Seminar 10

1. Evaluate $\displaystyle\iint_A x\sqrt{1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$, if

$$A := \{\,(x,y)\in\mathbb{R}^2 \mid \sqrt{3}x-3y\geq 0,\ 1\leq 4(x^2+y^2)\leq 4\,\}.$$

2. Evaluate $\displaystyle\iint_A \frac{x^2}{x^2+3y^2}\,\mathrm{d}x\mathrm{d}y$, if

$$A := \{\,(x,y)\in\mathbb{R}^2 \mid 1\leq x^2+y^2\leq 3,\ x+y\geq 0,\ y\geq 0\,\}.$$

3. **(Homework)** Evaluate $\displaystyle\iint_A \frac{y}{x+y+\sqrt{x^2+y^2}}\,\mathrm{d}x\mathrm{d}y$, if

$$A := \{\,(x,y)\in\mathbb{R}^2 \mid 1\leq x^2+y^2\leq 4,\ x+y\geq 0,\ y\geq 0\,\}.$$

4. Let $a>0$ and let $A := \{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 2ax\,\}$. Calculate

$$\iint_A \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y.$$

5. Let $a>0$ and let $A := \{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 2ax,\ x^2+y^2\leq 2ay\,\}$. Calculate

$$\iint_A (x^2+y^2)\,\mathrm{d}x\mathrm{d}y.$$

6. Compute $\displaystyle\int_0^2\left(\int_0^{\sqrt{2x-x^2}}\sqrt{x^2+y^2}\,\mathrm{d}y\right)\mathrm{d}x.$

7. **(Homework)** Compute $\displaystyle\int_0^{1/2}\left(\int_{\sqrt{3}y}^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\,\mathrm{d}x\right)\mathrm{d}y.$

8. Calculate $\displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-2)^2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\leq 1\,\}.$$

9. Calculate $\displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+z^2+3}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1,\ z \geq 0\}.$$

10. Calculate $\displaystyle\iiint_A \frac{z}{(x^2+y^2+1)^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1,\ z \geq 0\}.$$

11. Calculate $\displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+(3-z)^2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{\,(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \leq 1,\ 0 \leq z \leq 2\,\}.$$

12. Determine the area of the plane set A which is bounded by the parabolas $y^2 = ax$ and $y^2 = bx$ $(0 < a < b)$ and by the hyperbolas $xp = p$ and $xy = q$ $(0 < p < q)$.

13. **(Homework)** Determine the area of the plane set A which is bounded by the parabolas of equations $x^2 = ay$, $x^2 = by$, $y^2 = px$, $y^2 = qx$, where $0 < a < b$, $0 < p < q$.

14. **(Homework)** Calculate $\displaystyle\iint_A \arcsin\sqrt{x+y}\,\mathrm{d}x\mathrm{d}y$, if A is the plane set bounded by the lines $x+y = 0$, $x+y = 1$, $y = -1$ and $y = 1$, respectively.

Solutions

1. Pass to polar coordinates: $x = \rho\cos\theta$, $y = \rho\sin\theta$, where (see figure 1)

$$\rho \in \left[\frac{1}{2}\,,\,1\right] \quad \text{and} \quad \theta \in \left[0,\frac{\pi}{6}\right] \cup \left[\frac{7\pi}{6}\,,\,2\pi\right] \quad \text{or} \quad \theta \in \left[-\frac{5\pi}{6}\,,\,\frac{\pi}{6}\right].$$

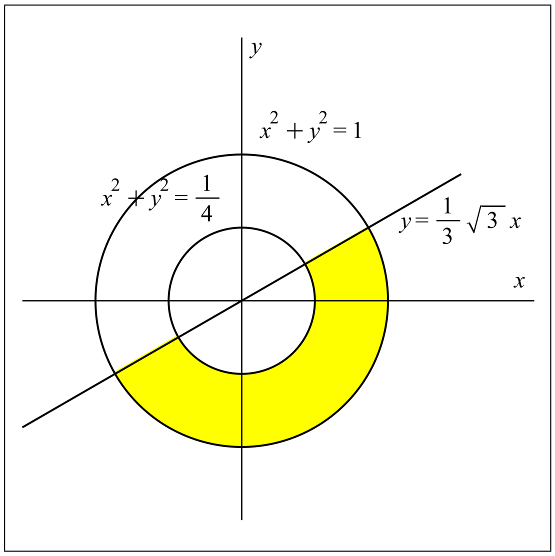

Figure 1:

We obtain

$$\begin{aligned}
&\iint_A x\sqrt{1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y \\
&\qquad = \int_{\rho=1/2}^{\rho=1}\int_{\theta=-5\pi/6}^{\theta=\pi/6} \rho\cos\theta\sqrt{1-\rho^2\cos^2\theta-\rho^2\sin^2\theta}\cdot\rho\,\mathrm{d}\rho\mathrm{d}\theta \\
&\qquad = \int_{\rho=1/2}^{\rho=1}\int_{\theta=-5\pi/6}^{\theta=\pi/6} \rho^2\cos\theta\sqrt{1-\rho^2}\,\mathrm{d}\rho\mathrm{d}\theta \\
&\qquad = \left(\int_{1/2}^{1}\rho^2\sqrt{1-\rho^2}\,\mathrm{d}\rho\right)\left(\int_{-5\pi/6}^{\pi/6}\cos\theta\,\mathrm{d}\theta\right) \\
&\qquad = \frac{\sqrt{3}}{64}+\frac{\pi}{24}\,.
\end{aligned}$$

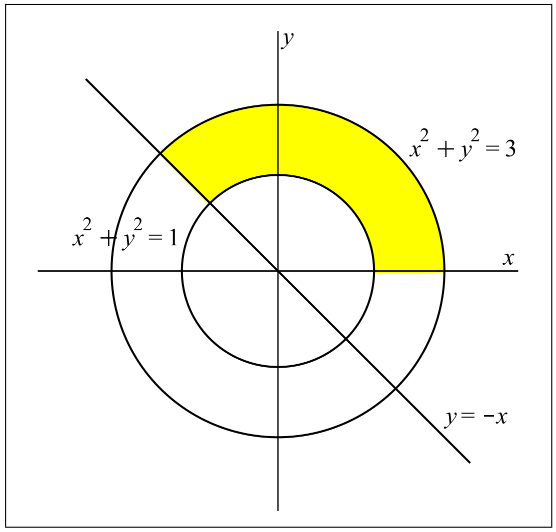

Figure 2:

2. Pass to polar coordinates: $x=\rho\cos\theta$, $y=\rho\sin\theta$, where $\rho\in[1,\sqrt{3}\,]$ and $\theta\in\left[0,\frac{3\pi}{4}\right]$ (see figure 2).

We obtain

$$\begin{aligned}
I &:= \iint_A \frac{x^2}{x^2+3y^2}\,dxdy = \int_{\rho=1}^{\rho=\sqrt{3}}\int_{\theta=0}^{\theta=3\pi/4} \frac{\rho^2\cos^2\theta}{\rho^2\cos^2\theta+3\rho^2\sin^2\theta}\cdot\rho\,\mathrm{d}\rho\mathrm{d}\theta \\
&= \int_{\rho=1}^{\rho=\sqrt{3}}\int_{\theta=0}^{\theta=3\pi/4} \rho\,\frac{\cos^2\theta}{\cos^2\theta+3\sin^2\theta}\,\mathrm{d}\rho\mathrm{d}\theta \\
&= \left(\int_1^{\sqrt{3}}\rho\,\mathrm{d}\rho\right)\left(\int_0^{3\pi/4}\frac{\cos^2\theta}{\cos^2\theta+3\sin^2\theta}\,\mathrm{d}\theta\right).
\end{aligned}$$

The integral with respect to θ can be calculated by means of the substitution $\operatorname{ctg}\theta=t$. We have

$$\begin{aligned}
I &= \int_{0+0}^{3\pi/4}\frac{\cos^2\theta}{\cos^2\theta+3\sin^2\theta}\,\mathrm{d}\theta = \int_{\infty}^{-1}\frac{\frac{t^2}{t^2+1}}{\frac{t^2}{t^2+1}+\frac{3}{t^2+1}}\left(-\frac{1}{t^2+1}\right)\mathrm{d}t \\
&= \int_{-1}^{\infty}\frac{t^2}{(t^2+1)(t^2+3)}\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{\infty}\left(\frac{3}{t^2+3}-\frac{1}{t^2+1}\right)\mathrm{d}t \\
&= \frac{1}{2}\left(\sqrt{3}\arctan\frac{t}{\sqrt{3}}-\arctan t\right)\Bigg|_{-1}^{\infty} = \frac{\pi}{24}\,(8\sqrt{3}-9).
\end{aligned}$$

3. Answer: $\dfrac{3\pi}{16} + \dfrac{1}{4}\ln(2-\sqrt{2})$.

4. ***Method 1.*** We pass to polar coordinates taking into account that A is the disk about $(a, 0)$, whose radius is a (see figure 3):

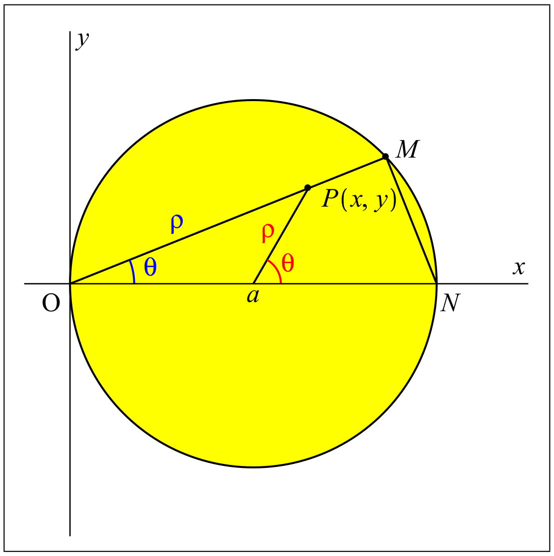

Figure 3:

$$x = a + \rho\cos\theta, \qquad \rho \in [0, a],$$
$$y = \rho\sin\theta, \qquad \theta \in [0, 2\pi].$$

The Jacobian determinant of the coordinate conversion is ρ. We obtain

$$\iint_A \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi} \sqrt{(a+\rho\cos\theta)^2+\rho^2\sin^2\theta}\cdot\rho\,\mathrm{d}\rho\mathrm{d}\theta$$
$$= \int_{\rho=0}^{\rho=a}\int_{\theta=0}^{\theta=2\pi} \rho\sqrt{a^2+\rho^2+2a\rho\cos\theta}\,\mathrm{d}\rho\mathrm{d}\theta.$$

This double integral does not look promising at all. Neither of the two iterated integrals is easy to calculate.

ABANDON DU TRAVAIL.

Method 2. We apply the usual conversion to polar coordinates:

$$x = \rho\cos\theta, \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$y = \rho\sin\theta, \qquad \rho \in [0, 2a\cos\theta].$$

Now ρ represents the distance OP and not the distance between P and the center of the disk. The maximum value that ρ can take for a given angle θ is OM. The length of $[OM]$ can be determined from the right-angle triangle OMN: $OM = ON\cos\theta = 2a\cos\theta$. Hence $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\rho \in [0, 2a\cos\theta]$. We have

$$\iint_A \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left(\int_{\rho=0}^{\rho=2a\cos\theta} \sqrt{\rho^2\cos^2\theta + \rho^2\sin^2\theta}\cdot\rho\,\mathrm{d}\rho\right)\mathrm{d}\theta$$
$$= \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \frac{\rho^3}{3}\bigg|_{\rho=0}^{\rho=2a\cos\theta}\mathrm{d}\theta = \frac{8a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\theta\,\mathrm{d}\theta = \frac{32a^3}{9}\,.$$

5. Pass to polar coordinates: $x = \rho\cos\theta$, $y = \rho\sin\theta$. We have $\theta \in \left[0, \frac{\pi}{2}\right]$ (see figure 4).

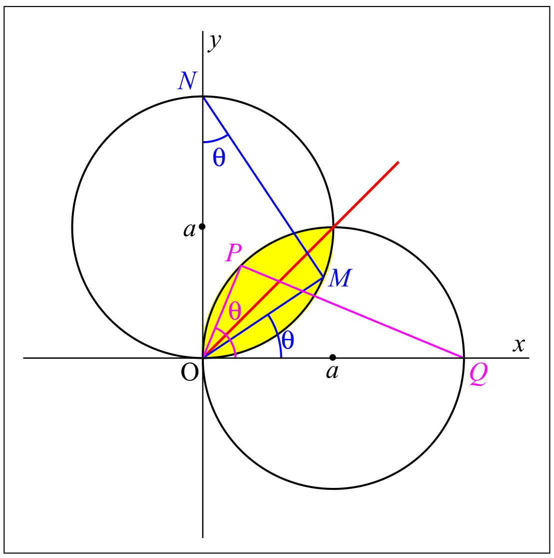

Figure 4:

If $\theta \in \left[0, \frac{\pi}{4}\right]$, then the maximum value that ρ can take is OM. The length of the segment $[OM]$ can be easily determined from the right-angle triangle

OMN: $OM = ON\sin\theta = 2a\sin\theta$. If $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, then the maximum value that ρ can take is OP. The length of $[OP]$ can be determined from the right-angle triangle OPQ: $OP = OQ\cos\theta = 2a\cos\theta$. Therefore, we have

$$\rho \in [0, 2a\sin\theta], \quad \text{dacă } \theta \in \left[0, \frac{\pi}{4}\right],$$
$$\rho \in [0, 2a\cos\theta], \quad \text{dacă } \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right].$$

Passing to polar coordinates, we get

$$\begin{aligned}
&\iint_A (x^2+y^2)\,\mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\theta=0}^{\theta=\frac{\pi}{4}} \left(\int_{\rho=0}^{\rho=2a\sin\theta} \rho^2 \cdot \rho\,\mathrm{d}\rho\right)\mathrm{d}\theta + \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{2}} \left(\int_{\rho=0}^{\rho=2a\cos\theta} \rho^2 \cdot \rho\,\mathrm{d}\rho\right)\mathrm{d}\theta \\
&\quad = 4a^4 \int_0^{\frac{\pi}{4}} \sin^4\theta\,\mathrm{d}\theta + 4a^4 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\,\mathrm{d}\theta = 8a^4 \int_0^{\frac{\pi}{4}} \sin^4\theta\,\mathrm{d}\theta \\
&\quad = \left(\frac{3\pi}{4} - 2\right)a^4.
\end{aligned}$$

6. Let $I := \displaystyle\int_0^2 \left(\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\right)\mathrm{d}x$. The iterated integral I comes from the calculation of the double integral $\displaystyle\iint_A \sqrt{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$, where

$$A := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 0 \le x \le 2,\, 0 \le y \le \sqrt{2x-x^2}\right\}.$$

The image of the curve of equation $y = \sqrt{2x-x^2}$ is the semi-circle located above the Ox axis of the circle $y^2 = 2x - x^2 \;\Leftrightarrow\; (x-1)^2+y^2 = 1$. Therefore, A is the plane set bounded by Ox and by the above mentioned semi-circle. Proceeding as in the solution of problem **4**, we find $I = 16/9$.

7. Answer: $\dfrac{\pi}{18}$.

8. Let $I := \displaystyle\iiint_A \frac{1}{\sqrt{x^2+y^2+(z-2)^2}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$. The set A is the closed unit ball in $\mathbb{R}^3$. The projection of A onto the plane Oxy is the disk

$$A_0 := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1\}.$$

Trying to calculate I by using Fubini's theorem, we obtain

$$\begin{aligned}
I &= \iint_{A_0} \left(\int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} \frac{\mathrm{d}z}{\sqrt{x^2+y^2+(z-2)^2}} \right) \mathrm{d}x\mathrm{d}y \\
&= \iint_{A_0} \ln\left(z-2+\sqrt{(z-2)^2+x^2+y^2}\right) \Big|_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} \mathrm{d}x\mathrm{d}y \\
&= \iint_{A_0} \ln\left(\sqrt{1-x^2-y^2}-2+\sqrt{5-4\sqrt{1-x^2-y^2}}\right) \mathrm{d}x\mathrm{d}y \\
&\quad - \iint_{A_0} \ln\left(-\sqrt{1-x^2-y^2}-2+\sqrt{5+4\sqrt{1-x^2-y^2}}\right) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

HORRIBLY ! (convince yourself by trying to continue the calculations)

Therefore, we will evaluate the triple integral I by passing to spherical coordinates instead of using Fubini's theorem (see figure 5).

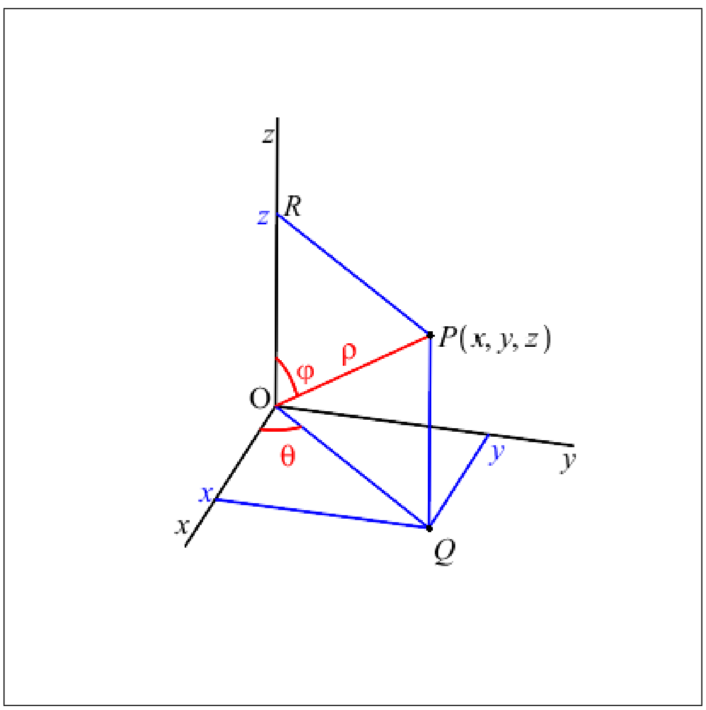

Figure 5:

The spherical coordinates of an arbitrary point $P(x, y, z)$ are the distance ρ from P to origin, the polar angle θ of the projection of P onto the plane Oxy, and the angle φ, formed by OP with the positive direction of Oz axis. The connection between Cartesian and spherical coordinates is the following (see figure 5):

$$\begin{aligned} x &= OQ\cos\theta = PR\cos\theta = \rho\sin\varphi\cos\theta, \\ y &= OQ\sin\theta = PR\sin\theta = \rho\sin\varphi\sin\theta, \\ z &= OP\cos\varphi = \rho\cos\varphi. \end{aligned}$$

The Jacobian determinant of the coordinate conversion is

$$\begin{aligned} \frac{D(x,y,z)}{D(\rho,\varphi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} \\ &= \rho^2\sin\varphi. \end{aligned}$$

The ranges for the new variables (corresponding to all points P in the unit ball in $\mathbb{R}^3$) are

$$\rho \in [0,1], \quad \varphi \in [0,\pi], \quad \theta \in [0, 2\pi].$$

We obtain

$$\begin{aligned} I &= \int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi}\int_{\theta=0}^{\theta=2\pi} \frac{\rho^2\sin\varphi}{\sqrt{\rho^2-4\rho\cos\varphi+4}}\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \left(\int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi} \frac{\rho^2\sin\varphi}{\sqrt{\rho^2-4\rho\cos\varphi+4}}\,\mathrm{d}\rho\mathrm{d}\varphi\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right) \\ &= 2\pi\int_{\rho=0}^{\rho=1}\left(\int_{\varphi=0}^{\varphi=\pi} \frac{\rho^2\sin\varphi}{\sqrt{\rho^2-4\rho\cos\varphi+4}}\,\mathrm{d}\varphi\right)\mathrm{d}\rho. \end{aligned}$$

We use the change of variable $\sqrt{\rho^2-4\rho\cos\varphi+4} = t$. Then

$$\rho^2-4\rho\cos\varphi+4 = t^2 \quad\Rightarrow\quad \rho\sin\varphi\,\mathrm{d}\varphi = \frac{1}{2}\,t\,\mathrm{d}t.$$

We have

$$I = 2\pi\int_{\rho=0}^{\rho=1}\left(\int_{t=2-\rho}^{t=2+\rho}\frac{\rho}{t}\cdot\frac{1}{2}\,t\,\mathrm{d}t\right)\mathrm{d}\rho = \pi\int_0^1 2\rho^2\,\mathrm{d}\rho = \frac{2\pi}{3}\,.$$

9. Let $I := \iiint_A \frac{1}{\sqrt{x^2+y^2+z^2+3}}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

Method 1. The projection of A onto the plane Oxy is the disk

$$A_0 := \{(x,y) \in \mathbb{R}^2 \;\mid\; x^2+y^2 \le 1\}.$$

Applying Fubini's theorem, we have

$$\begin{aligned} I &= \iint_{A_0} \left(\int_{z=0}^{z=\sqrt{1-x^2-y^2}} \frac{\mathrm{d}z}{\sqrt{x^2+y^2+z^2+3}} \right) \mathrm{d}x\mathrm{d}y \\ &= \iint_{A_0} \ln\left(z+\sqrt{x^2+y^2+z^2+3}\right)\Big|_{z=0}^{z=\sqrt{1-x^2-y^2}} \mathrm{d}x\mathrm{d}y \\ &= \iint_{A_0} \ln\left(2+\sqrt{1-x^2-y^2}\right)\mathrm{d}x\mathrm{d}y - \iint_{A_0} \frac{1}{2}\ln\left(x^2+y^2+3\right)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

The above double integrals can be easily calculated by passing to polar coordinates **(Homework)**.

Method 2. By passing to spherical coordinates, we obtain

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi,$$

where $\rho \in [0,1]$, $\varphi \in \left[0, \frac{\pi}{2}\right]$, $\theta \in [0, 2\pi]$. We have

$$\begin{aligned} I &= \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \frac{1}{\sqrt{\rho^2+3}}\,\rho^2 \sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \\ &= \left(\int_0^1 \frac{\rho^2}{\sqrt{\rho^2+3}}\,\mathrm{d}\rho\right)\left(\int_0^{\frac{\pi}{2}} \sin\varphi\,\mathrm{d}\varphi\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right) \\ &= 2\pi\left(1-\frac{3}{4}\ln 3\right). \end{aligned}$$

10. Let $I := \iiint_A \frac{z}{(x^2+y^2+1)^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

Method 1. The projection of A onto the plane Oxy is the disk

$$A_0 := \{(x,y) \in \mathbb{R}^2 \;\mid\; x^2+y^2 \le 1\}.$$

Applying Fubini's theorem, we have

$$\begin{aligned} I &= \iint_{A_0} \left(\int_{z=0}^{z=\sqrt{1-x^2-y^2}} \frac{z}{(x^2+y^2+1)^2} \, \mathrm{d}z \right) \mathrm{d}x\mathrm{d}y \\ &= \iint_{A_0} \frac{1}{(x^2+y^2+1)^2} \cdot \left. \frac{z^2}{2} \right|_{z=0}^{z=\sqrt{1-x^2-y^2}} \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2} \iint_{A_0} \frac{1-x^2-y^2}{(x^2+y^2+1)^2} \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

The above double integral can be easily calculated by passing to polar coordinates **(Homework)**.

Method 2. By passing to spherical coordinates, we obtain

$$\begin{aligned} I &= \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \frac{\rho^3 \sin\varphi \cos\varphi}{(\rho^2 \sin^2\varphi + 1)^2} \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \\ &= \left(\int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \frac{\rho^3 \sin\varphi \cos\varphi}{(\rho^2 \sin^2\varphi + 1)^2} \, \mathrm{d}\rho \mathrm{d}\varphi \right) \left(\int_0^{2\pi} \mathrm{d}\theta \right) \\ &= 2\pi \int_{\rho=0}^{\rho=1} \left(\int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \frac{\rho}{2} \cdot \frac{2\rho^2 \sin\varphi \cos\varphi}{(\rho^2 \sin^2\varphi + 1)^2} \, \mathrm{d}\varphi \right) \mathrm{d}\rho. \end{aligned}$$

Substituting $\rho^2 \sin^2\varphi + 1 = t$, we have $2\rho^2 \sin\varphi \cos\varphi \, \mathrm{d}\varphi = \mathrm{d}t$, hence

$$\begin{aligned} I &= 2\pi \int_{\rho=0}^{\rho=1} \left(\int_{t=1}^{t=\rho^2+1} \frac{\rho}{2} \cdot \frac{\mathrm{d}t}{t^2} \right) \mathrm{d}\rho = \pi \int_{\rho=0}^{\rho=1} \left. -\frac{\rho}{t} \right|_{t=1}^{t=\rho^2+1} \mathrm{d}\rho \\ &= \pi \int_0^1 \left(\rho - \frac{\rho}{\rho^2+1} \right) \mathrm{d}\rho = \frac{\pi}{2}(1 - \ln 2). \end{aligned}$$

11. Let $I := \iiint_A \frac{1}{\sqrt{x^2+y^2+(3-z)^2}} \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$

Method 1. The projection of A onto the plane Oxy is the closed unit disk A_0 in $\mathbb{R}^2$. Applying Fubini's theorem, we have

$$\begin{aligned}
I &= \iint_{A_0} \left(\int_{z=0}^{z=2} \frac{\mathrm{d}z}{\sqrt{(3-z)^2+x^2+y^2}} \right) \mathrm{d}x\mathrm{d}y \\
&= \iint_{A_0} -\ln\left(3-z+\sqrt{(3-z)^2+x^2+y^2}\right) \Big|_{z=0}^{z=2} \mathrm{d}x\mathrm{d}y \\
&= \iint_{A_0} \left(\ln\left(\sqrt{x^2+y^2+9}+3\right) - \ln\left(\sqrt{x^2+y^2+1}+1\right) \right) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

To evaluate the double integral, we pass to polar coordinates. We obtain

$$\begin{aligned}
I &= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=2\pi} \left(\ln\left(\sqrt{\rho^2+9}+3\right) - \ln\left(\sqrt{\rho^2+1}+1\right) \right) \cdot \rho \,\mathrm{d}\rho\mathrm{d}\theta \\
&= \left(\int_0^{2\pi} \mathrm{d}\theta \right) \left(\int_0^1 \rho \ln\left(\sqrt{\rho^2+9}+3\right) \mathrm{d}\rho - \int_0^1 \rho \ln\left(\sqrt{\rho^2+1}+1\right) \mathrm{d}\rho \right).
\end{aligned}$$

Substituting $\sqrt{\rho^2+9}=t$ and $\sqrt{\rho^2+1}=t$, respectively, we get

$$\begin{aligned}
I &= 2\pi \left(\int_3^{\sqrt{10}} t\ln(t+3)\,\mathrm{d}t - \int_1^{\sqrt{2}} t\ln(t+1)\,\mathrm{d}t \right) \\
&= 2\pi \left(\frac{3\sqrt{10}-\sqrt{2}-8}{2} + \frac{1}{2} \ln \frac{3+\sqrt{10}}{1+\sqrt{2}} \right).
\end{aligned}$$

Method 2. We pass to cylindrical coordinates. The cylindrical coordinates of an arbitrary point $P(x,y,z)$ are the distance ρ from P to the Oz axis, the polar angle θ of the projection Q of P onto the plane Oxy and the z-coordinate of P (z is a coordinate in both Cartesian and cylindrical system). The connection between Cartesian and cylindrical coordinates is the following (see figure 6):

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z.$$

The Jacobian determinant of the coordinate conversion is

$$\frac{D(x,y,z)}{D(\rho,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

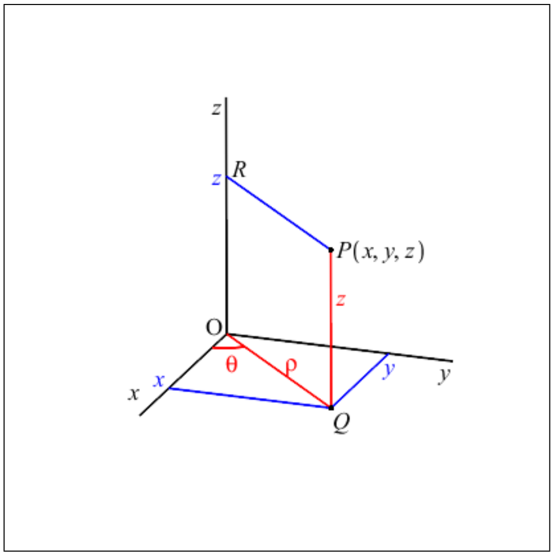

Figure 6:

The ranges for the new variables (corresponding to all points P in the cylinder A) are $\rho \in [0,1]$, $\theta \in [0,2\pi]$, $z \in [0,2]$. We obtain

$$
\begin{aligned}
I &= \int_{\rho=0}^{\rho=1} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=2} \frac{1}{\sqrt{\rho^2+(3-z)^2}}\, \rho \, \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}z \\
&= \left(\int_0^{2\pi} \mathrm{d}\theta \right) \left(\int_{\rho=0}^{\rho=1} \int_{z=0}^{z=2} \frac{\rho}{\sqrt{\rho^2+(3-z)^2}} \, \mathrm{d}\rho \mathrm{d}z \right) \\
&= 2\pi \int_{z=0}^{z=2} \left(\int_{\rho=0}^{\rho=1} \frac{\rho}{\sqrt{\rho^2+(3-z)^2}} \, \mathrm{d}\rho \right) \mathrm{d}z \\
&= 2\pi \int_{z=0}^{z=2} \sqrt{\rho^2+(3-z)^2} \Big|_{\rho=0}^{\rho=1} \mathrm{d}z \\
&= 2\pi \int_0^2 \left(\sqrt{(3-z)^2+1} - (3-z) \right) \mathrm{d}z
\end{aligned}
$$

The change of variable $3 - z = t$ leads to

$$
I = 2\pi \int_1^3 \left(\sqrt{t^2+1} - t \right) \mathrm{d}t = 2\pi \left(\frac{3\sqrt{10} - \sqrt{2} - 8}{2} + \frac{1}{2} \ln \frac{3+\sqrt{10}}{1+\sqrt{2}} \right).
$$

12. Denote by $\mathcal{A}(A)$ the area of A. We have $\mathcal{A}(A) = \iint_A \mathrm{d}x\mathrm{d}y$. To evaluate the double integral, we use the change of variables defined by (see figure 7)

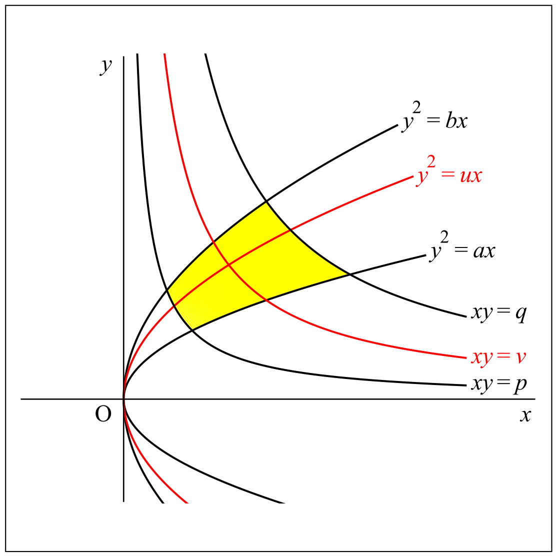

Figure 7:

$$\begin{cases} y^2 = ux \\ xy = v \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = u^{-1/3}v^{2/3}, & u \in [a,b], \\ y = u^{1/3}v^{1/3}, & v \in [p,q]. \end{cases}$$

The Jacobian determinant of the coordinate conversion is

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} -\dfrac{1}{3}\,u^{-4/3}v^{2/3} & \dfrac{2}{3}\,u^{-1/3}v^{-1/3} \\ \dfrac{1}{3}\,u^{-2/3}v^{1/3} & \dfrac{1}{3}\,u^{1/3}v^{-2/3} \end{vmatrix} = -\frac{1}{3u}\,.$$

We have

$$\mathcal{A}(A) = \int_{u=a}^{u=b}\int_{v=p}^{v=q} \frac{1}{3u}\,\mathrm{d}u\mathrm{d}v = \frac{q-p}{3}\,\ln\frac{b}{a}\,.$$