cdot x_k^6}{x_k \cdot m x_k^2} = \lim_{k\to\infty} \frac{1 + m^3 x_k^3}{m} = \frac{1}{m}. \tag{2}$$

From (1) and (2) we have

$$\lim_{k\to\infty} f(x_k, y_k) = 0 \neq \frac{1}{m} = \lim_{k\to\infty} f(x_k, z_k).$$

Thus, the limit does not exist.

Excercise 5: Consider the set

$$A = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$$

and the function

$$f : A \to \mathbb{R}, \quad f(x,y) = x \sin\frac{1}{y} + y \sin\frac{1}{x}, \quad \forall (x,y) \in A.$$

Notice that $(0,0) \in A'$. **Prove that**

a) $\forall y \in \mathbb{R}, \quad \nexists \lim_{x\to 0} f(x,y)$.

b) $\exists \lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution:

a) Choose $y \in \mathbb{R}^*$ randomly. We will prove that $\nexists \lim_{x\to 0} f(x,y)$, with the help of sequences. Since we have the function sin in the expression of the function, we make use of its particular values.

Consider the sequences $(x_k) \subseteq \mathbb{R}$, and $(z_k) \subseteq \mathbb{R}$ with the general terms

$$x_k = \frac{1}{2k\pi}, \quad , z_k := \frac{1}{2k\pi + \frac{\pi}{2}} \quad \forall k \in \mathbb{N}.$$

Then

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} z_k = 0,$$

but

$$\lim_{k\to\infty} f(x_k,y) = \lim_{k\to\infty} \frac{1}{2k\pi}\sin\frac{1}{y} + y\sin(2k\pi) = 0\cdot\sin\frac{1}{y} + y\cdot 0 = 0,$$

while

$$\lim_{k\to\infty} f(z_k,y) = \lim_{k\to\infty} \frac{1}{2k\pi+\frac{\pi}{2}}\sin\frac{1}{y} + y\sin\left(2k\pi+\frac{\pi}{2}\right) = 0\cdot\sin\frac{1}{y} + y\cdot 1 = y.$$

Since $y \neq 0$ it follows that

$$\lim_{k\to\infty} f(x_k,y) \neq \lim_{k\to\infty} f(z_k,y).$$

Thus

$$\nexists \lim_{x\to 0} f(x,y).$$

Recall the fact that y was randomly chosen, therefore, statement $a)$ is proved.

$b)$ We prove that

$$\exists \lim_{(x,y)\to(0,0)} f(x,y),$$

by using the sandwich theorem.

Let $x, y \in \mathbb{R}^*$ be randomly chosen then, the following chain of inequalities is satisfied:

$$0 \le \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \le |x|\cdot\left|\sin\frac{1}{y}\right| + |y|\cdot\left|\sin\frac{1}{x}\right| \le |x|+|y|.$$

Notice that $\lim_{(x,y)\to(0,0)} |x|+|y| = 0+0 = 0$, hence, due to the sandwich theorem,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Excercise 6: Consider the set

$$A = \{(x,y)\in\mathbb{R}^2 : y > 0\}$$

and the function

$$f : A \to \mathbb{R} \qquad \textbf{by } f(x,y) = x\ln y.$$

Note that $(0,0)\in A'$. Study

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

Solution:

Consider a random sequence $(y_k) \subset (0,\infty)$ such that $\lim_{k\to\infty} y_k = 0$.

Define for each $k \in \mathbb{N}$,

$$x_k := \frac{1}{\ln y_k}.$$

Then

$$\lim_{k\to\infty} x_k = \frac{1}{-\infty} = 0.$$

We are interesting in determining

$$\lim_{k\to\infty} f(x_k, y_k)$$

Before considering the limit, we compute for a random $k \in \mathbb{N}$

$$f(x_k, y_k) = \frac{1}{\ln y_k} \cdot \ln y_k = 1.$$

So, $(f(x_k, y_k))_{k\in\mathbb{N}}$ is actually the constant sequence 1, therefore

$$\lim_{k\to\infty} f(x_k, y_k) = 1$$

Define for each $k \in \mathbb{N}$,

$$t_k := \frac{1}{\ln^2 y_k}.$$

Then

$$\lim_{k\to\infty} x_k = \frac{1}{\infty} = 0.$$

We are interesting in determining

$$\lim_{k\to\infty} f(t_k, y_k)$$

Before considering the limit, we compute for a random $k \in \mathbb{N}$

$$f(t_k, y_k) = \frac{1}{\ln^2 y_k} \cdot \ln y_k = \lim_{k\to\infty} \frac{1}{\ln y_k} = \frac{1}{-\infty} = 0$$

So, $(f(t_k, y_k))_{k\in\mathbb{N}}$ is actually the constant sequence 1, therefore

$$\lim_{k\to\infty} f(t_k, y_k) = 0$$

.

Thus

$$\lim_{k\to\infty} (x_k, y_k) = \lim_{k\to\infty} (t_k, y_k) = 0, \quad \text{but} \quad \lim_{k\to\infty} f(x_k, y_k) = 1 \neq 0 = \lim_{k\to\infty} f(t_k, y_k).$$

Thus

$$\lim_{(x,y)\to(0,0)} f(x, y) \quad \text{does not exist.}$$

For further details, take a look at the solved exercises on Seminar4-ttrif