FINITE DIFFERENCES METHOD - TYPICAL PROBLEMS (part III)

NUMERICAL ALGORITHMS: PRIMITIVE VARIABLES

The dimensionless incompressible Navier Stokes equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{8-56}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2 + p) + \frac{\partial}{\partial y}(uv) = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{8-57}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2 + p) = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{8-58}$$

Obviously, the system of equations given by (8-56) through (8-58) includes three unknowns: u, v, and p. If one considers a simple explicit formulation, it would seem logical to use Equation (8-57) to solve for u (where p and v are lagged), whereas one would solve Equation (8-58) for v. Now, one is left with Equation (8-56) to solve for the pressure, but unfortunately pressure does not appear in that equation!

This difficulty is overcome, however, by manipulation of the continuity equation to include the pressure term. Two procedures have been introduced for this purpose. One procedure involves the manipulation of the momentum equation along with the continuity equation. The mathematical details were described previously which resulted in the Poisson equation for pressure. A second procedure incorporates the addition of a time-dependent pressure term to the continuity equation, i.e., to Equation (8-56). This approach is generally known as the artificial compressibility method which is investigated in the following section.

(See Course 7) Poisson equation for pressure:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=-\frac{\partial D}{\partial t}-\frac{\partial^2}{\partial x^2}(u^2)-2\frac{\partial^2}{\partial x \partial y}(uv)-\frac{\partial^2}{\partial y^2}(v^2)+\frac{1}{Re}\left[\frac{\partial^2}{\partial x^2}(D)+\frac{\partial^2}{\partial y^2}(D)\right] \tag{8-46}$$

where

$$D=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

is known as dilatation.

Boundary condition for Poisson equation for pressure:

(Janos Benk, How to Solve The Navier-Stokes Equation, Joint Advanced Student School (JASS), St. Petersburg - Sunday, March 25 through Wednesday, April 4, 2007, http://wwwmayr.informatik.tu-muenchen.de/konferenzen/Jass07/courses/2/)

First we choose the normal component: (Neumann)

$$n \cdot \nabla P \equiv \partial P / \partial n = \frac{1}{\mathrm{Re}} \nabla^2 u_n - \left((\partial u_n / \partial t) + u \cdot \nabla u_n \right) \qquad \text{on } \boldsymbol{\Gamma}$$

The BC with the tangential component (Dirichlet):

$$\tau \cdot \nabla P \equiv \partial P / \partial \tau = \frac{1}{\mathrm{Re}} \nabla^2 u_\tau - \left((\partial u_\tau / \partial t) + u \cdot \nabla u_\tau \right) \qquad \text{on } \boldsymbol{\Gamma}$$

It has been proven that these 2 conditions are equivalent.

We can calculate the pressure up to an arbitrary additive constant!!!

How it works:

(Benjamin Seibold, A compact and fast Matlab code solving the incompressible Navier-Stokes equations on rectangular domains, www-math.mit.edu/~seibold)

We consider the incompressible Navier-Stokes equations in two space dimensions

$$u_t + p_x = -(u^2)_x - (uv)_y + \frac{1}{Re}(u_{xx} + u_{yy}) \tag{1}$$

$$v_t + p_y = -(uv)_x - (v^2)_y + \frac{1}{Re}(v_{xx} + v_{yy}) \tag{2}$$

$$u_x + v_y = 0 \tag{3}$$

on a rectangular domain $\Omega = [0, l_x] \times [0, l_y]$. The four domain boundaries are denoted ***N**orth*, ***S**outh*, ***W**est*, and ***E**ast*. The domain is fixed in time, and we consider no-slip boundary conditions on each wall, i.e.

$$u(x, l_y) = u_N(x) \qquad v(x, l_y) = 0 \tag{4}$$

$$u(x, 0) = u_S(x) \qquad v(x, 0) = 0 \tag{5}$$

$$u(0, y) = 0 \qquad v(0, y) = v_W(y) \tag{6}$$

$$u(l_x, y) = 0 \qquad v(l_x, y) = v_E(y) \tag{7}$$

Solve by projection approach:

In each time step

I. Solve $\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} = \frac{1}{\mathrm{Re}} \nabla^2 \underline{u}$

$$\frac{U^* - U^n}{\Delta t} = -(U^n \cdot \nabla)U^n + \frac{1}{\mathrm{Re}} \nabla^2 U^n$$

Note: $\nabla \cdot U^* \neq 0$

II. Project on divergence-free velocity field

$$\frac{U^{n+1} - U^*}{\Delta t} = -\nabla p$$

What is $p : 0 \stackrel{!}{=} \nabla \cdot U^{n+1} = \nabla \cdot U^* - \Delta t \nabla^2 P$

$$\Rightarrow \nabla^2 p = \frac{1}{\Delta t} \nabla \cdot U^*$$ Poisson equation for pressure

Discretization:

Solution:

$u = v = 0,$

$p = \text{constant}$

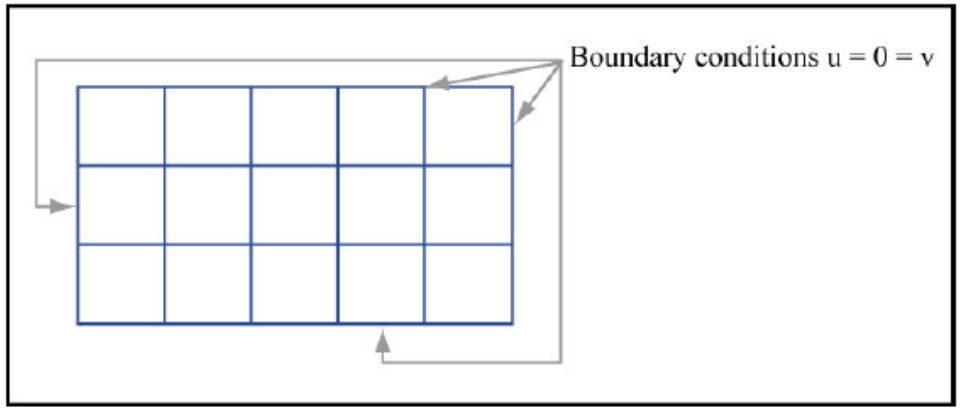
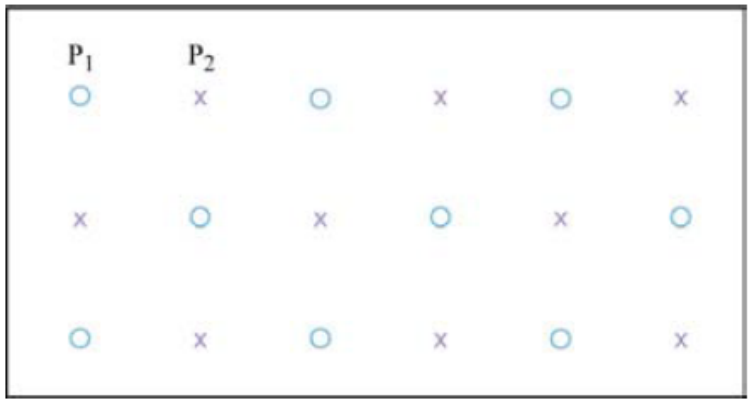


Image by MIT OpenCourseWare.

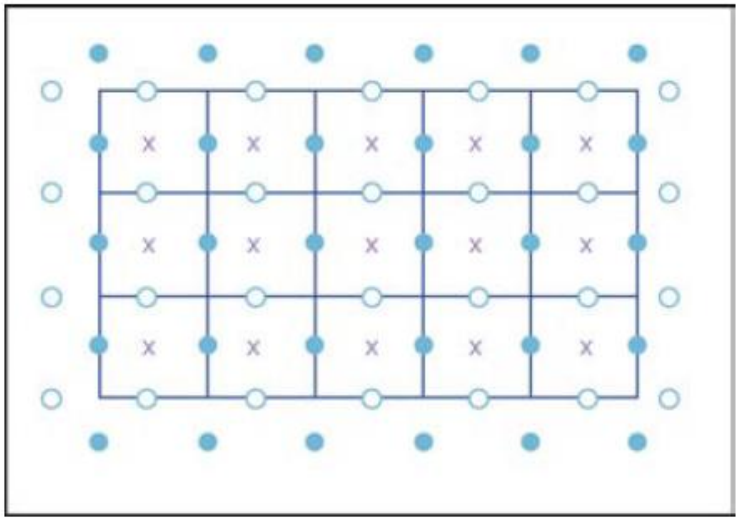
But: Central differences on grid allow solution

$U_{ij} = V_{ij} = 0,$

$$P_{ij} = \left\{ \begin{array}{ll} P_1 & \\ P_2 & \text{for} \end{array} \begin{array}{l} i+j \text{ even} \\ i+j \text{ odd} \end{array} \right\}$$

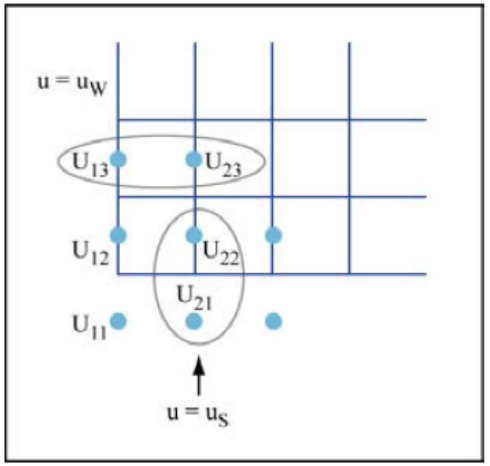


Fix: Staggered grid



× pressure p
● velocity u
○ velocity v

Boundary Conditions:



$$U_{13} = u_w$$

$$\frac{U_{21} + U_{22}}{2} = u_s \Rightarrow U_{21} + U_{22} = 2u_s$$

Numerical Method:

I. a) Treat Nonlinear Terms

$$uu_x + vu_y = (u^2)_x + (uv)_y$$
$$uv_x + vv_y = (uv)_x + (v^2)_y \qquad (\text{use } u_x + v_y = 0)$$

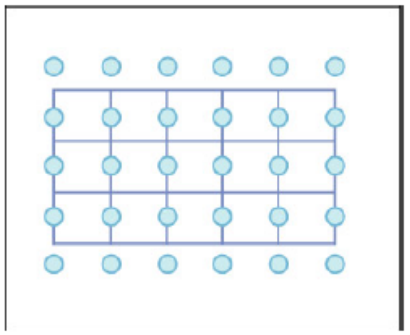
$$\left[\frac{\partial(U^2)}{\partial x}\right]_{ij} = \frac{(U_{i+\frac{1}{2},j})^2 - (U_{i-\frac{1}{2},j})^2}{\Delta x}$$

$$\left[\frac{\partial(UV)}{\partial y}\right]_{ij} = \frac{U_{i,j+\frac{1}{2}}V_{i,j+\frac{1}{2}} - U_{i,j-\frac{1}{2}}V_{i,j-\frac{1}{2}}}{\Delta y}$$

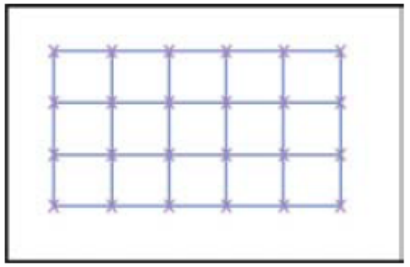
$$\left[\frac{\partial(UV)}{\partial x}\right]_{ij} = \frac{U_{i+\frac{1}{2},j}V_{i+\frac{1}{2},j} - U_{i-\frac{1}{2},j}V_{i-\frac{1}{2},j}}{\Delta x}$$

$$\left[\frac{\partial(V^2)}{\partial y}\right]_{ij} = \frac{(V_{i,j+\frac{1}{2}})^2 - (V_{i,j-\frac{1}{2}})^2}{\Delta y}$$

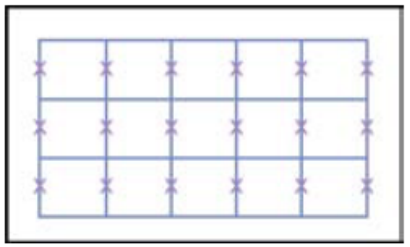
U_{ij}



$U_{i,j+\frac{1}{2}}$



$\left[\frac{\partial(UV)}{\partial y}\right]_{ij}$



where $U_{i+\frac{1}{2},j} = \dfrac{U_{i,j} + U_{i+1,j}}{2}$, $U_{i,j+\frac{1}{2}} = \dfrac{U_{i,j} + U_{i,j+1}}{2}$

$$\frac{U^*_{i,j} - U^n_{i,j}}{\Delta t} = -\left[\frac{\partial(U^2)}{\partial x}\right]^n_{i,j} - \left[\frac{\partial(UV)}{\partial y}\right]^n_{i,j}$$

$$\frac{{V_{i,j}}^* - {V_{i,j}}^n}{\Delta t} = -\left[\frac{\partial(UV)}{\partial x}\right]^n_{i,j} - \left[\frac{\partial(V^2)}{\partial y}\right]^n_{i,j}$$

I. b) Implicit Diffusion

$$\frac{U^{**} - U^*}{\Delta t} = \frac{1}{Re}(U^{**}_{xx} + U^{**}_{yy})$$

$$\frac{V^{**} - V^*}{\Delta t} = \frac{1}{Re}(V^{**}_{xx} + V^{**}_{yy})$$

II. Pressure Correction

We correct the intermediate velocity field (U^{**}, V^{**}) by the gradient of a pressure P^{n+1} to enforce incompressibility.

$$\frac{U^{n+1} - U^{**}}{\Delta t} = -(P^{n+1})_x \tag{16}$$

$$\frac{V^{n+1} - V^{**}}{\Delta t} = -(P^{n+1})_y \tag{17}$$

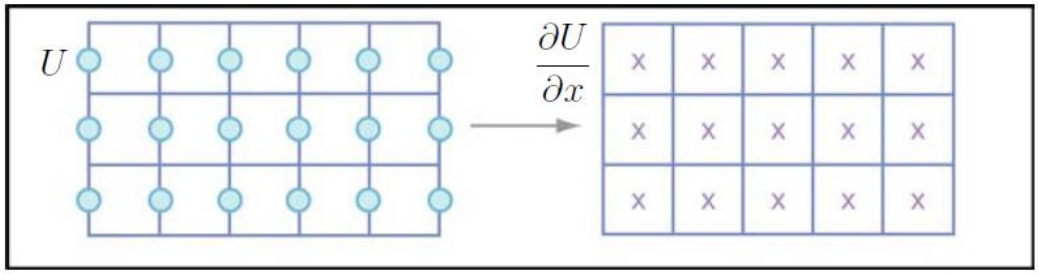
The pressure is denoted P^{n+1}, since it is only given implicitly. It is obtained by solving a linear system. In vector notation the correction equations read as

$$\frac{1}{\Delta t}\mathbf{U}^{n+1} - \frac{1}{\Delta t}\mathbf{U}^{n} = -\nabla P^{n+1} \tag{18}$$

Applying the divergence to both sides yields the linear system

$$-\Delta P^{n+1} = -\frac{1}{\Delta t}\nabla \cdot \mathbf{U}^{n} \tag{19}$$

Hence, the pressure correction step is



(a) Compute $F^n = \nabla \cdot \mathbf{U}^n$

(b) Solve Poisson equation $-\Delta P^{n+1} = -\frac{1}{\Delta t} F^n$

(c) Compute $\mathbf{G}^{n+1} = \nabla P^{n+1}$

(d) Update velocity field $\mathbf{U}^{n+1} = \mathbf{U}^n - \Delta t \mathbf{G}^{n+1}$

The question, which boundary conditions are appropriate for the Poisson equation for the pressure P, is complicated. A standard approach is to prescribe homogeneous Neumann boundary conditions for P wherever no-slip boundary conditions are prescribed for the velocity field. For the lid driven cavity problem this means that homogeneous Neumann boundary conditions are prescribed everywhere. This implies in particular that the pressure P is only defined up to a constant, which is fine, since only the gradient of P enters the momentum equation.

In addition to the solution steps, we have the visualization step, in which the stream function Q^n is computed. Similarly to the pressure is is obtained by the following steps

1. Compute $F^n = (V^n)_x - (U^n)_y$
2. Solve Poisson equation $-\Delta Q^n = -F^n$

We prescribe homogeneous Dirichlet boundary conditions.

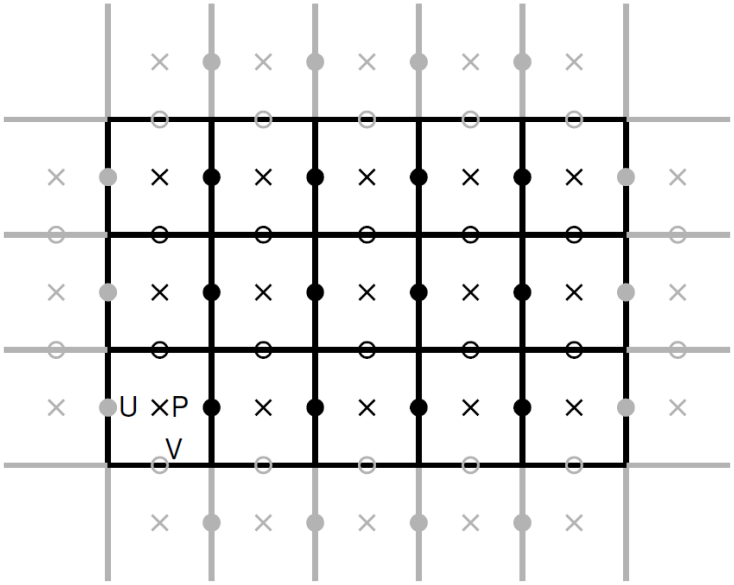
Spacial Discretization

The spacial discretization is performed on a staggered grid with the pressure P in the cell midpoints, the velocities U placed on the vertical cell interfaces, and the velocities V placed on the horizontal cell interfaces. The stream function Q is defined on the cell corners.

Consider to have $n_x \times n_y$ cells. Figure 1 shows a staggered grid with $n_x = 5$ and $n_y = 3$. When speaking of the fields P, U and V (and Q), care has to be taken about interior and boundary points. Any point truly inside the domain is an interior point, while points on or outside boundaries are boundary points. Dark markers in Figure 1 stand for interior points, while light markers represent boundary points. The fields have the following sizes:

field quantity	interior resolution	resolution with boundary points
pressure P	$n_x \times n_y$	$(n_x + 2) \times (n_y + 2)$
velocity component U	$(n_x - 1) \times n_y$	$(n_x + 1) \times (n_y + 2)$
velocity component V	$n_x \times (n_y - 1)$	$(n_x + 2) \times (n_y + 1)$
stream function Q	$(n_x - 1) \times (n_y - 1)$	$(n_x + 1) \times (n_y + 1)$

The values at boundary points are no unknown variables. For Dirichlet boundary conditions they are prescribed, and for Neumann boundry conditions they can be expressed in term of interior points. However, boundary points of U and V are used for the finite difference approximation of the nonlinear advection terms. Note that the boundary points in the four corner are never used.



- **Second derivatives**
  Finite differences can approximate second derivatives in a grid point by a centered stencil. At an interior point $U_{i,j}$ we approximate the Laplace operator by

$$\Delta U_{i,j} = (U_{xx})_{i,j} + (U_{yy})_{i,j} \approx \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h_x^2} + \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h_y^2}$$

- **First derivatives**
  A first derivative in a grid point can be approximated by a centered stencil.

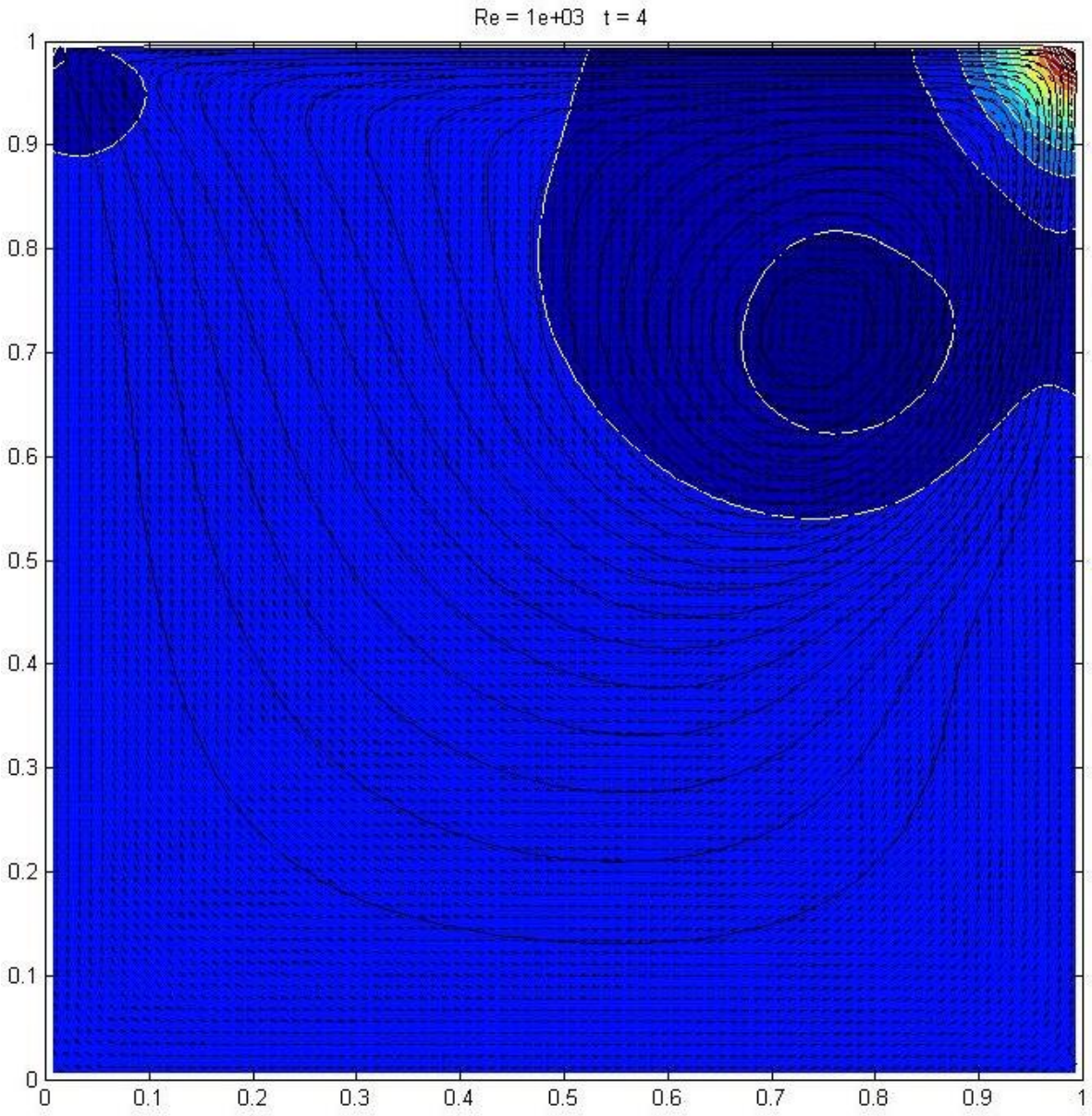
$$(U_x)_{i,j} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2h_x}$$

This, however, can yield instabilities, as shown in many textbooks on numerical analysis. Here the staggered grid comes into play. Assume, we are not interested in the value of U_x in the position of $U_{i,j}$, but instead we want the value in the middle between the points $U_{i+1,j}$ and $U_{i,j}$. Then the approximation

$$(U_x)_{i+\frac{1}{2},j} \approx \frac{U_{i+1,j} - U_{i,j}}{h_x}$$

is a stable centered approximation to U_x in the middle between the two points. In the staggered grid this position happens to be the position of $P_{i,j}$.

See more details:
(http://math.mit.edu/~gs/cse/codes/mit18086_navierstokes.pdf)
(http://math.mit.edu/~gs/cse/codes/mit18086_navierstokes.m)


Re = 1e+03 t = 4
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Artificial compressibility method:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2 + p) + \frac{\partial}{\partial y}(uv) = \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2 + p) = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Artificial Compressibility

The application of the scheme to the steady incompressible Navier-Stokes equations was introduced by Chorin (Ref. [8-5]). The continuity equation is modified by inclusion of a time-dependent term and is given by

$$\frac{\partial p}{\partial t} + \frac{1}{\tau}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

where τ can be interpreted as the "artificial compressibility" of the fluid. Following the equation of state the compressibility can be related to a pseudo-speed of sound

and to an artificial density by the following relations

$$\tau = \frac{1}{a^2}$$

$$a^2 = \frac{p}{\rho}$$

where all the quantities are in nondimensional form.

Thus, the steady incompressible Navier-Stokes equations (two-dimensional Cartesian coordinates) are expressed in a pseudotransient form as

$$\frac{\partial p}{\partial t} + a^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2 + p) + \frac{\partial}{\partial y}(uv) = \frac{1}{Re} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2 + p) = \frac{1}{Re} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

Solution on Regular Grid

To facilitate the application of the finite difference formulations, the conservative form of the governing equations from Equations (8-60) through (8-62) are written in a flux vector form as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{1}{Re}[N]\nabla^2 Q$$

where

$$Q = \begin{bmatrix} p \\ u \\ v \end{bmatrix} \quad , \quad E = \begin{bmatrix} a^2 u \\ u^2 + p \\ uv \end{bmatrix} \quad , \quad F = \begin{bmatrix} a^2 v \\ uv \\ v^2 + p \end{bmatrix}$$

and

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Liniarization

$$E^{n+1} = E^n + \frac{\partial E}{\partial Q}\Delta Q + O(\Delta t)^2$$

$$\Delta Q = Q^{n+1} - Q^n$$

Terms such as $\frac{\partial E}{\partial Q}$ are known as flux Jacobian matrices. The Jacobian matrices $\frac{\partial E}{\partial Q}$ and $\frac{\partial F}{\partial Q}$ will be denoted by A and B, respectively. The Jacobian matrix A is

$$A = \frac{\partial [E_1, E_2, E_3]}{\partial [Q_1, Q_2, Q_3]} = \begin{bmatrix} \frac{\partial E_1}{\partial Q_1} & \frac{\partial E_1}{\partial Q_2} & \frac{\partial E_1}{\partial Q_3} \\ \frac{\partial E_2}{\partial Q_1} & \frac{\partial E_2}{\partial Q_2} & \frac{\partial E_2}{\partial Q_3} \\ \frac{\partial E_3}{\partial Q_1} & \frac{\partial E_3}{\partial Q_2} & \frac{\partial E_3}{\partial Q_3} \end{bmatrix}$$

Recall that the vectors Q and E are

$$Q = \begin{bmatrix} p \\ u \\ v \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} a^2 u \\ u^2 + p \\ uv \end{bmatrix} = \begin{bmatrix} a^2 Q_2 \\ Q_2^2 + Q_1 \\ Q_2 Q_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Thus,

$$\frac{\partial E_1}{\partial Q_1} = \frac{\partial (a^2 Q_2)}{\partial Q_1} = 0$$

$$\frac{\partial E_1}{\partial Q_2} = \frac{\partial (a^2 Q_2)}{\partial Q_2} = a^2$$

$$\frac{\partial E_1}{\partial Q_3} = \frac{\partial (a^2 Q_2)}{\partial Q_3} = 0$$

The remaining elements are determined in a similar fashion resulting in

$$A = \begin{bmatrix} 0 & a^2 & 0 \\ 1 & 2u & 0 \\ 0 & v & u \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & v & u \\ 1 & 0 & 2v \end{bmatrix}$$

Crank-Nicolson Implicit

Recall that the Crank-Nicolson formulation requires an averaging of the terms at time levels of n and $n+1$.

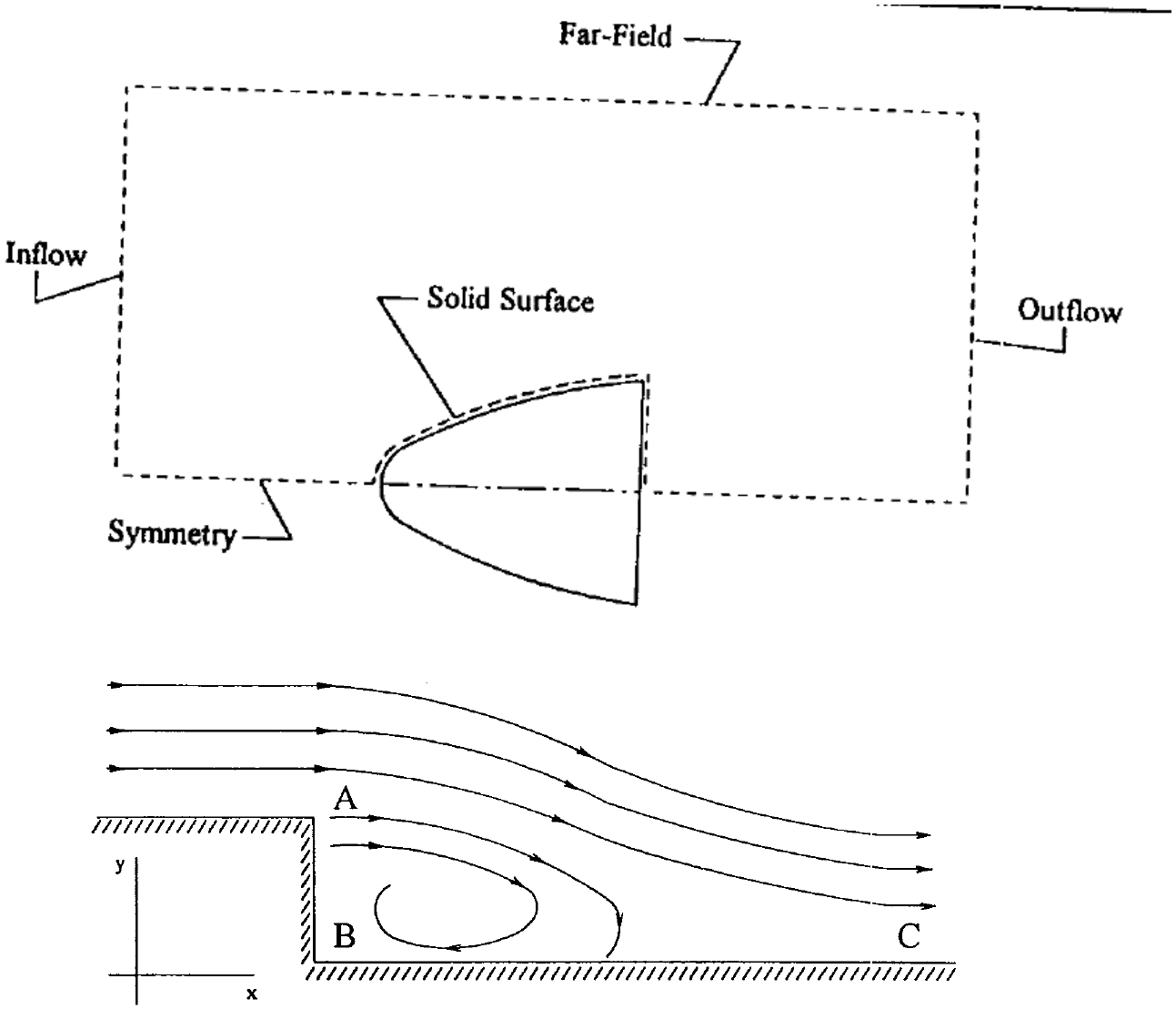
$$\frac{\Delta Q}{\Delta t} + \frac{1}{2}\left[\left(\frac{\partial E}{\partial x}\right)^n + \left(\frac{\partial E}{\partial x}\right)^{n+1}\right] + \frac{1}{2}\left[\left(\frac{\partial F}{\partial y}\right)^n + \left(\frac{\partial F}{\partial y}\right)^{n+1}\right]$$

$$= \frac{1}{2}\frac{1}{Re}[N]\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(Q^n + Q^{n+1})$$

or

$$\left\{I + \frac{\Delta t}{2}\left[\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} - \frac{N}{Re}\left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right)\right]\right\}\Delta Q = \text{RHS}$$

Boundary Conditions



Generally speaking, the Neumann-type boundary condition is imposed for the pressure. For this purpose a relation involving the normal pressure gradient is obtained from the appropriate momentum equation. For example, the following expression can be utilized along the solid boundary aligned parallel to the y-coordinate:

$$\frac{\partial p}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

This pressure boundary condition may be specified along the boundary AB of the classical steady flow over step Similarly, the condition

$$\frac{\partial p}{\partial y} = \frac{1}{Re}\frac{\partial^2 v}{\partial y^2}$$

would be imposed along the boundary BC. These boundary conditions are implemented in the solution of the Poisson equation for pressure.

Far-Field

Specification of boundary conditions on the far-field boundary is very much problem dependent. If the boundary is located far away such that the flow properties on the boundary are not influenced by the interior solution, then it is indeed far-field and usually the freestream conditions are imposed. On the other hand, if the boundary is located relatively close to the "action," the boundary can no longer be considered as far-field and must be dealt with as an inflow and/or outflow boundary. Whether the boundary is considered as an outflow boundary or an inflow boundary depends on the sign of the velocity component normal to the boundary. If the velocity is into the domain, that portion is considered as inflow boundary; otherwise it is considered as an outflow and appropriate boundary conditions must be incorporated.

Symmetry

For applications where the configuration and the domain of solution are symmetrical, the axis of symmetry (or surface of symmetry) may be used as a boundary. The boundary location may be defined in two fashions. First, the boundary is set on the axis of symmetry as shown in Fig. 8-3.

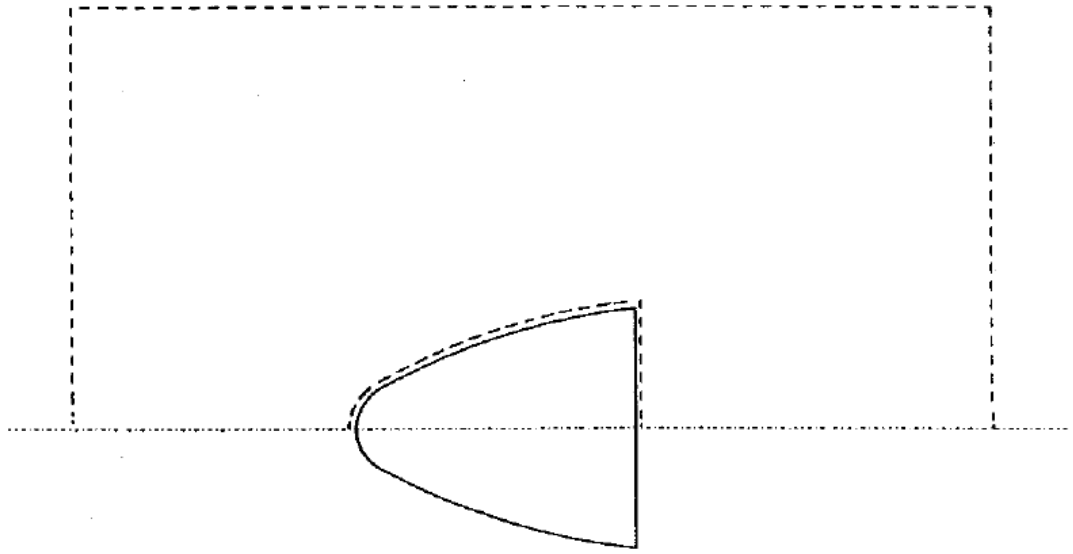


Figure 8-3. Illustration of boundary set on the axis of symmetry.

In this case the net flow across the symmetry line is zero. Therefore, the component of the velocity normal to the boundary is set to zero. Furthermore, the shear stress along the axis of symmetry may be zero in some applications. Thus, the velocity gradient is set to zero. Second, the boundary may be set below the axis of symmetry as shown in Fig. 8-4, in which case the symmetry of flow variables are used as the required boundary conditions.

Inflow

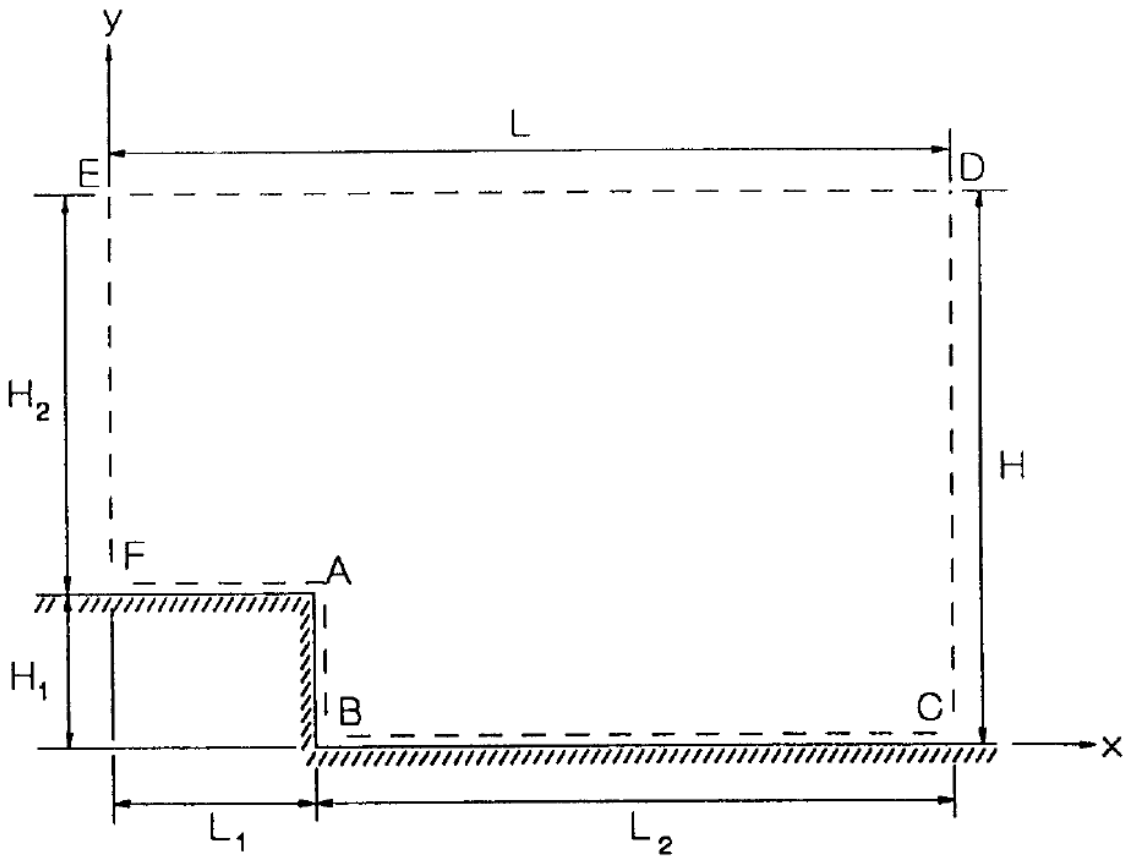
Usually two boundary conditions are required at the inflow. For most applications, pressure and one component of the velocity are provided. Typically then $u(y) = u_o$ and p are provided. The y-component of the velocity may be set to zero or may be determined by setting the velocity gradient $\partial v/\partial x$ to zero. If one uses a first-order approximation, then $v_{1,j} = v_{2,j}$. Higher order approximations can be used as well; for example a second-order approximation yields

$$v_{1,j} = 2v_{2,j} - v_{3,j}$$

Outflow

Generally speaking the value of the velocity and/or pressure are not known at the outflow boundary. Therefore, for most applications Neumann type boundary conditions are imposed. The specification of zero velocity gradient at the outflow may be appropriate for most applications.

Example for BC



Boundary EF (Inflow):

$$H_1 < y < H \qquad \begin{cases} u = u_\infty \\ v = 0 \ \text{ or } \ \dfrac{\partial v}{\partial x} = 0 \\ p = p_\infty \ \text{ or } \ \dfrac{\partial p}{\partial x} = \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial x^2} \end{cases}$$

Boundary FA (Solid surface):

$$0 < x < L_1 \qquad \begin{cases} u = 0 \\ v = 0 \\ \dfrac{\partial p}{\partial y} = \dfrac{1}{Re}\dfrac{\partial^2 v}{\partial y^2} \end{cases}$$

Boundary AB (Solid surface):

$$0 < y < H_1 \qquad \begin{cases} u = 0 \\ v = 0 \\ \dfrac{\partial p}{\partial x} = \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial x^2} \end{cases}$$

Boundary BC (Solid surface):

$$L_1 < x < L \qquad \begin{cases} u = 0 \\ v = 0 \\ \dfrac{\partial p}{\partial y} = \dfrac{1}{Re}\dfrac{\partial^2 v}{\partial y^2} \end{cases}$$

Boundary CD (Outflow):

$$0 < y < H \qquad \begin{cases} \dfrac{\partial u}{\partial x} = 0 \\ \dfrac{\partial v}{\partial y} = 0 \\ p = p_\infty \quad \text{or} \quad \dfrac{\partial p}{\partial x} = 0 \end{cases}$$

Boundary ED (Far-Field):

$$0 < x < L \qquad \begin{cases} u = u_\infty \\ v = 0 \\ p = p_\infty \end{cases}$$