Seminar 11

1. Find the volume bounded by the cone $z = \sqrt{x^2 + y^2}$ and by the sphere $x^2 + y^2 + z^2 = 1$.

2. Find the volume bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $x + y + z = \frac{1}{2}$.

3. Find the mass of a square plate of side $2a$, if the density varies as the distance from the center of the plate.

4. Find the mass of a ball of radius a, if its density varies directly proportional to the distance from a fixed point O, lying on the boundary of the ball.

5. Let A be the set bounded by the planes $z = 0$ and $z = 4$, lying inside the cone $z^2 = x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 1$. Find the moment of inertia with respect to the Oz axis of a homogeneous solid body of density 1, occupying the region A.

Solutions

$\boxed{\textbf{1.}}$ Let $A := \{(x,y,z) \in \mathbb{R}^3 \;\mid\; z \geq \sqrt{x^2+y^2},\, x^2+y^2+z^2 \leq 1\}$ and let V denote the volume of A. We have $V = \iiint_A \mathrm{d}x\mathrm{d}y\mathrm{d}z$. To evaluate the triple integral, we use the spherical coordinates

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho\cos\varphi,$$

where (see figure 1)

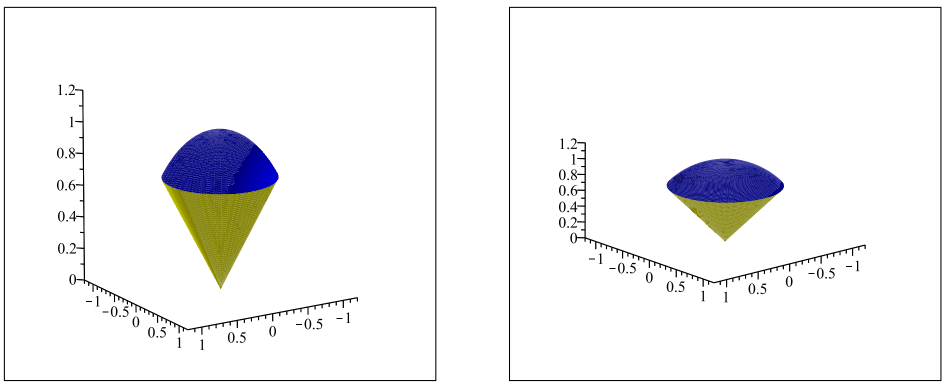

Figure 1:

$$\rho \in [0,1], \quad \varphi \in \left[0, \frac{\pi}{4}\right], \quad \theta \in [0, 2\pi].$$

We obtain

$$\begin{aligned} V &= \int_0^1 \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \rho^2 \sin\varphi \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \\ &= \left(\int_0^1 \rho^2 \, \mathrm{d}\rho\right)\left(\int_0^{\frac{\pi}{4}} \sin\varphi \, \mathrm{d}\varphi\right)\left(\int_0^{2\pi} \mathrm{d}\theta\right) = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right). \end{aligned}$$

$\boxed{\textbf{2.}}$ Let $A := \{(x,y,z) \in \mathbb{R}^3 \;\mid\; x^2+y^2 \leq z \leq \frac{1}{2} - x - y\}$ be the solid whose volume is required to be evaluated and let C be the curve of intersection of the paraboloid with the plane (see figure 2).

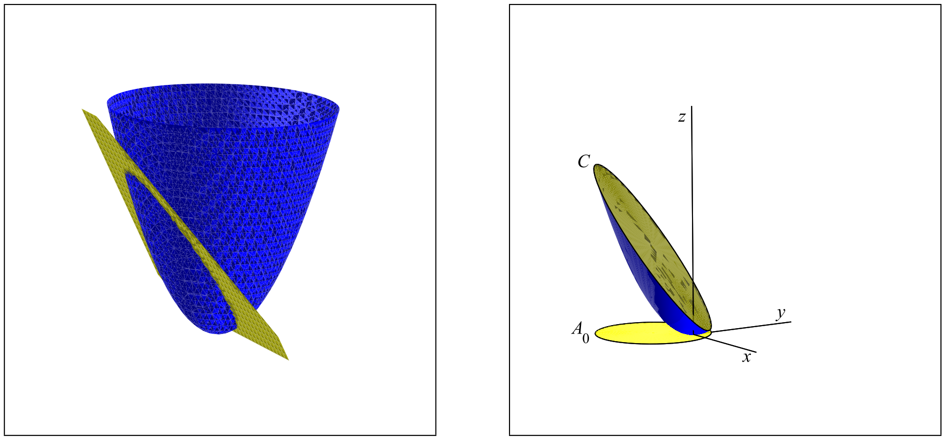

Figure 2:

If $P(x,y,z)$ is an arbitrary point of C, then we have

$$z = x^2 + y^2 = \frac{1}{2} - x - y,$$

hence P belongs to the cylinder

$$x^2 + y^2 = \frac{1}{2} - x - y \quad \Leftrightarrow \quad \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 1.$$

Therefore, the projection of A onto the plane Oxy is the disk

$$A_0 := \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \leq 1 \right\}.$$

The volume of A is given by

$$V = \iiint_A \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{A_0} \left(\frac{1}{2} - x - y - x^2 - y^2\right) \mathrm{d}x\mathrm{d}y.$$

To evaluate the double integral, we use the polar coordinates:

$$\begin{aligned} x &= -\frac{1}{2} + \rho\cos\theta, & \rho &\in [0,1], \\ y &= -\frac{1}{2} + \rho\sin\theta, & \theta &\in [0,2\pi]. \end{aligned}$$

We get

$$V = \int_0^1 \int_0^{2\pi} \left(\rho - \rho^3\right) \mathrm{d}\rho \mathrm{d}\theta = \left(\int_0^1 \left(\rho - \rho^3\right) \mathrm{d}\rho \right) \left(\int_0^{2\pi} \mathrm{d}\theta \right) = \frac{\pi}{2}.$$

$\boxed{\mathbf{3.}}$ Choose a Cartesian system with the origin at the center of the plate such that the coordinate axes are parallel to the plate sides. The plate is divided into four squares A_1, A_2, A_3, A_4, each having side a (see figure 3).

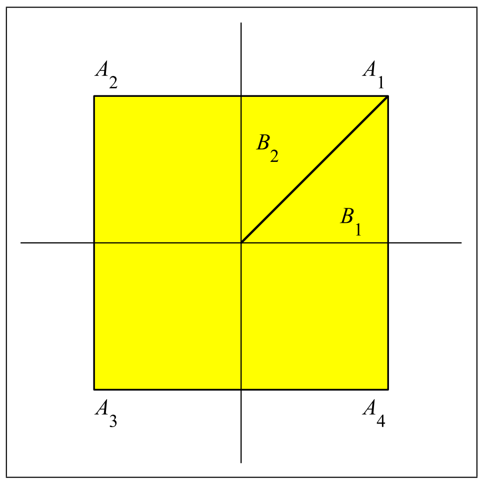

Figure 3:

Let $A := [-a, a] \times [-a, a]$, let $\bar{\rho}(x, y) := c\sqrt{x^2 + y^2}$ be the superficial density of the plate at an arbitrary point $(x, y) \in A$, and let m denote the mass of the plate. We have

$$m = \iint_A \bar{\rho}(x, y) \, \mathrm{d}x\mathrm{d}y = c \iint_A \sqrt{x^2 + y^2} \, \mathrm{d}x\mathrm{d}y = c \sum_{j=1}^{4} \iint_{A_j} \sqrt{x^2 + y^2} \, \mathrm{d}x\mathrm{d}y.$$

Due to symmetry reasons, the above four double integrals are all equal. Indeed, the change of variables $x = -u, y = v$ leads to

$$\iint_{A_1} \sqrt{x^2 + y^2} \, \mathrm{d}x\mathrm{d}y = \iint_{A_2} \sqrt{u^2 + v^2} \, \mathrm{d}u\mathrm{d}v$$

etc. Therefore, we have

$$\begin{aligned} m &= 4c \iint_{A_1} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y \\ &= 4c \left(\iint_{B_1} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y + \iint_{B_2} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y \right). \end{aligned}$$

The change of variables $x = v, y = u$ shows that

$$\iint_{B_1} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \iint_{B_2} \sqrt{v^2+u^2}\,\mathrm{d}u\mathrm{d}v = \iint_{B_2} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y,$$

hence $m = 8c \iint_{B_1} \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$. To compute the double integral, we pass to polar coordinates. We obtain

$$\begin{aligned} m &= 8c \int_{\theta=0}^{\theta=\pi/4} \left(\int_{\rho=0}^{\rho=\frac{a}{\cos\theta}} \rho^2\,\mathrm{d}\rho \right) \mathrm{d}\theta = 8c \int_0^{\pi/4} \frac{a^3}{3\cos^3\theta}\,\mathrm{d}\theta \\ &= \frac{8a^3c}{3} \int_0^{\pi/4} \frac{\cos\theta\,\mathrm{d}\theta}{(1-\sin^2\theta)^2} = \frac{8a^3c}{3} \int_0^{1/\sqrt{2}} \frac{\mathrm{d}x}{(1-x^2)^2} = \\ &= \frac{4a^3c}{3} \left(\sqrt{2} + \ln(1+\sqrt{2}) \right). \end{aligned}$$

$\boxed{\textbf{4.}}$ Choose a Cartesian system with origin at O, such that the plane Oxy is tangent to the ball at O and the center of the ball is located on the Oz axis (at the point $(0,0,a)$). Denoting by m the mass of the ball, we have

$$m = \iiint_A c\sqrt{x^2+y^2+z^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$A := \{(x,y,z) \mid x^2+y^2+(z-a)^2 \le a^2\} = \{(x,y,z) \mid x^2+y^2+z^2 \le 2az\}.$$

Method 1. We use the change of variables

$$\begin{aligned} x &= \rho\sin\varphi\cos\theta, && \rho \in [0,a], \\ y &= \rho\sin\varphi\sin\theta, && \varphi \in [0,\pi], \\ z &= a + \rho\cos\varphi, && \theta \in [0,2\pi]. \end{aligned}$$

We obtain

$$
\begin{aligned}
m &= c\int_0^a \int_0^\pi \int_0^{2\pi} \sqrt{\rho^2+a^2+2a\rho\cos\varphi}\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\
&= c\left(\int_0^a \int_0^\pi \sqrt{\rho^2+a^2+2a\rho\cos\varphi}\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right) \\
&= 2\pi c\int_{\rho=0}^{\rho=a}\left(\int_{\varphi=0}^{\varphi=\pi} \sqrt{\rho^2+a^2+2a\rho\cos\varphi}\cdot\rho^2\sin\varphi\,\mathrm{d}\varphi\right)\mathrm{d}\rho.
\end{aligned}
$$

Substituting $\sqrt{\rho^2+a^2+2a\rho\cos\varphi}=t$, we get $\rho\sin\varphi\,\mathrm{d}\varphi=-\frac{t}{a}\,\mathrm{d}t$, whence

$$
\begin{aligned}
m &= 2\pi c\int_{\rho=0}^{\rho=a}\left(\int_{t=a+\rho}^{t=a-\rho} t\rho\left(-\frac{t}{a}\right)\mathrm{d}t\right)\mathrm{d}\rho = \frac{2\pi c}{a}\int_{\rho=0}^{\rho=a}\frac{\rho t^3}{3}\Big|_{t=a-\rho}^{t=a+\rho}\mathrm{d}\rho \\
&= \frac{2\pi c}{3a}\int_0^a 2\rho(3a\rho^2+\rho^3)\,\mathrm{d}\rho = \frac{8\pi}{5}\,a^4c.
\end{aligned}
$$

Method 2. We use the change of variables

$$
\begin{aligned}
x &= \rho\sin\varphi\cos\theta, & \varphi &\in \left[0,\frac{\pi}{2}\right], \\
y &= \rho\sin\varphi\sin\theta, & \theta &\in [0,2\pi], \\
z &= \rho\cos\varphi, & \rho &\in [0,2a\cos\varphi].
\end{aligned}
$$

We find

$$
\begin{aligned}
m &= c\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\left(\int_{\rho=0}^{\rho=2a\cos\varphi}\rho\cdot\rho^2\sin\varphi\,\mathrm{d}\rho\right)\mathrm{d}\theta\mathrm{d}\varphi \\
&= c\int_0^{2\pi}\int_0^{\pi/2} 4a^4\cos^4\varphi\sin\varphi\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= 4a^4c\left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^{\pi/2}\cos^4\varphi\sin\varphi\,\mathrm{d}\varphi\right) = \frac{8\pi}{5}\,a^4c.
\end{aligned}
$$

$\boxed{\mathbf{5.}}$ Let I denote the required moment of inertia. We have

$$
I = \iiint_A (x^2+y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.
$$

To evaluate the triple integral, we pass to cylindrical coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z,$$

where $\theta \in [0, 2\pi]$, $z \in [0, 4]$ și (see figure 4)

$$\begin{array}{ll} \rho \in [0, z] & \text{if } z \in [0, 1], \\ \rho \in [0, 1] & \text{if } z \in [1, 4]. \end{array}$$

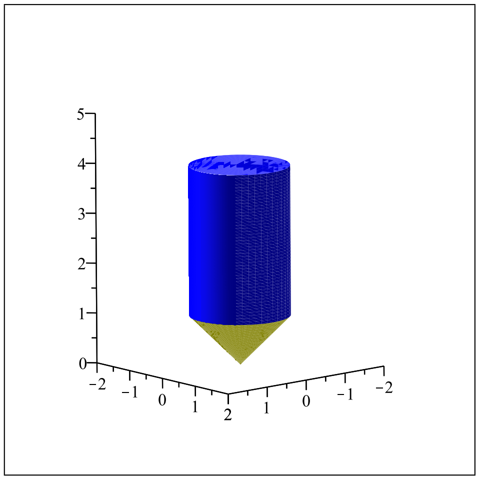

Figure 4:

We get

$$\begin{aligned} I &= \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \left(\int_{\rho=0}^{\rho=z} \rho^3 \, \mathrm{d}\rho \right) \mathrm{d}\theta \mathrm{d}z + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=4} \int_{\rho=0}^{\rho=1} \rho^3 \, \mathrm{d}\theta \mathrm{d}z \mathrm{d}\rho \\ &= \int_0^{2\pi} \int_0^1 \frac{z^4}{4} \, \mathrm{d}\theta \mathrm{d}z + \left(\int_0^{2\pi} \mathrm{d}\theta \right) \left(\int_1^4 \mathrm{d}z \right) \left(\int_0^1 \rho^3 \, \mathrm{d}\rho \right) = \frac{8\pi}{5} \, . \end{aligned}$$