Calculus on $\mathbb{R}^n$
Seminar 3 **2020**

Sets: compact, bounded, closed

To begin with, let us recall some main concepts.

According to the definition a set $M \subseteq \mathbb{R}^n$ is said to be

- **open, if**

$$\exists r > 0 \quad \textbf{s.t. } B(x,r) \subseteq B \Longleftrightarrow \quad \forall x \in M, \quad M \in \mathcal{V}(x).$$

- **closed, if**

$$\mathbb{R}^n \setminus M \quad \textbf{is open}.$$

If $(x_k) \subseteq \mathbb{R}^n$ is a sequence, then $x_0 \in \mathbb{R}^n$ is its limit if

$$\forall V \in \mathcal{V}(x_0) \quad \exists k_0 \in \mathbb{N} \quad \textbf{s.t.} \quad \forall k \geq k_0, \quad x_k \in V \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \quad \textbf{s.t.} \quad \forall k \geq k_0, \quad \|x_k - x_0\| < \varepsilon.$$

According to a characterization theorem

A set $M \subseteq \mathbb{R}^n$ is closed if and only if for all sequences $(x_k) \subseteq M$, which are convergent (so $\exists x_0 \in \mathbb{R}^n$ such that $lim_{k\to\infty} x_k = x_0$), the limit must be in M (thus $x_0 \in M$.)

Let $I \subseteq \mathbb{N}$, let $A_i \subseteq \mathbb{R}^n, \forall i \in I$ be a collection of sets. Then the set of all those sets $\{A_i : i \in I\}$ forms a covering of M if

$$M \subseteq \bigcup_{i \in I} A_i.$$

According to the definition

A set $M \subseteq \mathbb{R}^n$ is said to be compact, if from every covering, we may emphasize a finite covering, thus if $\{A_i : i \in I\}$ is a covering, exists $J \subseteq I$, J with a finite number of elements, such that

$$M \subseteq \bigcup_{j \in J} A_j.$$

In practice, for certain exercises, we use the following characterization for compact sets:

The set $M \subseteq \mathbb{R}^n$ is compact, if each sequence $(x_k) \subseteq M$, has a convergent subsequence in M (which actually means that there exits also $x_0 \in M$, which is the limit of that subsequence). Mathematically, this is expressed as:

$$\forall (x_k) \subseteq M, \quad \exists (x_{k_j})_{j \geq 1}, \quad \exists x_0 \in M \quad \textbf{s.t.} \ \lim_{j \to \infty} x_{k_j} = x_0.$$

Exercise 1:Let $(x_k) \subseteq \mathbb{R}^n$ be a sequence, and let

$$x = \lim_{k \to \infty} x_k \in R^n$$

be its limit. Prove that the set

$$A = \{x\} \bigcup \{x_k : k \in \mathbb{N}\}$$

is compact.

Solution: We prove that the set A is compact, by showing that from an open cover, we can determine a finite subcover.

Let $(A_i)_{i \in I}$ be an open cover of A. This actually means that each set A_i is open, and

$$A \subseteq \bigcup_{i \in I} A_i.$$

Let $i_0 \in I$ be such that $x \in A_{i_0}$. Since A_{i_0} is open, it follows that

$$A_{i_0} \in \mathcal{V}(x).$$

Recall now that $x = \lim_{k \to \infty} x_k$, thus,

$$\exists k_0 \in \mathbb{N} \quad s.d. \quad \forall k \geq k_0, \quad x_k \in A_0.$$

For each $i \in 1, ..., k_0 - 1$, denote by

$$A_i \text{ a set which contains the element } x_i.$$

Therefore $A \subseteq A_0 \cup A_1 \cup ... \cup A_{k_0}$, so we have found a finete subcover $(A_i)_{i\in\{0,...,k_0-1\}}$ which is finite.

Therefore, A is compact.

Exercise 2: Given $A, B \subseteq \mathbb{R}^n$

$$A + B = \{x \in \mathbb{R}^n : \exists a \in A, \exists b \in B \quad \textbf{s.t. } x = a + b\}.$$

a) Prove that if A is closed and B is compact then

$$A + B \quad \textbf{is closed.}$$

b) Give examples of two closed sets C and D, for which $C + D$ is not closed.

Solution:

a) We will prove that $A + B$ is closed by showing that

$$\forall (a_k) \subseteq A + B \quad \text{with} \quad \lim_{k\to\infty} a_n = a \quad \Longrightarrow a \in A + B, \tag{1}$$

namely, each convergent sequence of elements from $A+B$, has the limit in $A+B$ as well.

Let now $(x_k) \subseteq A+B$ be a randomly chosen convergent sequence, thus $\exists x_0 \in \mathbb{R}^n$. We will prove that

$$x_0 \in A + B.$$

For a random $k \in \mathbb{N}$, we have $x_k \in A + B$ which implies that

$$\exists a_k \in A, \quad \exists b_k \in B \quad \text{s.t.} \quad x_k = a_k + b_k. \tag{2}$$

This further implies the existence of two sequences $(a_k) \subseteq A$ and $(b_k) \subseteq B$ such that

$$\lim_{k\to\infty} (a_k + b_k) = x_0. \tag{3}$$

Due to the fact that the set B is compact, there exists a subsequence

$$(b_{k_j})_{j\in\mathbb{N}}$$

and $\exists b_0 \in B$ such that

$$\lim_{j\to\infty} b_{k_j} = b_0. \tag{4}$$

From (3) and (4) we get

$$\lim_{j\to\infty} (x_{k_j} - b_{k_j}) = x_0 - b_0$$

Using now (2) we obtain

$$\exists \lim_{j\to\infty} a_{k_j} = x_0 - b_0.$$

Since A is a closed set and (a_{k_j}) is a subsequence of A which is convergent, it implies that the limit belongs to A, namely

$$x_0 - b_0 \in A,$$

which, by simply adding $b_0 \in B$, we get the desired conclusion that

$$x_0 \in A + b_0 \subseteq A + B.$$

Recall that the sequence (x_k) was chosen randomly, so the proof is complete.

b) It is quite clear, according to what we proved at a), that, in order to be able to provide a good example, the two sets C and D cannot be one closed and the other one compact. This is why, we look for two closed sets which are both not compact (and we know that closed +bounded implies compact), so neither of them can be bounded.

Consider $C := \mathbb{Z}$ and

$$D := \left\{ n + \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1 + \frac{1}{1}, 2 + \frac{1}{2}, ..., i + \frac{1}{i} ... \right\}$$

For each $n \in \mathbb{N}$

$$\frac{1}{n} = -n + n + \frac{1}{n} \in C + D.$$

Thus $\left(\frac{1}{n}\right) \subset C + D$. Take into account that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \notin C + D,$$

it follows that $C + D$ is not closed.

Exercise 3: Given $A, B \subseteq \mathbb{R}^n$, the distance between the two sets is the nonnegative real number

$$d(A, B) = \inf\{d(a, b) : \quad a \in A \textbf{ and } \quad b \in B\}.$$

a) If $A = \{a\}$ and B is closed, then

$$\exists b \in B \quad \textbf{s.t.} \quad d(A, B) = d(a, b).$$

More precisely, the infimum becomes a minimum.

b) If A is compact and B is closed, then

$$\exists a \in A \quad \textbf{and } \exists b \in B \quad \textbf{s.t.} \quad d(A, B) = d(a, b).$$

More precisely, the infimum becomes a minimum.

c) Give an example of two sets for which the minimum is not attained.

Solution:

a) Since the distance is an infimum, this means that

$$\exists (b_k) \subseteq B \quad \text{ s.t. } \lim_{k\to\infty} d(a, b_k) = d(a, B.)$$

By using the characterization of the limit, for $\varepsilon = 1$

$$\exists k_0 \in \mathbb{N} \quad \text{ s.t. } \quad \forall k \geq k_0 \quad \|d(a, b_k) - d(a, B)\| < 1 \Longleftrightarrow \tag{5}$$

$$\Longleftrightarrow \forall k_0 \in \mathbb{N} \quad -1 \leq d(a, b_k) - d(a, B) \leq 1 \Longrightarrow$$
$$\forall k \geq k_0 \quad d(a, b_k) \leq d(A, B) + a.$$

Denote $r := d(a, B) + 1 > 0$. Then we obtain

$$\forall k \geq k_0, \quad b_k \in \overline{B}(a, r).$$

Recall that $(b_k) \subseteq B$, so we have that

$$\forall k \geq k_0 \quad b_k \in B \cap \overline{B}(a, r).$$

Both B and $\overline{B}(a, r)$ are close sets, which implies that

$$B \cap \overline{B}(a, r) \quad \text{ is closed.}$$

Moreover, since $\overline{B}(a, r)$ is bounded, it is clear that

$$B \cap \overline{B}(a, r) \quad \text{ is bounded.}$$

In conclusion, the set being both closed and bounded, is compact. Hence

$$B \cap \overline{B}(a, r) \quad \text{ is compact, and } \quad \forall k \geq k_0 \quad b_k \in B \cap \overline{B}(a, r).$$

Thus, there exists a convergent subsequence $(b_{k_j})_{j\in\mathbb{N}} \in B \cap \overline{B}(a, r)$, and there exists $b_0 \in B \cap \overline{B}(a, r)$ such that

$$\lim_{j\to\infty} b_{k_j} = b_0.$$

Hence

$$\lim_{j\to\infty} d(a, b_{k_j}) = d(a, b_0).$$

Recall that, being a subsequence, we have (from the beginning)

$$\lim_{j\to\infty} d(a, b_{k_j}) = d(a, B)$$

Thus

$$d(a, b_0) = d(a, B).$$

b) With an argument similar to the one in the proof of a), from the definition of the infimum, we know that

$$\forall k \in \mathbb{N}, \quad \exists a_k \in A, \quad \exists b_k \in B \quad s.t. \quad d(A, B) < d(a_k, b_k) < d(A, B) + \frac{1}{k}.$$

(a_k) is a sequence of the compact set A, this means that there exists a subsequence $(a_{k_j})_{j\in\mathbb{N}}$ and $a \in A$ such that
$$\lim_{j\to\infty} a_{k_j} = a,$$
and, due to this, there exists $R > 0$ such that $d(a_{k_j}, a) \le R$ for all $j \in \mathbb{N}$. By using the transitivity of the distance, we have for all $j \in \mathbb{N}$

$$d(b_{k_j}, a) \le d(a_{k_j}, b_{k_j}) - d(a_{k_j}, a) \le d(A, B) + \frac{1}{k} + R \le d(A, B) + 1 + R.$$

Denote now $r := d(A, B) + 1 + R > 0$. Then

$$(b_{k_j}) \subseteq B \cap \overline{B}(a, r).$$

Exactly like in the proof of a), $B \cap \overline{B}(a, r)$ is a compact set (being both closed and bounded). Hence, we can find (exists)a convergent sub sequence $(b_{k_{j_l}})_{l\in\mathbb{N}} \subset$, and exists $b \in B \cap \overline{B}(a, r)$ such that
$$\lim_{l\to\infty} b_{k_{j_l}} = b.$$
Therefore
$$\lim_{l\to\infty} d(a_{k_{j_l}}, b_{k_{j_l}=d(a,b)=d(A,B)}.$$
Hence, the infimum is attained.

c) Consider
$$A = \{(x, e^x) : x \in \mathbb{R}\}$$
and
$$B = \{(x, 0) : x \in \mathbb{R}\}.$$

Both sets are closed, however

$$\nexists a \in A, B \in B \quad \text{s.t.} \quad d(a, b) = 0.$$

Please draw the graph and notice that $d(A, B) = 0$, even though $A \cap B = \emptyset$.

Exercise 4: Let $f : R^n \to R^m$ be a continuous function, and let $M \subseteq R^m$ be a n open set. Prove that the set

$$f^{-1}(M) = \{x \in R^n : f(x) \in M\}$$

is an open set (in $\mathbb{R}^n$).

Solution:
We prove that $f^{-1}(M)$ is open, by using the definition, thus we prove that

$$\forall x \in f^{-1}(M) \quad \exists r > 0, \text{ s.t.} \quad B(x, r) \subseteq f^{-1}(M). \tag{6}$$

Let $x_0 \in f^{-1}(M)$ be a randomly chosen point. Be will determine a ball around it, which is in $f^{-1}(M)$.

$$x_0 \in f^{-1}(M) \Longleftrightarrow f(x_0) \in M \Longrightarrow M \in \mathcal{V}(x_0) \quad \text{from } M \text{ an open set} \Longleftrightarrow$$

$$\Longleftrightarrow \exists \varepsilon > 0 \text{ s.t. } B(x_0, \varepsilon) \subseteq M. \tag{7}$$

Now we have an $\varepsilon > 0$ whose existence is assured. For this given (fixed) number, we apply the characterization of the continuity of f at x_0. Thus,

$$f \text{ continuous at } x_O \Longrightarrow \text{ for } \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad \forall x \in \mathbb{R}^n \quad \text{with} \quad \|x - x_0\| < \delta,$$

$$\text{to hold } \|f(x) - f(x_0)\| < \varepsilon.$$

This meas that $\forall x \in B(x_0, \delta)$, it holds that $f(x) \in B(f(x_0), \varepsilon)$. By considering (7), this furhterm imples that $f(x) \in M$. Hence

$$\forall x \in B(x_0, \delta), \quad f(x) \in M \Longleftrightarrow B(x_0, \delta) \subseteq f^{-1}(M).$$

Recall that this last inclusion satisfies (6) for the randomly considered x_0. Thus the conclusion is proved.

Exercise 5: Prove that the assertions of Exercise 4 remain true, when we replace the word *open*, by closed.

Solution: We actually have to prove that for each $H \subseteq R^m$ closed, the set

$$f^{-1}(H) = \{x \in R^n : f(x) \in H\}$$

is closed.

According to the definitions:

$$f^{-1}(H) \text{ closed} \quad \Longleftrightarrow \quad \mathbb{R}^n \setminus f^{-1}(H) \text{ open} \quad \Longleftrightarrow$$

$$\forall a \in \mathbb{R}^n \setminus f^{-1}(H), \quad \exists \delta > 0 \text{ s.t. } B(a, \delta) \subseteq \mathbb{R}^n \setminus f^{-1}(H).$$

Let $a \in \mathbb{R}^n \setminus f^{-1}(H)$ be randomly chosen. This mean that

$$f(a) \in \mathbb{R}^m \setminus H.$$

Since $\mathbb{R}^m \setminus H$ is open, it follows that

$$\exists \varepsilon > 0 \text{ s.t. } B(f(a), \varepsilon) \subseteq R^m \setminus H. \tag{8}$$

For that given $\varepsilon > 0$, we apply the continuity of a. This means that

$$\exists \delta > 0 \text{ s.t. } \forall x \in B(a, \delta), \quad f(x) \in B(f(a), \varepsilon).$$

This actually means, by considering (8) that

$$f(B(a, \delta)) \subseteq R^m \setminus H \Longleftrightarrow \quad B(a, \delta) \subseteq R^n \setminus f^{-1}(H).$$

Since a was randomly chosen, the proof is complete.

Exercise 6: Let $f : [a, b] \to \mathbb{R}$ be a random function, and let

$$G_f = \{(x, f(x)) : x \in [a, b]\}$$

be its graph. Prove that f is continuous if and only if G_f si a compact subset of $\mathbb{R}^2$,

Solution:

$\Longrightarrow$ **The necessity.** We know that f is continuous on $[a, b]$ and we want to prove that G_f is a compact set.

In order to do that, we use the characterization that a set is compact, if and only if each sequence of points in G_f, posses a convergent subsequence, whose limit point belongs to G_f.

Let $(x_k, f(x_k))_{k\geq 1} \subseteq G_f$ be a random sequence. Notice that $(x_k) \subseteq [a, b]$, and take into account that $[a, b]$ is a compact set. This means that (x_k) posses a convergent subsequence $(x_{k_j})_{j\geq 1}$, for which

$$\exists x_0 \in [a, b] \quad \text{such that} \quad \lim_{j\to\infty} x_{k_j} = x_0.$$

Recall the definition of continuity which states that *if f is continuous at p, then for each given sequence $(p_k) \subset [a, b]$ with $lim_{k\to\infty} p_k = p$, it holds that $\lim_{k\to\infty} f(p_k) = f(p)$.*

Coming back to our problem we have that

$$x_0 \in [a, b], \quad \lim_{j\to\infty} x_{k_j} = x_0 \quad \text{and} \quad f \quad \text{is continuous at } x_0.$$

This implies automatically that

$$\lim_{j\to\infty} f(x_{k_j}) = f(x_0).$$

Hence

$$\lim_{j\to\infty} \left(x_{k_j}, f(x_{k_j})\right) = (x_0, f(x_0)) \in G_f.$$

Thus, we have determined a convergent subsequence of $(x_k, f(x_k))$, which is convergent, and the limit is in G_f. Due to the fact that $(x_k, f(x_k))$ was randomly chosen, the proof is complete.

$\Longleftarrow$ **The Sufficiency**

Now we know that G_f is compact, and we want to prove that f is continuous on $[a, b]$.

We prove that f is continuous at a random point in $x_0 \in [a, b]$ by showing that (according to the definition)

$$\forall (x_k) \subseteq [a, b] \quad \text{with} \quad \lim_{k\to\infty} x_k = x_0, \quad \text{it holds} \quad \lim_{k\to\infty} f(x_k) = f(x_0).$$

Thus, let us consider $x_0 \in [a, b]$ randomly chosen. Assume by contradiction that there exists a sequence $(x_k) \subset [a, b]$, with $\lim_{k\to\infty} x_k = x_0$ for which

$$\lim_{k\to\infty} f(x_k) \neq f(x_0). \tag{9}$$

This implies that $\exists \varepsilon > 0 \quad \text{s.t.} \quad \forall k \in \mathbb{N} \quad \exists k_0 \geq k \quad \text{s.t.} \quad \|f(x_k) - f(x_0)\| \geq \varepsilon$.

In particular, this implies that there exists a sequence of natural numbers

$$\exists k_1 < k_2 < .... < k_j < ...$$

with the property that

$$\|f(x_{k_j}) - f(x_0)\| \geq \varepsilon, \quad \forall j \geq 1. \tag{10}$$

Since $\left(x_{k_j}, f(x_{k_j})\right)_{j \geq 1} \subseteq G_f$, and since G_f is a compact set, there exists a convergent subsequence $\left(x_{k_{j_l}}, f(x_{k_{j_l}})\right)_{l \geq 1} \subseteq G_f$, so there exists $(t, f(t)) \in G_f$ such that

$$\lim_{l \to \infty} (x_{k_{j_l}}, f(x_{k_{j_l}}) = (t, f(t)) \in G_f \tag{11}$$

Since $(x_{k_{j_l}})$ is a subsequence of (x_k), whose limit is x_0 it follows that

$$t = x_0 \text{ and } f(t) = f(x_0)$$

Hence $\lim_{l \to \infty} f(x_{k_{j_l}}) = f(x_0)$. For the $\varepsilon > 0$ considered at the beginning, by applying the characterization of the limit, we get that

$$\exists l_0 \in \mathbb{N} \quad \text{such that} \quad \forall l \geq l_0 \quad \|f(x_{k_{j_{l_0}}}) - f(x_0) < \varepsilon. \tag{12}$$

From (10) and (12) it follows that

$$\varepsilon \leq \|f(x_{k_{j_{l_0}}}) - f(x_0) < \varepsilon,$$

which is a contradiction.

The proof is complete.