FINITE DIFFERENCES METHOD - TYPICAL PROBLEMS (part II)

FREE CONVECTIVE FLOW IN A DIFFERENTIALLY HEATED SQUARE CAVITY

1. Introduction

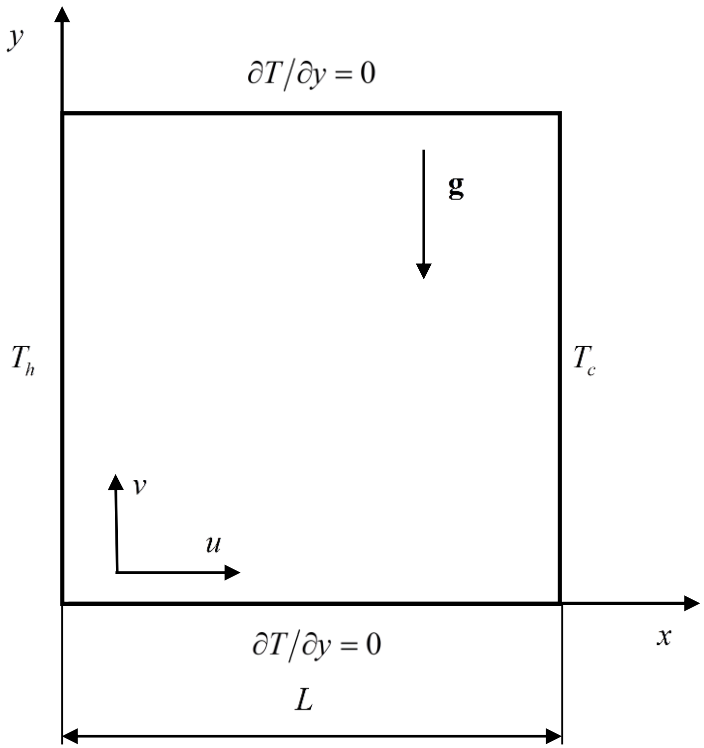


Fig. 1. Physical model and coordinate system.

With many industrial and environmental applications, the natural convection of enclosed fluids has been an important subject due to its particular transition to turbulence mechanism by destabilizing the buoyancy-driven flow and its high numerical computational requirements. A large section of the previous research done on this topic has been reviewed by Bejan [1]. Extensive work was done by De Vahl Davis [2] who presented the final form of the problem and computational results for high Rayleigh numbers.

2. Mathematical model

We consider the natural convection of a fluid in a rectangular cavity. The coordinate system is chosen such that the origin is set in the cavity's bottom left corner and the y-axis is parallel and of opposite direction with the gravitational acceleration vector g. In this paper the cavity is considered to be square hence the height is denoted by L (see Figure 1). The left hand wall is at given uniform temperature T_h, while the right hand one is subjected to a uniform temperature T_c, where $T_h > T_c$. Considering all the fluid properties except density in the buoyancy term (Boussinesq approximation) constant the governing equations written in Cartesian coordinates x, y are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left[\beta_T\,(T - T_0)\,\right]g \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

The boundary conditions of these equations are

$$
\begin{aligned}
x = 0: \quad & u = v = 0, \quad T = T_h \\
x = L: \quad & u = v = 0, \quad T = T_c \\
y = 0: \quad & u = v = 0, \quad \frac{\partial T}{\partial y} = 0 \\
y = L: \quad & u = v = 0, \quad \frac{\partial T}{\partial y} = 0
\end{aligned} \tag{5}
$$

Here u and v are the two dimensional fluid velocity components, p is the dynamic pressure, which is equal to the total pressure minus the hydrostatic component, T is the fluid temperature

Further, we introduce the following non-dimensional variables

$$
\begin{aligned}
& X = x/L, \quad Y = y/L, \quad U = u/U_0, V = v/U_0, \\
& \theta = (T - T_0)/(T_h - T_c), \; P = (\mu U_0 / L)^{-1} p
\end{aligned} \tag{6}
$$

where U_0 is defined as $U_0 = \alpha / L$ and T_0 is the characteristic temperature given by $T_0 = (T_h + T_c)/2$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial X}\left(U \cdot U_o\right) \cdot \frac{\partial X}{\partial x} = \frac{U_o}{L}\,\frac{\partial U}{\partial X}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial X}\left(\frac{U_o}{L} \cdot \frac{\partial U}{\partial X}\right) \cdot \frac{\partial X}{\partial x} = \frac{U_o}{L^2}\,\frac{\partial^2 U}{\partial X^2}$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial X}\left(\frac{\mu U_o}{L} \cdot P\right) \cdot \frac{\partial X}{\partial x} = \frac{\mu U_o}{L^2} \cdot \frac{\partial P}{\partial X^2}$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial X}\left(T_o + \theta\,(T_h - T_c)\right) \cdot \frac{\partial X}{\partial x} = \frac{T_h - T_c}{L} \cdot \frac{\partial \theta}{\partial X}$$

$$T = T_o + \theta\,(T_h - T_c)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial X}\left(\frac{T_h - T_c}{L} \cdot \frac{\partial \theta}{\partial X}\right) \cdot \frac{\partial X}{\partial x} = \frac{T_h - T_c}{L^2} \cdot \frac{\partial^2 \theta}{\partial X^2}$$

$$\frac{U_0}{L}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=0$$

$$U_0 U \frac{U_0}{L}\frac{\partial U}{\partial X}+U_0 V\frac{U_0}{L}\frac{\partial U}{\partial Y}=-\frac{1}{\rho}\frac{\mu U_0}{L^2}\frac{\partial P}{\partial X}+\frac{\nu U_0}{L^2}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)\ \Big|\ :\frac{U_0^2}{L}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{1}{\rho}\cdot\frac{\mu}{U_0 L}\frac{\partial P}{\partial X}+\frac{\nu}{U_0\cdot L}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$

($\frac{\mu}{\rho}=\nu$; $\frac{\mu}{\rho U_0 L}=\frac{\alpha}{L}$; $\frac{\nu}{U_0\cdot L}=\frac{\alpha}{L}$)

$$U\cdot\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-Pr\cdot\frac{\partial P}{\partial X}+Pr\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$

$$\frac{U_0^2}{L}\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right) = -\frac{\nu U_0}{L^2}\frac{\partial P}{\partial Y} + \frac{\nu U_0}{L^2}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) + \left[\beta_T (T_h-T_c)\cdot\theta\right]\cdot g \;\Big|\; : \frac{U_0^2}{L}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y} = -Pr\cdot\frac{\partial P}{\partial Y} + Pr\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) + \frac{\beta_T (T_h-T_c)\cdot g\cdot L}{U_0^2}\cdot\theta$$

$$\frac{\beta_T (T_h-T_c)\cdot g\cdot L}{\frac{\alpha^2}{L^2}} = \underbrace{\frac{\beta_T (T_h-T_c)\cdot g\cdot L^3}{\alpha\cdot\nu}}_{Ra}\cdot\underbrace{\frac{\nu}{\alpha}}_{Pr}$$

Ra: Rayleigh number; Pr: Prandtl number

$$U_0 U\cdot\frac{T_h-T_c}{L}\cdot\frac{\partial\theta}{\partial X} + U_0 V\cdot\frac{T_h-T_c}{L}\frac{\partial\theta}{\partial Y} = \alpha\frac{T_h-T_c}{L^2}\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right) \;\Big|\; : \frac{\alpha(T_h-T_c)}{L^2}$$

$$U_0 = \frac{\alpha}{L}$$

$$U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y} = \frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}$$

Using the dimensionless stream function ψ and vorticity function Ω, which are defined as usual $U = \partial\psi/\partial Y$, $V = -\partial\psi/\partial X$ and $\Omega = -\left(\partial^2\psi/\partial X^2 + \partial^2\psi/\partial Y^2\right)$, we obtain the following dimensionless system of equations:

$$\frac{\partial\Omega}{\partial X}\frac{\partial\psi}{\partial Y} - \frac{\partial\psi}{\partial X}\frac{\partial\Omega}{\partial Y} = \Pr\left(\frac{\partial^2\Omega}{\partial X^2} + \frac{\partial^2\Omega}{\partial Y^2}\right) + Ra\Pr\frac{\partial\theta}{\partial X} \qquad (7)$$

$$\frac{\partial\theta}{\partial X}\frac{\partial\psi}{\partial Y} - \frac{\partial\psi}{\partial X}\frac{\partial\theta}{\partial Y} = \frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} \qquad (8)$$

and the boundary conditions (5) become

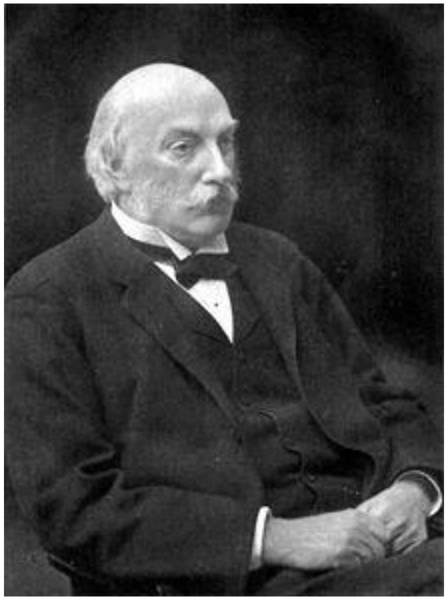
$$\begin{aligned}
X = 0:&\quad \psi = 0,\quad \theta = 1/2\\
X = 1:&\quad \psi = 0,\quad \theta = -1/2\\
Y = 0:&\quad \psi = 0,\quad \partial\theta/\partial y = 0\\
Y = 1:&\quad \psi = 0,\quad \frac{\partial\theta}{\partial y} = 0
\end{aligned} \qquad (9)$$

Here the parameters Ra and Pr are defined as

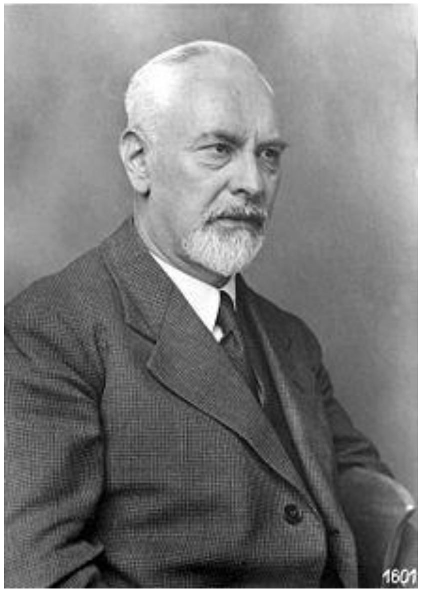
$$Ra = \frac{g\,\beta_T\left(T_h - T_c\right)L^3}{\alpha\,\nu}, \quad Pr = \frac{\nu}{\alpha} \qquad (10)$$

The physical quantity of interest is the mean Nusselt number Nu given by

$$Nu = -\int_0^1 \left(\frac{\partial \theta}{\partial x}\right)_{x=0} dy \qquad (11)$$



John William Strutt, 3rd Baron Rayleigh
(1842 –1919),



Ludwig Prandtl
(1875 – 1953)

Rayleigh number (Ra) for a fluid is a dimensionless number associated with buoyancy-driven flow, also known as free convection or natural convection. When the Rayleigh number is below a critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection.

The **Prandtl number** (Pr) or Prandtl group is a dimensionless number, named after the German physicist **Ludwig Prandtl**, defined as the ratio of momentum diffusivity to thermal diffusivity

The **Nusselt number** (**Nu**) is the ratio of convective to conductive heat transfer across (normal to) the boundary. In this context, convection includes both advection and diffusion. Named after **Wilhelm Nusselt**, it is a dimensionless number.

3 Numerical method

The numerical solution of system (7)-(9) was obtained using a central finite-difference scheme together with a Gauss-Seidel iteration technique. The unknowns θ, ϕ and Ω were iteratively computed until the following convergence criteria was fulfilled $\left|\max\left[f_{new}(i,j)-f_{old}(i,j)\right]\right|<\varepsilon$, where f represents the temperature, stream function or vorticity and ε(=10e-8) is the convergence criteria.

$$\frac{\partial\theta}{\partial x}\cdot\frac{\partial\Psi}{\partial y}-\frac{\partial\theta}{\partial y}\cdot\frac{\partial\Psi}{\partial x}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$$

$$\frac{\partial\Omega}{\partial x}\cdot\frac{\partial\Psi}{\partial y}-\frac{\partial\Omega}{\partial y}\cdot\frac{\partial\Psi}{\partial x}=Pr\left(\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}\right)+Ra\,Pr\,\frac{\partial\theta}{\partial x}$$

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-\Omega$$

$$\frac{\theta_{i+1,j}-\theta_{i-1,j}}{2\Delta x}\cdot\frac{\Psi_{i,j+1}-\Psi_{i,j-1}}{2\Delta y}-\frac{\theta_{i,j+1}-\theta_{i,j-1}}{2\Delta y}\cdot\frac{\Psi_{i+1,j}-\Psi_{i-1,j}}{2\Delta x}=$$

$$\frac{\theta_{i+1,j}-2\theta_{ij}+\theta_{i-1,j}}{(\Delta x)^2}+\frac{\theta_{i,j+1}-2\theta_{ij}+\theta_{i,j-1}}{(\Delta y)^2}$$

$$\frac{\Omega_{i+1,j}-\Omega_{i-1,j}}{2\Delta x}\cdot\frac{\Psi_{i,j+1}-\Psi_{i,j-1}}{2\Delta y}-\frac{\Omega_{i,j+1}-\Omega_{i,j-1}}{2\Delta y}\cdot\frac{\Psi_{i+1,j}-\Psi_{i-1,j}}{2\Delta x}=$$

$$=Pr\left[\frac{\Omega_{i+1,j}-2\Omega_{ij}+\Omega_{i-1,j}}{(\Delta x)^2}+\frac{\Omega_{i,j+1}-2\Omega_{ij}+\Omega_{i,j-1}}{(\Delta y)^2}\right]+Ra\,Pr\cdot\frac{\theta_{i+1,j}-\theta_{i-1,j}}{2\Delta x}$$

$$\frac{\Psi_{i+1,j}-2\Psi_{ij}+\Psi_{i-1,j}}{(\Delta x)^2}+\frac{\Psi_{i,j+1}-2\Psi_{ij}+\Psi_{i,j-1}}{(\Delta y)^2}=-\Omega_{ij}$$

$$\theta_{ij} = 0.0625\left[(\theta_{i,j+1} - \theta_{i,j-1})(\psi_{i+1,j} - \psi_{i-1,j}) - (\theta_{i+1,j} - \theta_{i-1,j})(\psi_{i,j+1} - \psi_{i,j-1})\right] + 0.25\left(\theta_{i+1,j} + \theta_{i-1,j} + \theta_{i,j+1} + \theta_{i,j-1}\right)$$

$$\Omega_{ij} = \frac{0.0625}{Pr}\left[(\quad)\cdot(\quad) - (\quad)(\quad)\right] + 0.25\cdot(\qquad) + Ra\cdot h\cdot 0.125\cdot\left(\theta_{i+1,j} - \theta_{i-1,j}\right)$$

$$\psi_{ij} = 0.25\left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1}\right) + 0.25\cdot h^2\cdot\Omega_{i,j}$$

```
format long g;
tic
N=101; h=1/(N-1);
xplot=0:h:1; yplot=0:h:1;

Ra=10000;Pr=0.71;
```

```
u=zeros(N,N);O=zeros(N,N);
uo=zeros(N,N);Oo=zeros(N,N);
T=zeros(N,N); To=zeros(N,N);
To(1,:)=0.5;To(N,:)=-0.5;

stop=1;
nr_it=0;

while stop==1
      nr_it=nr_it+1;
      for i=2:N-1
       for j=2:N-1
        T(i,j)=0.0625*((To(i,j+1)-T(i,j-1))*(uo(i+1,j)-u(i-1,j))-(To(i+1,j)-
             T(i-1,j))*(uo(i,j+1)-u(i,j-1)))+0.25*(To(i+1,j)+T(i-1,j)+
             To(i,j+1)+T(i,j-1));
        O(i,j)=0.0625/Pr*((Oo(i,j+1)-O(i,j-1))*(uo(i+1,j)-u(i-1,j))-(Oo(i+1,j)-
              O(i-1,j))*(uo(i,j+1)-u(i,j-1)))+0.25*(Oo(i+1,j)+O(i1,j)+Oo(i,j+1)+
              O(i,j-1))+0.125*Ra*h*(To(i+1,j)-T(i-1,j));
        u(i,j)=0.25*(uo(i+1,j)+u(i-1,j)+uo(i,j+1)+u(i,j-1))+0.25*h*h*O(i,j);
       end;
      end;
        O(1,:)=-2*(u(2,:)-u(1,:))/h/h;
        O(N,:)=2*(u(N,:)-u(N-1,:))/h/h;
        O(:,1)=-2*(u(:,2)-u(:,1))/h/h;
        O(:,N)=2*(u(:,N)-u(:,N-1))/h/h;
        T(1,:)=0.5;T(N,:)=-0.5;
        T(:,1)=T(:,2);T(:,N)=T(:,N-1);%adiabatic
```

```
      errU=norm(u-uo);errO=norm(O-Oo);errT=norm(T-To);
      if mod(nr_it,1000)==0
         fprintf('nr_it=%d  errU=%g  errO=%g errT=%g\n', nr_it, errU, errO,errT);
         if (errU<1e-8)&(errO<1e-8)&(errT<1e-8)
             stop=0;
         end
      end
      uo=u;Oo=O;To=T;
end;

figure(1)
 contour(xplot,yplot,u',50);
axis equal
 figure(2)
 contour(xplot,yplot,O',50);
 axis equal
figure(3)
 contour(xplot,yplot,T',50);
 axis equal

max(max(abs(u)))
Nu_loc=-(T(2,:)-T(1,:))/h;
Nu=trapz(yplot,Nu_loc)
toc
```

Table 1. Grid dependence study for $Ra = 1000$, $\Pr = 0.71$

Nodes	*Nu*	*CPU time* (seconds)
26x26	1.124468	3.53
51x51	1.121141	38.73
76x76	1.120023	179.55
101x101	1.119464	541.16
126x126	1.119128	1247.27
151x151	1.118905	3033.25
176x176	1.118746	5194.78
201x201	1.118626	9138.46

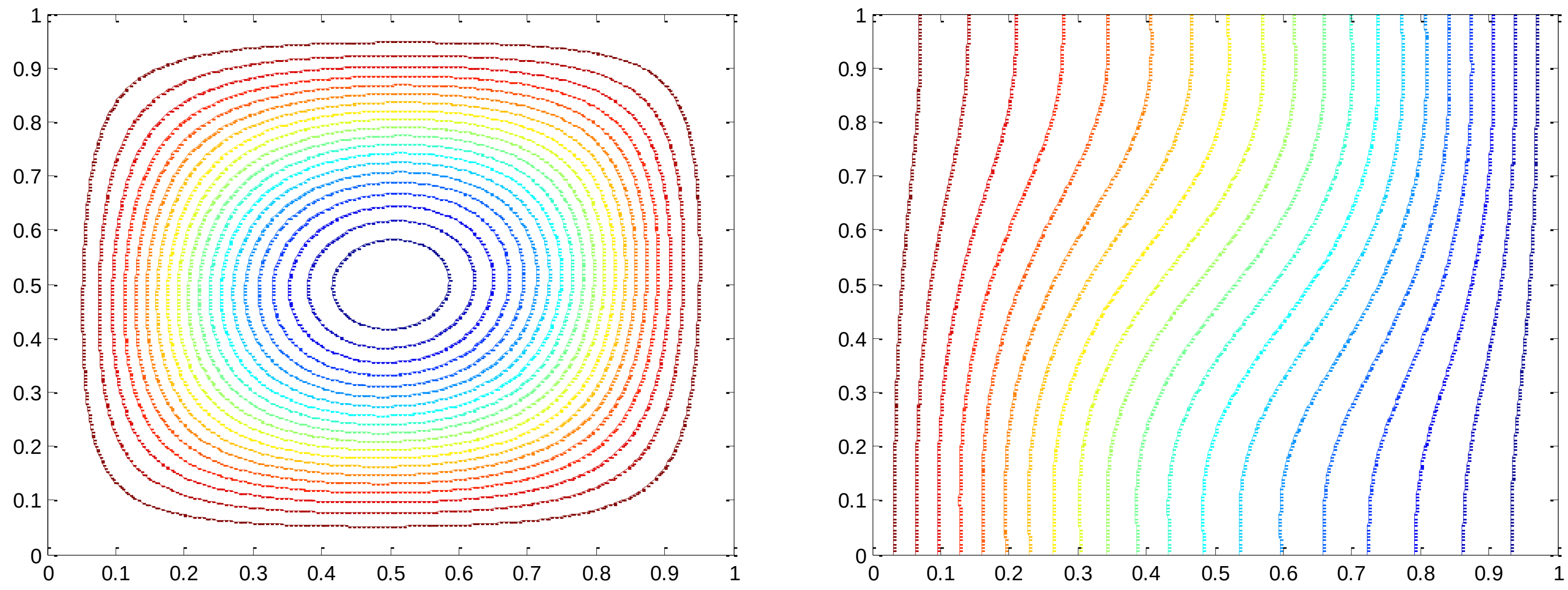
System

Processor:	Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz 3.60 GHz
Installed memory (RAM):	16.0 GB

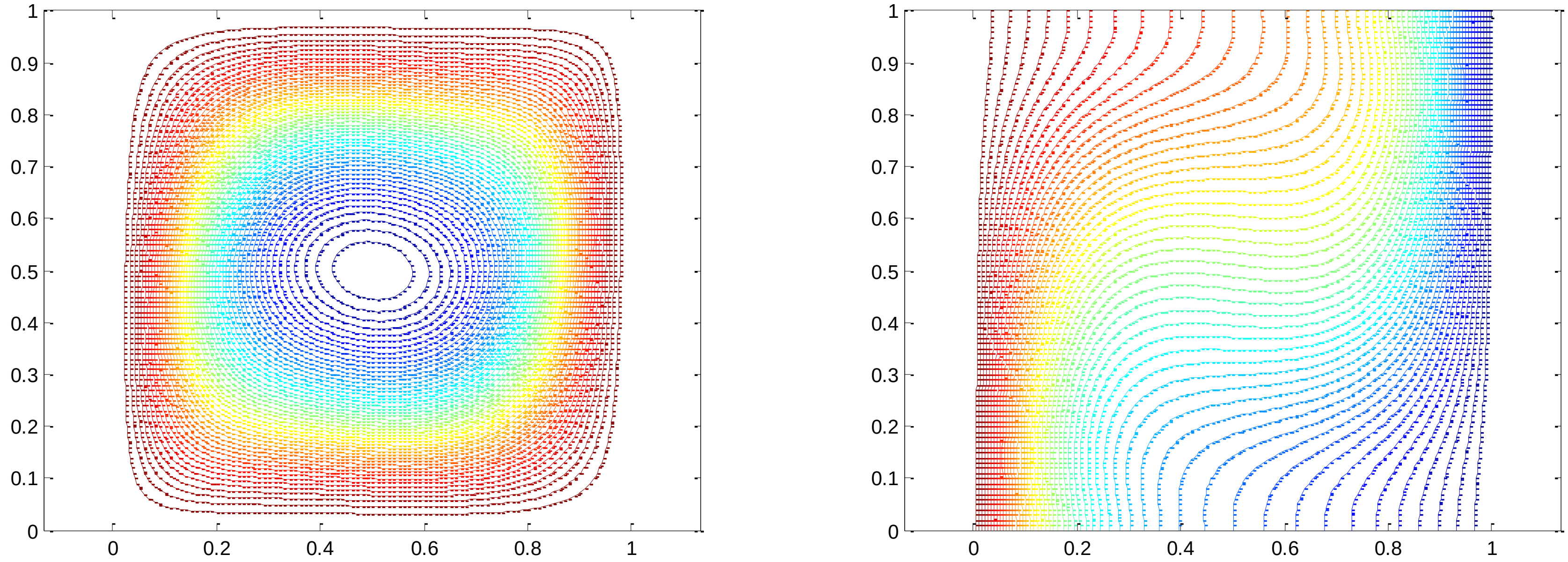
In order to determine the proper mesh size Table 2 clearly shows a good agreement for the mesh size 151x151 which was later on used in all our computations. Comparison results of other published paper (see De Vahl Davis [2]) are shown in Table 2. It is seen that the present results are in very good agreement with those determined in the paper mentioned above. Therefore we are confident that the present results are accurate.

Table 2. Validation of the code for *Nu* when $\Pr = 0.71$

Ra	De Vahl Davis [2]	Present results
1000	1.116	1.11852
10000	2.243	2.25579 (101 x 101)



Streamlines and Isotherms for Ra = 1000



Streamlines and Isotherms for Ra = 10000

References

[1]Bejan A., *Convection Heat Transfer*, John Wiley & Sons, New York, 2004.
[2]De Vahl Davis, G., Natural Convection of Air in a Square Cavity. A Benchmark Numerical Solution, *Int. J. Num. Methods Fluids 2* (1983), pp. 249-264