The Lidstone interpolation on tetrahedron

Teodora Cătinaș†

Abstract

We give a Lidstone interpolation formula, including its remainder, for a real function defined on a tetrahedron. These results extend the corresponding ones given for triangle by F.A. Costabile and F. Dell’Accio in a recent paper.

We also present some numerical examples.

Mathematics Subject Classification: 41A05, 41A80, 41A63.

Key words and phrases: Lidstone interpolation on triangle and tetrahedron, remainder term, Peano Theorem.

1 Preliminaries

The Lidstone interpolation was introduced in 1920 and is a two-point interpolation process utilizing even derivatives [7]. As it is mentioned in [5], there exist few results in the literature regarding the extension of approximation of univariate functions by means of Lidstone polynomials to functions of two variables over non rectangular domains.

In this paper we extend some univariate approximation formulas to functions of three variables defined on a tetrahedron. The extension to the bidimensional case has been considered in [5].

We recall first some classical results regarding Lidstone interpolation [1], [2].

The Lidstone polynomial is the unique polynomial $\Lambda_n$ of degree $2n+1$, $n \in \mathbb{N}$, defined on the interval $[0, 1]$ by

$$\Lambda_0(x) = x,$$
$$\Lambda_n''(x) = \Lambda_{n-1}(x),$$
$$\Lambda_n(0) = \Lambda_n(1) = 0, \quad n \geq 1.$$

For a given function $f$ possessing a sufficient number of derivatives, the Lidstone interpolation problem consists in finding a polynomial of degree $2n-1$ satisfying the Lidstone interpolation conditions,

$$(L_n f)^{(2k)}(0) = f^{(2k)}(0),$$
$$(L_n f)^{(2k)}(1) = f^{(2k)}(1), \quad 0 \leq k \leq n - 1$$

*This work has been supported by CNCSIS under Grant 8/139/2003.
†”Babeș-Bolyai” University, Faculty of Mathematics and Computer Science, Department of Applied Mathematics, M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania, e-mail: tcatinas@math.ubbcluj.ro.
According to [1], the Lidstone interpolant $L_n f$ uniquely exists and can be expressed as

$$(L_n f)(x) = \sum_{k=0}^{n-1} \left[ f^{(2k)}(0) \Lambda_k(1-x) + f^{(2k)}(1) \Lambda_k(x) \right]. \quad (3)$$

**Remark 1** The Lidstone operator $L_n$ is exact for the polynomials of degree not greater than $2n - 1$, $n \in \mathbb{N}$.

The Lidstone interpolation formula is

$$f = L_n f + R_n f,$$  

where $R_n f$ denotes the remainder.

For $f \in C^{2n}[0,1]$ one can apply Peano’s Theorem, [1], [2], and obtain

$$(R_n f)(x) = \int_0^1 g_n(x,s) f^{(2n)}(s) ds, \quad (5)$$

with

$$g_n(x,s) = \begin{cases} 
- \sum_{k=0}^{n-1} \Lambda_k(x) \frac{(1-s)^{n-2k-1}}{(n-2k-1)!}, & x \leq s \\
- \sum_{k=0}^{n-1} \Lambda_k(1-x) \frac{s^{n-2k-1}}{(n-2k-1)!}, & s \leq x.
\end{cases}$$

2 The Lidstone interpolation formula on tetrahedron

We consider the tetrahedron

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x + y + z \leq 1\},$$

and let $f$ be a real function from $C^{2n}(T)$, $n \in \mathbb{N}$.

We consider the Lidstone interpolation on a segment from $\mathbb{R}^3$. Similarly to [5, Lemma 3.1], where the segment was in $\mathbb{R}^2$, we obtain the following result:

**Lemma 2** Let $D$ be a convex domain in $\mathbb{R}^3$, $f \in C^{2n}(D)$, $n \in \mathbb{N}$, and consider two points $P(a,b,c), Q(u,v,w) \in D$. Let $l(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$, $\lambda \in [0,1]$, $l(0) = P$, $l(1) = Q$ be a linear parametrization of the segment $PQ$ and consider $h \equiv x'(\lambda), k \equiv y'(\lambda)$ and $p \equiv z'(\lambda)$, $\lambda \in [0,1]$. We have

$$f(x(\lambda), y(\lambda), z(\lambda)) = (L_n f)(x(\lambda), y(\lambda), z(\lambda)) + (R_n f)(x(\lambda), y(\lambda), z(\lambda)),$$

with

$$(L_n f)(x(\lambda), y(\lambda), z(\lambda)) = \sum_{i=0}^{n-1} \sum_{\alpha_1 + \alpha_2 + \alpha_3 = 2i} \frac{(2i)!}{\alpha_1! \alpha_2! \alpha_3!} h^{\alpha_1} k^{\alpha_2} p^{\alpha_3} \left[ \frac{\partial^{2i}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} f(a,b,c) \Lambda_i(1-\lambda) \right.$$

$$\quad \left. + \frac{\partial^{2i}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} f(u,v,w) \Lambda_i(\lambda) \right].$$
and

\[(R_n f)(x(\lambda), y(\lambda), z(\lambda)) = 0\]

\[= \sum_{\alpha_1 + \alpha_2 + \alpha_3 = 2n} \int_0^1 \frac{(2n)!}{\alpha_1! \alpha_2! \alpha_3!} h^{\alpha_1} k^{\alpha_2} p^{\alpha_3} \frac{\partial^{2n}}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} f(x(s), y(s), z(s)) g_n(\lambda, s) ds.\]  

(7)

**Proof.** Let us denote \(\varphi(\lambda) = (f \circ l)(\lambda)\). We apply the interpolation formula (4) for the function \(\varphi\):

\[\varphi(\lambda) = (L_n \varphi)(\lambda) + (R_n \varphi)(\lambda),\]

with

\[(L_n \varphi)(\lambda) = \sum_{i=0}^{n-1} \left[ \varphi^{(2i)}(0) \Lambda_i(1 - \lambda) + \varphi^{(2i)}(1) \Lambda_i(\lambda) \right] \]

(8) and by (5)

\[(R_n \varphi)(\lambda) = \int_0^1 g_n(\lambda, s) \varphi^{(2n)}(s) ds.\]

(9)

We have

\[\varphi^{(k)}(\lambda) = \sum_{\alpha_1 + \alpha_2 + \alpha_3 = k} \frac{k!}{\alpha_1! \alpha_2! \alpha_3!} h^{\alpha_1} k^{\alpha_2} p^{\alpha_3} \frac{\partial^k}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}} (f \circ l)(\lambda),\]

(10)

for \(1 \leq k \leq 2n\). Replacing (10) in (8) and (9), we obtain (6) and (7), respectively.

We shall restrict further to the cases when the segment \(PQ\) is in one of the three coordinate planes, i.e., when \(c = w = 0\) or \(b = v = 0\) or \(a = u = 0\). Therefore, we consider the derivatives of \(f\) in the directions of \(\nu_1\left(\frac{h}{\sqrt{h^2 + k^2}}, \frac{k}{\sqrt{h^2 + k^2}}, 0\right), \nu_2\left(0, \frac{b}{\sqrt{b^2 + p^2}}, \frac{p}{\sqrt{b^2 + p^2}}\right)\) and respectively \(\nu_3\left(0, \frac{k}{\sqrt{k^2 + p^2}}, \frac{p}{\sqrt{k^2 + p^2}}\right)\). The polynomial given by (6) satisfies the following interpolation conditions corresponding to the previous three cases:

\[\frac{\partial^k}{\partial x^k} (L_n f)(x(\lambda), y(\lambda), 0)|_{\lambda=0} = \frac{\partial^k}{\partial x^k} f(a, b, 0),\]

\[\frac{\partial^k}{\partial x^k} (L_n f)(x(\lambda), y(\lambda), 0)|_{\lambda=1} = \frac{\partial^k}{\partial x^k} f(u, v, 0),\]

\[\frac{\partial^k}{\partial x^k} (L_n f)(x(\lambda), 0, z(\lambda))|_{\lambda=0} = \frac{\partial^k}{\partial x^k} f(a, 0, c),\]

\[\frac{\partial^k}{\partial x^k} (L_n f)(x(\lambda), 0, z(\lambda))|_{\lambda=1} = \frac{\partial^k}{\partial x^k} f(u, 0, w),\]

\[\frac{\partial^k}{\partial x^k} (L_n f)(0, y(\lambda), z(\lambda))|_{\lambda=0} = \frac{\partial^k}{\partial x^k} f(0, b, c),\]

\[\frac{\partial^k}{\partial x^k} (L_n f)(0, y(\lambda), z(\lambda))|_{\lambda=1} = \frac{\partial^k}{\partial x^k} f(0, v, w), \quad k = 0, ..., n - 1,\]

with the remainders of the interpolation formulas given by:

\[(R_n f)(x(\lambda), y(\lambda), 0) = (h^2 + k^2)^n \int_0^1 \frac{\partial^{2n}}{\partial x^{2n}} f(x(\lambda), y(\lambda), 0) g_n(\lambda, s) ds,\]

(12)

\[(R_n f)(x(\lambda), 0, z(\lambda)) = (h^2 + p^2)^n \int_0^1 \frac{\partial^{2n}}{\partial x^{2n}} f(x(\lambda), 0, z(\lambda)) g_n(\lambda, s) ds,\]

\[(R_n f)(0, y(\lambda), z(\lambda)) = (k^2 + p^2)^n \int_0^1 \frac{\partial^{2n}}{\partial y^{2n}} f(0, y(\lambda), z(\lambda)) g_n(\lambda, s) ds.\]
These results follow from chain rule (or deriving (6) and the conditions (2)) and taking into account the definitions of the derivatives in a direction,

\[
\frac{\partial^{2n}}{\partial x^{2i} \partial y^{l} \partial z^j} f(x(\lambda), y(\lambda), 0) = \frac{1}{(k+n!)^2} \sum_{j=0}^{2n} \left(\frac{2n}{j}\right) h^{2n-j} k^j \frac{\partial^{2n}}{\partial x^{2i} \partial y^l \partial z^j} f(x(\lambda), y(\lambda), 0)
\]

\[
\frac{\partial^{2n}}{\partial y^{2l} \partial z^j} f(x(\lambda), 0, z(\lambda)) = \frac{1}{(k+n!)^2} \sum_{j=0}^{2n} \left(\frac{2n}{j}\right) h^{2n-j} p^j \frac{\partial^{2n}}{\partial y^{2l} \partial z^j} f(x(\lambda), y(\lambda), 0)
\]

\[
\frac{\partial^{2n}}{\partial z^{2j}} f(0, y(\lambda), z(\lambda)) = \frac{1}{(k+n!)^2} \sum_{j=0}^{2n} \left(\frac{2n}{j}\right) k^{2n-j} p^j \frac{\partial^{2n}}{\partial z^{2j}} f(0, y(\lambda), z(\lambda)).
\]

We give now the main result of this paper.

**Theorem 3** Let \( n \in \mathbb{N}, \nu = (-1, 1, 0) \) and \( f \) be a real function from \( C^{4n-2}(T) \).

The Lidstone interpolation conditions

\[
\frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} (L_n^T f)(0, 0, 0) = \frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} f(0, 0, 0), \quad i = 0, \ldots, n - 1, \quad j = 0, \ldots, 2i
\]

and

\[
\frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} (L_n^T f)(0, 0, 1) = \frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} f(0, 0, 1),
\]

\[
\frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} (L_n^T f)(1, 0, 0) = \frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} f(1, 0, 0),
\]

\[
\frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} (L_n^T f)(0, 1, 0) = \frac{\partial^{2i}}{\partial x^{2i} \partial y^l \partial z^j} f(0, 1, 0), \quad i = 0, \ldots, n - 1, \quad j = 0, 2, \ldots, 2i
\]

are satisfied by the following rational function

\[
(L_n^T f)(x, y, z) = \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-i-1} \sum_{\ell=0}^{n-i-k-1} \left(\frac{2^i}{i!}\right) (-1)^{i} \left(\frac{x+y}{x+y+z}\right)^{2i}
\]

\[
\cdot 
\left[
\Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k, j, 2l)}(0, 0, 0)
\right.
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(x + y + z\right) f^{(2i-j+2k, j, 2l)}(0, 0, 1)
\]

\[
+ \Lambda_k \left(x+y\right) \Lambda_l \left(x+y\right) f^{(2i-j+2k, j, 2l)}(0, 0, 0)
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(x + y + z\right) f^{(2i-j+2k, j, 2l)}(1, 0, 0)
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k, j, 2l)}(0, 0, 0)
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k, j, 2l)}(0, 0, 0)
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k+2l, j, 2l)}(0, 0, 0)
\]

\[
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k+2l, j, 2l)}(0, 0, 0)
\]

\[
\left[\Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(1 - x - y - z\right) f^{(2i-j+2k, j, 2l)}(0, 0, 0)
\right]
\]

The remainder of the Lidstone interpolation formula on tetrahedron,

\[
f(x, y, z) = (L_n^T f)(x, y, z) + (R_n^T f)(x, y, z),
\]
is given by

\[(R_n^T f)(x, y, z) = \sum_{i=0}^{n-1} \sum_{k=0}^{n-i-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i}.\]

First, we consider Lidstone interpolation on the triangle $OMN$, $O(0, 0, 0)$, $M(1, 0, 0)$, $N(0, 1, 0)$ according to [5]. The corresponding interpolation formula is

\[f = L_n^{\Delta OMN} f + R_n^{\Delta OMN} f, \quad (17)\]

with

\[(L_n^{\Delta OMN} f)(x, y, 0) = \sum_{i=0}^{n-1} \sum_{j=0}^{2i-1} \sum_{k=0}^{n-i-1} \binom{2i}{j} (-1)^j (x+y)^{2i} \cdot \frac{\partial^{2i-j+2k} f}{\partial x^{2i-j+2k} \partial y^j} (0, 0, 0)
\]

\[+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i-1} \sum_{k=0}^{n-i-1} \binom{2i}{j} (-1)^j (x+y)^{2i} \cdot \frac{\partial^{2i-j+2k} f}{\partial x^{2i-j+2k} \partial y^j} (1, 0, 0)
\]

\[+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i-1} \sum_{k=0}^{n-i-1} \binom{2i}{j} (-1)^j (x+y)^{2i} \cdot \frac{\partial^{2i-j+2k} f}{\partial x^{2i-j+2k} \partial y^j} (0, 0, 0)
\]

\[+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i-1} \sum_{k=0}^{n-i-1} \binom{2i}{j} (-1)^j (x+y)^{2i} \cdot \frac{\partial^{2i-j+2k} f}{\partial x^{2i-j+2k} \partial y^j} (1, 0, 0).
\]
and \( R_{n}^{\Delta O M N} f \) is the remainder and it is given by:

\[
(R_{n}^{\Delta O M N} f)(x, y, z) = \\
= \sum_{i=0}^{n-1} 2^i (x + y)^{2i} \left[ \Lambda_i \left( \frac{x}{x+y} \right) \int_{0}^{1} \frac{\partial^{2n}}{\partial x^{2n-i} \partial z^{i}} f(s, 0) g_{n-i}(x + y, s) ds \\
+ \Lambda_i \left( \frac{y}{x+y} \right) \int_{0}^{1} \frac{\partial^{2n}}{\partial y^{2n-i} \partial z^{i}} f(0, s) g_{n-i}(x + y, s) ds \right] \\
+ 2^n (x + y)^{2n} \int_{0}^{1} \frac{\partial^{2n}}{\partial z^{2n}} f((1 - s)(x + y), s(x + y), 0) g_n \left( \frac{y}{x+y}, s \right) ds.
\]

By means of the affine transformation \( h : \mathbb{R}^3 \to \mathbb{R}, \)

\[
h(x, y, z) = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ -a & -a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix},
\]

with \( a = x + y + z, \) we can obtain from (18) the interpolation formula on the triangle \( ABC, A(x + y + z, 0, 0), B(0, x + y + z, 0), C(0, 0, x + y + z) \) with \( x + y + z = \alpha, \alpha \in (0, 1): \)

\[
f = L_{n}^{\Delta A B C} f + R_{n}^{\Delta A B C} f,
\]

\[
(L_{n}^{\Delta A B C} f)(x, y, z) = \\
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \sum_{k=0}^{j} (-1)^{j} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{x}{x+y} \right) f^{(2i-j+2k,0)}(0,0,x+y+z) \\
+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \sum_{k=0}^{j} (-1)^{j} \Lambda_k \left( \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{x}{x+y} \right) f^{(2i-j+2k,0)}(x+y+z,0,0) \\
+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \sum_{k=0}^{j} (-1)^{j} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{y}{x+y} \right) f^{(2i-j+2k,0)}(0,0,x+y+z) \\
+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \sum_{k=0}^{j} (-1)^{j} \Lambda_k \left( \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{y}{x+y} \right) f^{(2i-j+2k,0)}(0,0,x+y+z).
\]

and

\[
(R_{n}^{\Delta A B C} f)(x, y, z) = \\
= \sum_{i=0}^{n-1} \left( \frac{x+y}{x+y+z} \right)^{2i} \left[ \Lambda_i \left( \frac{x}{x+y} \right) \int_{0}^{1} \frac{\partial^{2n}}{\partial z^{2n-i} \partial y^{i}} f((x + y + z)s, 0, (x + y + z)(1 - s)) g_{n-i}(x + y + z, s) ds \\
+ \Lambda_i \left( \frac{y}{x+y} \right) \int_{0}^{1} \frac{\partial^{2n}}{\partial y^{2n-i} \partial z^{i}} f(0, (x + y + z)s, (x + y + z)(1 - s)) g_{n-i}(x + y + z, s) ds \right] \\
+ 2^n (x + y)^{2n} \int_{0}^{1} \frac{\partial^{2n}}{\partial z^{2n}} f((1 - s)(x + y), s(x + y), z) g_n \left( \frac{y}{x+y}, s \right) ds.
\]

By making the notation

\[
A_{ij}(x, y, z) = \binom{2i}{j}(-1)^{j} \left( \frac{x+y}{x+y+z} \right)^{2i},
\]

(20)
we obtain that
\[
(L_n^{\Delta ABC} f)(x, y, z) =
\]
\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1-j} A_{ij}(x, y, z)\Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{x}{x+y}\right)^{f(2i-j+2k,0)}(0, 0, x + y + z)
\]
\[
+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1-j} A_{ij}(x, y, z)\Lambda_k \left(\frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{y}{x+y}\right)^{f(2i-j+2k,0)}(x + y, z, 0)
\]
\[
+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1-j} A_{ij}(x, y, z)\Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{y}{x+y}\right)^{f(2i-j+2k,0)}(0, 0, x + y + z)
\]
\[
+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1-j} A_{ij}(x, y, z)\Lambda_k \left(\frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{y}{x+y}\right)^{f(2i-j+2k,0)}(0, x + y + z, 0).
\]

By Lemma 2 we obtain the interpolant on the segment with endpoints 
(0, 0, 0) and (1, 0, 0):
\[
(L_n^{[1]} f)(x, 0, 0) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial x^{2i}} f(0, 0, 0) \Lambda_i (1 - x) + \frac{\partial^{2i}}{\partial y^{2i}} f(1, 0, 0) \Lambda_i (x) \right),
\]
\[
(L_n^{[1]} f)(x + y + z, 0, 0) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial x^{2i}} f(0, 0, 0) \Lambda_i (1 - x - y - z) + \frac{\partial^{2i}}{\partial y^{2i}} f(1, 0, 0) \Lambda_i (x + y + z) \right).
\]

We have the interpolation formula
\[
f(x + y + z, 0, 0) = (L_n^{[1]} f)(x + y + z, 0, 0) + (R_n^{[1]} f)(x + y + z, 0, 0),
\]
where the remainder $R_n^{[1]} f(x + y + z, 0, 0)$ is obtained from (12):
\[
(R_n^{[1]} f)(x + y + z, 0, 0) = \int_0^1 \frac{\partial^{2n}}{\partial x^{2n}} f(s, 0, 0) g_n(x + y + z, s) ds.
\]

Hence,
\[
f^{(2i-j+2k,0)}(x + y + z, 0, 0) = \sum_{i=0}^{n-1} \left( f(2i-j+2k,0)(0, 0, 0) \Lambda_i (1 - x - y - z) + f(2i-j+2k,0)(1, 0, 0) \Lambda_i (x + y + z) \right)
\]
\[
+ \int_0^1 \frac{\partial^{2n}}{\partial x^{2n} x^{2k}} f(s, 0, 0) g_n(x + y + z, s) ds.
\]

Again, by Lemma 2, we obtain the interpolant on the segment with endpoints 
(0, 0, 0) and (0, 1, 0):
\[
(L_n^{[2]} f)(0, y, 0) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial y^{2i}} f(0, 0, 0) \Lambda_i (1 - y) + \frac{\partial^{2i}}{\partial x^{2i}} f(0, 1, 0) \Lambda_i (y) \right),
\]
\[
(L_n^{[2]} f)(0, x + y + z, 0) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial y^{2i}} f(0, 0, 0) \Lambda_i (1 - x - y - z) + \frac{\partial^{2i}}{\partial x^{2i}} f(0, 1, 0) \Lambda_i (x + y + z) \right).
\]
We have the interpolation formula
\[ f(0, x + y + z, 0) = (L_n^2 f)(0, x + y + z, 0) + (R_n^2 f)(0, x + y + z, 0), \]
where the remainder \((R_n^2 f)(0, x + y + z, 0)\) is obtained from (12):
\[ (R_n^2 f)(0, x + y + z, 0) = \int_0^1 \frac{\partial^{2n}}{\partial y^n} f(0, s, 0) g_n(x + y + z, s) ds. \]
Hence,
\[ f^{(2i-j,j+2k,0)}(0, x + y + z, 0) = \]
\[ = L_n^2 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f(0, x + y + z, 0) \right) + R_n^2 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f \right) (0, x + y + z, 0) \]
\[ = \sum_{i=0}^{n-1} \left( f^{(2i-j,j+2k,0)}(0, 0, 0) \Lambda_i(1 - x - y - z) + f^{(2i-j,j+2k+2l,0)}(0, 0, 1) \Lambda_i(x + y + z) \right) \]
\[ + \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i} \partial y^{2k} \partial z^{n}} f(0, s, 0) g_n(x + y + z, s) ds. \]
Finally, using once more Lemma 2, we obtain the interpolant on the segment with endpoints \((0, 0, 0)\) and \((0, 0, 1)\):
\[ (L_n^3 f)(0, 0, z) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial y^i} f(0, 0, 0) \Lambda_i(1 - z) + \frac{\partial^{2i}}{\partial y^i} f(0, 0, 1) \Lambda_i(z) \right) \]
\[ (L_n^3 f)(0, 0, x + y + z) = \sum_{i=0}^{n-1} \left( \frac{\partial^{2i}}{\partial y^i} f(0, 0, 0) \Lambda_i(1 - x - y - z) + \frac{\partial^{2i}}{\partial y^i} f(0, 0, 1) \Lambda_i(x + y + z) \right). \]
We have the interpolation formula
\[ f(0, x + y + z) = (L_n^3 f)(0, 0, x + y + z) + (R_n^3 f)(0, 0, x + y + z), \]
where the remainder \((R_n^3 f)(0, 0, x + y + z)\) is obtained from (12):
\[ (R_n^3 f)(0, 0, x + y + z) = \int_0^1 \frac{\partial^{2n}}{\partial y^n} f(0, 0, s) g_n(x + y + z, s) ds. \]
Hence,
\[ f^{(2i-j,j+2k,0)}(0, 0, x + y + z) = \]
\[ = L_n^3 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f \right) (0, 0, x + y + z) + R_n^3 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f \right) (0, 0, x + y + z) \]
\[ = \sum_{i=0}^{n-1} \left( f^{(2i-j,j+2k,0)}(0, 0, 0) \Lambda_i(1 - x - y - z) + f^{(2i-j,j+2k+2l,0)}(0, 0, 1) \Lambda_i(x + y + z) \right) \]
\[ + \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i} \partial y^{2k} \partial z^{n}} f(0, s, 0) g_n(x + y + z, s) ds. \]
and
\[ f^{(2i-j+2k,j,0)}(0, 0, x + y + z) = \]
\[ = L_n^3 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f \right) (0, 0, x + y + z) + R_n^3 \left( \frac{\partial^{2i+2k}}{\partial x^{2i-j} \partial y^{2k}} f \right) (0, 0, x + y + z) \]
\[ = \sum_{i=0}^{n-1} \left( f^{(2i-j+2k,j,0)}(0, 0, 0) \Lambda_i(1 - x - y - z) + f^{(2i-j+2k+2l,0)}(0, 0, 1) \Lambda_i(x + y + z) \right) \]
\[ + \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i} \partial y^{2k} \partial z^{n}} f(0, s, 0) g_n(x + y + z, s) ds. \]
Next, replacing (22), (23), (24) and (25) in (21) we obtain

\[(L^n f)(x, y, z) = (26)\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} A_{ij}(x, y, z) \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(\frac{x}{x+y}\right) \\
. \left( f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (1 - x - y - z) + f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (x + y + z) \right) \\
+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} A_{ij}(x, y, z) \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(\frac{x}{x+y}\right) \\
. \left( f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (1 - x - y - z) + f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (x + y + z) \right) \\
+ \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} A_{ij}(x, y, z) \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_l \left(\frac{x}{x+y}\right) \\
. \left( f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (1 - x - y - z) + f^{(2i-j+2k, 2l)}(0, 0, 0) \Lambda_i (x + y + z) \right),
\]

which, taking into account the notation for $A_{ij}$, results in (15).

Replacing (22), (23), (24) and (25) in (21) we also obtain the remainder term of the interpolation formula on tetrahedron:

\[(R^n f)(x, y, z) = (27)\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{2i} \sum_{k=0}^{n-1} \sum_{l=0}^{k-1} \left( \begin{array}{c}
\Lambda_k \\
(1 - \frac{x+y}{x+y+z}) \Lambda_i \\
(\frac{x}{x+y}) \\
(\frac{y}{x+y})
\end{array} \right) \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2l} \partial y^{2j+2k}} f(0, 0, s) g_n(x + y + z, s) ds \\
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{x}{x+y}\right) \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2l} \partial y^{2j+2k}} f(s, 0, 0) g_n(x + y + z, s) ds \\
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{y}{x+y}\right) \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2l} \partial y^{2j+2k}} f(0, 0, s) g_n(x + y + z, s) ds \\
+ \Lambda_k \left(1 - \frac{x+y}{x+y+z}\right) \Lambda_i \left(\frac{y}{x+y}\right) \int_0^1 \frac{\partial^{2n+2i+2k}}{\partial x^{2i-j+2l} \partial y^{2j+2k}} f(s, 0, 0) g_n(x + y + z, s) ds \\
+ (R^n_{\text{ABC}} f)(x, y, z).
\]

Applying in (27) the formula

\[
\sum_{j=0}^{2i} \left( \begin{array}{c}
(2i)
\end{array} \right) (-1)^j \frac{\partial^{2i-j}}{\partial x^{2i-j} \partial y^j} f = 2^i \frac{\partial^{2i}}{\partial x^{2i}} f
\]

and replacing the expression of $R^n_{\text{ABC}} f(x, y, z)$ from (19) we obtain (17).

Next we prove that $L^n f$ is verifying the Lidstone interpolation condition given by (13) and (14). We have $\frac{\partial}{\partial y} = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$. Taking into account that

\[
\sum_{j=0}^{2i} \left( \begin{array}{c}
(2i)
\end{array} \right) (-1)^{2i-j} \frac{\partial^{2i}}{\partial x^{2i-j} \partial y^j} f = 2^i \frac{\partial^{2i}}{\partial x^{2i}} f
\]
after we replace (20) in (15) we obtain the following expression

\[(L^2) f(x, y, z) =
\]

\[= \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(1 - x - y - z) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 0)\]

\[+ \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(x + y + z) \frac{\partial^{2i+2k+2l}}{\partial z^i \partial y^k \partial x^l} f(0, 0, 1)\]

\[+ \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(1 - x - y - z) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 0)\]

\[+ \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(x + y + z) \frac{\partial^{2i+2k+2l}}{\partial y^i \partial z^k \partial x^l} f(0, 0, 0)\]

\[+ \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(1 - x - y - z) \frac{\partial^{2i+2k+2l}}{\partial y^i \partial x^k \partial z^l} f(0, 0, 0)\]

\[+ \sum_{i=0}^{n-1} \sum_{k=0}^{n-1-i} \sum_{l=0}^{n-1-i-k-1} 2^i \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(x + y + z) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial z^k \partial y^l} f(0, 0, 0)\]

It is not difficult to verify that

\[\frac{\partial}{\partial \nu} \left( \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_l(1 - x - y - z) \right) = 0,
\]

\[\frac{\partial}{\partial \nu} \left( \left( \frac{x+y}{x+y+z} \right)^{2i} \Lambda_l(x + y + z) \right) = 0
\]

and

\[\frac{\partial}{\partial \nu} \left( \Lambda_k(1 - \frac{x+y}{x+y+z}) \Lambda_l(\frac{x}{x+y}) \right) = \Lambda_k(1 - \frac{x+y}{x+y+z}) \frac{\partial}{\partial \nu} \Lambda_l(\frac{x}{x+y}), \quad (29)
\]

\[\frac{\partial}{\partial \nu} \left( \Lambda_k(\frac{x+y}{x+y+z}) \Lambda_l(\frac{x}{x+y}) \right) = \Lambda_k(\frac{x+y}{x+y+z}) \frac{\partial}{\partial \nu} \Lambda_l(\frac{x}{x+y}),
\]

for \(i = 0, ..., n - 1\). We have

\[\frac{\partial}{\partial \nu} \Lambda_l(\frac{x}{x+y}) = -\frac{1}{\sqrt{2}} \Lambda_{l-1}(\frac{x}{x+y})
\]

\[\frac{\partial^2}{\partial \nu^2} \Lambda_l(\frac{x}{x+y}) = \frac{1}{2} \Lambda'_{l-1}(\frac{x}{x+y})
\]

We know from (1) that \(\Lambda_n''(x) = \Lambda_{n-1}(x)\), thus

\[\frac{\partial^2}{\partial \nu^2} \Lambda_l(\frac{x}{x+y}) = \frac{1}{2} \Lambda_{l-1}(\frac{x}{x+y}) \Lambda_{l-1}(\frac{x}{x+y}) \quad (30)
\]

\[\frac{\partial^2}{\partial \nu^2} \Lambda_l(\frac{x}{x+y}) = \frac{1}{2} \Lambda_{l-1}(\frac{x}{x+y}) \Lambda_{l-1}(\frac{x}{x+y})
\]

and in analogous way we obtain

\[\frac{\partial^m}{\partial \nu^m} \Lambda_l(\frac{x}{x+y}) = \frac{1}{m!} \Lambda_{l-m}(\frac{x}{x+y}) \quad (31)
\]

10
for each $m = 0, \ldots, n - 1$, $i = 0, \ldots, n - 1$, $k = 0, \ldots, n - i - 1$. Taking into account (29), (30) and (31) we obtain

\[
\frac{\partial^{2m}}{\partial x^m} \left( \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{x}{x+y} \right) \right) = \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \frac{1}{2m(x+y)^{2m}} \Lambda_{i-m} \left( \frac{x}{x+y} \right),
\]

(32)

\[
\frac{\partial^{2m}}{\partial x^m} \left( \Lambda_k \left( \frac{x+y}{x+y+z} \right) \Lambda_i \left( \frac{x}{x+y} \right) \right) = \Lambda_k \left( \frac{x+y}{x+y+z} \right) \frac{1}{2m(x+y)^{2m}} \Lambda_{i-m} \left( \frac{x}{x+y} \right).
\]

Hence,

\[
\frac{\partial^{2m}}{\partial x^m} \left( L_T^f \right)(x, y, z) = \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( 1 - x - y - z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 0)
\]

+ \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( x + y + z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 1)
\]

+ \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( x + y + z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(1, 0, 0)
\]

+ \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( 1 - x - y - z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 0)
\]

+ \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( x + y + z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 0, 1)
\]

+ \sum_{i=m}^{n-1} \sum_{k=0}^{n-i-1} \sum_{l=0}^{i-k-1} 2^m \left( \frac{1}{x+y+z} \right)^{2i} (x+y)^{2i-2m} \Lambda_k \left( 1 - \frac{x+y}{x+y+z} \right) \Lambda_i \left( 1 - x - y - z \right) \frac{\partial^{2i+2k+2l}}{\partial x^i \partial y^k \partial z^l} f(0, 1, 0).
\]

It is not difficult to see that

\[
\left. \frac{\partial^{2p+2m}}{\partial x^m \partial y^p} \left( L_T^f \right)(x, y, z) \right|_{(0,0,0)} = \left. \frac{\partial^{2p+2m}}{\partial x^m \partial y^p} \left( L_T^f \right)(0, y, z) \right|_{(0,0,0)}, \quad (33)
\]

\[
\left. \frac{\partial^{2p+2m}}{\partial z^m \partial y^p} \left( L_T^f \right)(x, y, z) \right|_{(0,0,1)} = \left. \frac{\partial^{2p+2m}}{\partial z^m \partial y^p} \left( L_T^f \right)(0, y, z) \right|_{(0,0,1)}, \quad (34)
\]

for all $m$ and $p$ chosen such that $m+p$ is not greater than $n - 1$. After we make
\[ i - m \rightarrow i \text{ we obtain} \]

\[
\frac{\partial^{2m}}{\partial r^m} (L^T_n f)(0, y, z) =
\]

\[= \sum_{i=0}^{n-n_i-1} \sum_{k=0}^{i-k-1} \sum_{l=0}^{i-l-1} 2^i \left( \frac{1}{y+z} \right)^2 y^{2i} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2i+2m+2k+2l}}{\partial x^{2i} \partial y^{2m} \partial z^{2l}} f(0, 0, 0) + \sum_{i=0}^{n-n_i-1} \sum_{k=0}^{i-k-1} \sum_{l=0}^{i-l-1} 2^i \left( \frac{1}{y+z} \right)^2 y^{2i} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2i+2m+2k+2l}}{\partial x^{2i} \partial y^{2m} \partial z^{2l}} f(0, 0, 1) + \sum_{i=0}^{n-n_i-1} \sum_{k=0}^{i-k-1} \sum_{l=0}^{i-l-1} 2^i \left( \frac{1}{y+z} \right)^2 y^{2i} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2i+2m+2k+2l}}{\partial x^{2i} \partial y^{2m} \partial z^{2l}} f(0, 0, 0) + \sum_{i=0}^{n-n_i-1} \sum_{k=0}^{i-k-1} \sum_{l=0}^{i-l-1} 2^i \left( \frac{1}{y+z} \right)^2 y^{2i} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2i+2m+2k+2l}}{\partial x^{2i} \partial y^{2m} \partial z^{2l}} f(1, 0, 0) + \sum_{i=0}^{n-n_i-1} \sum_{k=0}^{i-k-1} \sum_{l=0}^{i-l-1} 2^i \left( \frac{1}{y+z} \right)^2 y^{2i} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2i+2m+2k+2l}}{\partial x^{2i} \partial y^{2m} \partial z^{2l}} f(0, 1, 0).
\]

Taking into account the relations (1) we obtain

\[
\frac{\partial^{2m}}{\partial r^m} (L^T_n f)(0, y, z) = \sum_{k=0}^{n-n_i-1} \sum_{l=0}^{i-l-1} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2m+2k+2l}}{\partial y^{2m} \partial z^{2l}} f(0, 0, 0) + \sum_{k=0}^{n-n_i-1} \sum_{l=0}^{i-l-1} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2m+2k+2l}}{\partial y^{2m} \partial z^{2l}} f(0, 0, 1) + \sum_{k=0}^{n-n_i-1} \sum_{l=0}^{i-l-1} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2m+2k+2l}}{\partial y^{2m} \partial z^{2l}} f(0, 0, 0) + \sum_{k=0}^{n-n_i-1} \sum_{l=0}^{i-l-1} \Lambda_k \left( 1 - \frac{y}{y+z} \right) \Lambda_{i-k} (1 - y - z) \frac{\partial^{2m+2k+2l}}{\partial y^{2m} \partial z^{2l}} f(0, 1, 0).
\]

From [5] we have that

\[
\frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, y, z) \big|_{(0,0,0)} = \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, 0, z) \big|_{(0,0,0)}, \quad (36)
\]

\[
\frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, y, z) \big|_{(0,0,1)} = \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, 0, z) \big|_{(0,0,1)},
\]

\[
\frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(x, 0, z) \big|_{(1,0,0)} = \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(x, 0, z) \big|_{(1,0,0)},
\]

\[
\frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, y, z) \big|_{(0,1,0)} = \frac{\partial^{2m}}{\partial y^{2m}} \frac{\partial^{2m}}{\partial z^{2m}} (L^T_n f)(0, y, z) \big|_{(0,1,0)}.
\]
We have
\[ \frac{\partial^m}{\partial z^m} (L^n f)(0,0,z) = \sum_{i=0}^{n-1} \sum_{l=0}^{l_n-i-1} \Lambda_k (1-z) \left( \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0) \right. \\
+ \sum_{i=0}^{n-1} \sum_{l=0}^{l_n-i-1} \Lambda_k (1) \Lambda_l (1-x-z) \left( \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,0,1) \right) \\
+ \sum_{i=0}^{n-1} \sum_{l=0}^{l_n-i-1} \Lambda_k (0) \Lambda_l (1-z) \left( \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0) \right) \\
+ \sum_{i=0}^{n-1} \sum_{l=0}^{l_n-i-1} \Lambda_k (0) \Lambda_l (0) \left( \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,1,0) \right). \]

Taking into account the relations (1) we obtain
\[ \frac{\partial^m}{\partial z^m} (L^n f)(0,0,z) = \sum_{i=0}^{n-1} \sum_{l=0}^{l_n-i-1} \Lambda_k (1-z) \frac{\partial^{2m+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0) + \sum_{i=0}^{n-1} \Lambda_l (1-x-z) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,0,1) \\
= L_n \left( \frac{\partial^m}{\partial y^m} f(0,0,\cdot) \right)(z), \]

whence, taking into account the Lidstone interpolation conditions (2) and conditions (36), we obtain
\[ \frac{\partial^{p+2m}}{\partial z^p \partial y^z \partial z^m} (L^n f)(0,0,0) = \frac{\partial^p}{\partial z^p} L_n \left( \frac{\partial^m}{\partial y^m} f(0,0,\cdot) \right)(0) = \frac{\partial^{p+2m}}{\partial z^p \partial y^z \partial z^m} f(0,0,0) \]
\[ \frac{\partial^{p+2m}}{\partial z^p \partial y^z \partial z^m} (L^n f)(0,0,1) = \frac{\partial^p}{\partial z^p} L_n \left( \frac{\partial^m}{\partial y^m} f(0,0,\cdot) \right)(1) = \frac{\partial^{p+2m}}{\partial z^p \partial y^z \partial z^m} f(0,0,1). \]

We know that
\[ \frac{\partial^{2m+2l}}{\partial y^m \partial y^z \partial z^l} (L^n f)(x,y,z) \bigg|_{(1,0,0)} = \frac{\partial^{2m+2l}}{\partial x^m \partial y^z \partial z^l} (L^n f)(x,0,z) \bigg|_{(1,0,0)}. \]

In the same way we calculate \( \frac{\partial^m}{\partial z^m} (L^n f)(x,0,z) \) and making \( i - m \to i \), we get
\[ \frac{\partial^m}{\partial z^m} (L^n f)(x,0,z) = \]
\[ = \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-x-z) \Lambda_l (1) \Lambda_i (1-x-z) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,0) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-z) \Lambda_l (1) \Lambda_i (1-z) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0,1) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-x) \Lambda_l (1-x-z) \Lambda_i (1-x-z) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,0) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-x) \Lambda_l (0) \Lambda_i (1-x-z) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,0,1) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-z) \Lambda_l (0) \Lambda_i (0) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-x) \Lambda_l (0) \Lambda_i (0) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,0,1) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-z) \Lambda_l (0) \Lambda_i (0) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^z \partial z^l} f(0,0,0) \\
+ \sum_{i=0}^{n-m-1} \sum_{l=0}^{l_n-i-1} \sum_{m=0}^{m_n-i-1} \frac{1}{x+z} \Lambda_k (1-x) \Lambda_l (0) \Lambda_i (0) \frac{\partial^{2m+2k+2l}}{\partial y^m \partial y^x \partial z^l} f(0,1,0). \]

13
We obtain
\[
\frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} (L_n^T f)(x, 0, 0) = \sum_{i=0}^{n-1} A_i(1-x) \frac{\partial^{2m+2i}}{\partial x^{2m+2i}} f(0, 0, 0) + \sum_{i=0}^{n-1} A_i(x) \frac{\partial^{2m+2i}}{\partial x^{2m+2i}} f(1, 0, 0)
\]
whence, taking into account conditions (2) and (36), we obtain
\[
\frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} (L_n^T f)(1, 0, 0) = \frac{\partial^{2m}}{\partial y^{2m}} L_n \left( \frac{\partial^{2m}}{\partial y^{2m}} f(\cdot, 0, 0) \right) (1) = \frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} f(1, 0, 0).
\]  
(37)

Similarly, we get
\[
\frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} (L_n^T f)(0, 0, 0) = \frac{\partial^{2m+2n}}{\partial y^{2m} \partial x^{2n}} f(0, 0, 0).
\]  
(38)

We have from (35) the expression of \(\frac{\partial^{2m}}{\partial y^{2m}} L_n^T (f)(0, y, z)\) and we obtain
\[
\frac{\partial^{2m}}{\partial y^{2m}} (L_n^T f)(0, y, 0) = \sum_{i=0}^{n-1} A_i(1-y) \frac{\partial^{2m+2i}}{\partial y^{2m+2i}} f(0, 0, 0) + \sum_{i=0}^{n-1} A_i(y) \frac{\partial^{2m+2i}}{\partial y^{2m+2i}} f(0, 1, 0)
\]
whence, taking into account that
\[
\frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} (L_n^T f)(x, y, z) \big|_{(0,1,0)} = \frac{\partial^{2m+2n}}{\partial y^{2m} \partial x^{2n}} (L_n^T f)(0, y, z) \big|_{(0,1,0)},
\]
relations (36) and the Lidstone interpolation conditions (2) we obtain that
\[
\frac{\partial^{2m+2n}}{\partial y^{2m} \partial x^{2n}} (L_n^T f)(0, 1, 0) = \frac{\partial^{2m}}{\partial y^{2m}} L_n \left( \frac{\partial^{2m}}{\partial y^{2m}} f(0, \cdot, 0) \right) (1) = \frac{\partial^{2m+2n}}{\partial y^{2m} \partial x^{2n}} f(0, 1, 0).
\]  
(39)

Analogous, we get
\[
\frac{\partial^{2m+2n}}{\partial y^{2m} \partial x^{2n}} (L_n^T f)(0, 0, 0) = \frac{\partial^{2m+2n}}{\partial x^{2m} \partial y^{2n}} f(0, 0, 0).
\]  
(40)

From (33), (34), (37) and (39) the interpolation conditions (14) are satisfied.

Further, following the procedure described in [5], we prove that the interpolation condition (13) also holds. By (28) we have
\[
2^j \frac{\partial^{2j}}{\partial x^{2j} \partial y^2} f = \sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial x^{2j-k} \partial y^2} f,
\]
\[
2^j \frac{\partial^{2j}}{\partial y^{2j} \partial x^2} f = \sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial y^{2j-k} \partial x^2} f.
\]
By replacing these results in (38) and (40), we get
\[
\begin{cases}
\sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial x^{2j-k} \partial y^2} f(0, 0, 0) = \sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial x^{2j-k} \partial y^2} (L_n^T f)(0, 0, 0) \\
\sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial y^{2j-k} \partial x^2} f(0, 0, 0) = \sum_{k=0}^{2j} \binom{2j}{k} (-1)^{2j-k} \frac{\partial^{2j-k}}{\partial y^{2j-k} \partial x^2} (L_n^T f)(0, 0, 0),
\end{cases}
\]
for \(j = 0, ..., i\). This is a nonzero \((2i+1) \times (2i+1)\) linear system with the unknowns \(\frac{\partial^{2j}}{\partial x^{2j} \partial y^2} f(0, 0, 0), k = 0, ..., 2i\). The unique solution of this system is
\[
\frac{\partial^{2j}}{\partial x^{2j} \partial y^2} f(0, 0, 0) = \frac{\partial^{2j}}{\partial x^{2j} \partial y^2} L_n^T f(0, 0, 0),
\]
for \(j = 0, ..., 2i\) and for all \(i = 0, ..., n-1\). So the interpolation condition (13) is proved.
3 Numerical examples

We consider the functions $f : T \to \mathbb{R}$ and $g : T \to \mathbb{R}$ given by

$$f(x, y, z) = x^3 + y^3 + z^3$$

and

$$g(x, y, z) = e^{-(x^2 + y^2 + z^2)}.$$

In the following tables we present the values of the functions and of the Lidstone approximations at some points of the tetrahedron.

<table>
<thead>
<tr>
<th>point</th>
<th>$f$</th>
<th>$L_2^f$</th>
<th>point</th>
<th>$g$</th>
<th>$L_2^g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>0.250000</td>
<td>0.250000</td>
<td>$(\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>0.606531</td>
<td>0.367879</td>
</tr>
<tr>
<td>(0, 0, \frac{1}{2})</td>
<td>0.125000</td>
<td>0.125000</td>
<td>(0, 0, \frac{1}{2})</td>
<td>0.778801</td>
<td>0.762955</td>
</tr>
<tr>
<td>$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$</td>
<td>0.046875</td>
<td>0.116319</td>
<td>$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$</td>
<td>0.829029</td>
<td>0.722597</td>
</tr>
<tr>
<td>$(\frac{1}{2}, 0, \frac{1}{2})$</td>
<td>0.140625</td>
<td>0.421875</td>
<td>$(\frac{1}{2}, 0, \frac{1}{2})$</td>
<td>0.731616</td>
<td>0.680667</td>
</tr>
<tr>
<td>$(\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>0.304296</td>
<td>0.189425</td>
<td>$(\frac{1}{2}, \frac{1}{2}, 0)$</td>
<td>0.616039</td>
<td>0.516703</td>
</tr>
</tbody>
</table>

Remark 4 We note that, according to the steps in the proof, in checking the interpolation conditions for example at $(0, 0, 1)$, one must take first $x = 0$ in the expressions occurring in $L_2^T$, then set $y = 0$ and finally $z = 1$.

References