3 Correlation and Regression

So far we have been discussing a number of descriptive techniques for describing one variable only. However, a very important part of Statistics is describing the association between two (or more) variables, whether or not they are independent, and if they are not, what is the nature of their dependence. One of the most fundamental concepts in statistical research is the concept of correlation.

Correlation is a measure of the relationship between one dependent variable, called the *response* variable and one or more independent variables, called *predictor* variables (or, simply, predictors). If two variables are correlated, this means that one can use information about one variable to predict the values of the other variable. **Regression** is then the method or statistical procedure that is used to establish that relationship.

3.1 Correlation, Curves of Regression

We will restrict our discussion to the case of two characteristics, X and Y. If X and Y have the same length, we can get a first idea of the relationship between the two, by plotting them in a **scattergram**, or **scatterplot**, , which is a plot of the points with coordinates $(x_i, y_i)_{i=\overline{1,k}}$, $x_i \in X,\ y_i \in Y,\ i = \overline{1,k}$. We group the N primary data into mn classes and denote by (x_i, y_j) the class mark and by f_{ij} the absolute frequency of the class (i, j), $i = \overline{1,m}$, $j = \overline{1,n}$. Then we represent the two-dimensional characteristic (X, Y) in a *correlation table*, or *contingency table*, as shown in Table 1.

$X \setminus Y$	y_1	$\ldots$	y_j	$\ldots$	y_n	
x_1	f_{11}	$\ldots$	f_{1j}	$\ldots$	f_{1n}	$f_{1.}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	f_{i1}	$\ldots$	f_{ij}	$\ldots$	f_{in}	$f_{i.}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_m	f_{m1}	$\ldots$	f_{mj}	$\ldots$	f_{mn}	$f_{m.}$
	$f_{.1}$	$\ldots$	$f_{.j}$	$\ldots$	$f_{.n}$	$f_{..} = N$

Table 1: Correlation Table

Notice that

$$\sum_{j=1}^{n} f_{ij} = f_{i.},\quad \sum_{i=1}^{m} f_{ij} = f_{.j},\quad \sum_{i=1}^{m} f_{i.} = \sum_{j=1}^{n} f_{.j} = f_{..} = N.$$

Now we can define numerical characteristics associated with (X, Y).

Definition 3.1. *Let (X, Y) be a two-dimensional characteristic whose distribution is given by Table 1 and let $k_1, k_2 \in \mathbb{N}$.*

(1) *The **(initial) moment of order (k_1, k_2)** of (X, Y) is the value*

$$\overline{\nu}_{k_1k_2} = \frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}x_i^{k_1}y_j^{k_2}. \tag{3.1}$$

(2) *The **central moment of order (k_1, k_2)** of (X, Y) is the value*

$$\overline{\mu}_{k_1k_2} = \frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(x_i - \overline{x})^{k_1}(y_j - \overline{y})^{k_2}, \tag{3.2}$$

where $\overline{x} = \overline{\nu}_{10} = \frac{1}{N}\sum_{i=1}^{m} f_{i.}x_i$ and $\overline{y} = \overline{\nu}_{01} = \frac{1}{N}\sum_{j=1}^{n} f_{.j}y_j$ are the means of X and Y, respectively.

Remark 3.2. Just as the means of the two characteristics X and Y can be expressed as moments of (X, Y), so can their variances:

$$\begin{aligned} \overline{\sigma}_X^2 &= \overline{\mu}_{20} &= \overline{\nu}_{20} - \overline{\nu}_{10}^2, \\ \overline{\sigma}_Y^2 &= \overline{\mu}_{02} &= \overline{\nu}_{02} - \overline{\nu}_{01}^2. \end{aligned}$$

Definition 3.3. *Let (X, Y) be a two-dimensional characteristic whose distribution is given by Table 1.*

(1) *The **covariance** ($\boxed{\text{cov}}$) of (X, Y) is the value*

$$\operatorname{cov}(X, Y) = \overline{\mu}_{11} = \frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(x_i - \overline{x})(y_j - \overline{y}). \tag{3.3}$$

(2) *The **correlation coefficient** ($\boxed{\text{corrcoef}}$) of (X, Y) is the value*

$$\overline{\rho} = \overline{\rho}_{XY} = \frac{\operatorname{cov}(X, Y)}{\sqrt{\overline{\mu}_{20}}\sqrt{\overline{\mu}_{02}}} = \frac{\overline{\mu}_{11}}{\overline{\sigma}_X\overline{\sigma}_Y}. \tag{3.4}$$

These two notions have been mentioned before, for two random variables. They are defined similarly for sets of data and they have the same properties. The covariance gives a rough idea of the relationship between X and Y. As before, if X and Y are independent (so there is no relationship, no correlation between them), then the covariance is 0. If large values of X are associated with large values of Y, then the covariance will have a positive value, if, on the contrary, large values of X are associated with small values of Y, then the covariance will have a negative value. Also, an easier computational formula for the covariance is $\operatorname{cov}(X, Y) = \overline{\nu}_{11} - \overline{x} \cdot \overline{y}$.

The correlation coefficient is then

$$\overline{\rho} = \frac{\overline{\nu}_{11} - \overline{x} \cdot \overline{y}}{\overline{\sigma}_X \overline{\sigma}_Y}$$

and, as before, it satisfies the inequality

$$-1 \leq \overline{\rho} \leq 1. \tag{3.5}$$

By its variation between -1 and 1, its value measures the linear relationship between X and Y. If $\overline{\rho}_{XY} = 1$, there is a *perfect positive correlation* between X and Y, if $\overline{\rho}_{XY} = -1$, there is a *perfect negative correlation* between X and Y. In both cases, the linearity is "perfect", i.e there exist $a, b \in \mathbb{R},\ a \neq 0$, such that $Y = aX + b$. If $\overline{\rho}_{XY} = 0$, then there is no linear correlation between X and Y, they are said to be *(linearly) uncorrelated*. However, in this case, they may not be independent, some other type of relationship (not linear) may exist between them.

In our task of finding a relationship between X and Y, we may go the following path: knowing the value of one of the characteristics, try to find a probable, an "expected" value for the other. If the two characteristics are related in any way, then there should be a pattern developing, i.e., the expected value of one of them, *conditioned* by the other one taking a certain value, should be a function of that value that the other variable assumes. In other words, we should consider *conditional means*, defined similarly to regular means, only taking into account the condition.

Definition 3.4. *Let (X, Y) be a two-dimensional characteristic whose distribution is given by Table 1.*

(1) *The **conditional mean** of Y, given $X = x_i$, is the value*

$$\overline{y}_i = \overline{y}(x_i) = \frac{1}{f_{i\cdot}} \sum_{j=1}^{n} f_{ij} y_j,\ i = \overline{1, m}. \tag{3.6}$$

(2) *The **conditional mean** of X, given $Y = y_j$, is the value*

$$\overline{x}_j = \overline{x}(y_j) = \frac{1}{f_{\cdot j}} \sum_{i=1}^{m} f_{ij} x_i,\ j = \overline{1, n}. \tag{3.7}$$

Definition 3.5. *Let (X, Y) be a two-dimensional characteristic.*

(1) *The curve $y = f(x)$ formed by the points with coordinates $(x_i, \overline{y}_i)$, $i = \overline{1, m}$, is called the **curve of regression** of Y on X.*

(2) *The curve $x = g(y)$ formed by the points with coordinates $(y_j, \overline{x}_j)$, $j = \overline{1, n}$, is called the **curve of regression** of X on Y.*

Remark 3.6. The curve of regression of a characteristic Y with respect to another characteristic X is then the mean value of Y, $\overline{y}(x)$, given $X = x$. The curve of regression is determined so that it approximates best the scatterplot of (X, Y).

3.2 Least Squares Estimation, Linear Regression

One of the most popular ways of finding curves of regression is the *least squares method*.
Assume the curve of regression of Y on X is of the form

$$y = y(x) = f(x; a_1, \ldots, a_s).$$

We determine the unknown parameters $a_1, \ldots, a_s$ so that the *sum of squares error* (SSE) (the sum of the squares of the differences between the responses y_j and their fitted values $y(x_i)$, each counted with the corresponding frequency)

$$S = SSE = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} \Big(y_j - y(x_i) \Big)^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} \Big(y_j - f(x_i; a_1, \ldots, a_s) \Big)^2$$

is minimum (hence, the name of the method).
We find the point of minimum $(\overline{a}_1, \ldots, \overline{a}_s)$ of S by solving the system

$$\frac{\partial S}{\partial a_k} = 0, \ k = \overline{1, s},$$

i.e.

$$-2 \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} \Big(y_j - f(x_i; a_1, \ldots, a_s) \Big) \frac{\partial f(x_i; a_1, \ldots, a_s)}{\partial a_k} = 0, \tag{3.8}$$

for every $k = \overline{1, s}$.
Then the equation of the curve of regression of Y on X is

$$y = f\left(x; \overline{a}_1, \ldots, \overline{a}_s \right).$$

Linear regression

Let us consider the case of *linear regression* and find the equation of the *line of regression* of Y on X.

We are finding a curve

$$y = ax + b,$$

for which

$$S(a,b) = \sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\Big(y_j - ax_i - b\Big)^2$$

is minimum. We have to solve the 2×2 system

$$\begin{aligned} \frac{\partial S(a,b)}{\partial a} &= 0 \\ \frac{\partial S(a,b)}{\partial b} &= 0, \end{aligned}$$

i.e.

$$\begin{aligned} -2\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\Big(y_j - ax_i - b\Big)x_i &= 0 \\ -2\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\Big(y_j - ax_i - b\Big) &= 0, \end{aligned}$$

which becomes

$$\left\{ \begin{array}{lclcl} \left(\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}x_i^2\right)a & + & \left(\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}x_i\right)b & = & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}x_iy_j \\ \left(\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}x_i\right)a & + & \left(\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}\right)b & = & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}y_j \end{array} \right.$$

and after dividing both equations by N,

$$\left\{ \begin{array}{lclcl} \overline{\nu}_{20}a & + & \overline{\nu}_{10}b & = & \overline{\nu}_{11} \\ \overline{\nu}_{10}a & + & \overline{\nu}_{00}b & = & \overline{\nu}_{01}. \end{array} \right.$$

Its solution is

$$\overline{a} \;=\; \frac{\overline{\nu}_{11} - \overline{\nu}_{10}\overline{\nu}_{01}}{\overline{\nu}_{20} - \overline{\nu}_{10}^2} \;=\; \frac{\overline{\nu}_{11} - \overline{x}\cdot\overline{y}}{\overline{\sigma}_X^2} \;=\; \frac{\overline{\nu}_{11} - \overline{x}\cdot\overline{y}}{\overline{\sigma}_X\overline{\sigma}_Y}\cdot\frac{\overline{\sigma}_Y}{\overline{\sigma}_X} \;=\; \overline{\rho}\,\frac{\overline{\sigma}_Y}{\overline{\sigma}_X},$$

$$\overline{b} \;=\; \overline{\nu}_{01} - \overline{\nu}_{10}\overline{a} \;=\; \overline{y} - \overline{a}\cdot\overline{x}.$$

So the equation of the line of regression of Y on X is

$$y - \overline{y} = \overline{\rho}\,\frac{\overline{\sigma}_Y}{\overline{\sigma}_X}\,(x - \overline{x}) \tag{3.9}$$

and, by analogy, the equation of the line of regression of X on Y is

$$x - \overline{x} = \overline{\rho}\,\frac{\overline{\sigma}_X}{\overline{\sigma}_Y}\,(y - \overline{y})\,. \tag{3.10}$$

Remark 3.7.
1. The point of intersection of the two lines of regression, $(\overline{x},\overline{y})$, is called the *centroid* of the distribution of the characteristic (X, Y).
2. The slope $\overline{a}_{Y|X} = \overline{\rho}\,\dfrac{\overline{\sigma}_Y}{\overline{\sigma}_X}$ of the line of regression of Y on X is called the *coefficient of regression* of Y on X. Similarly, $\overline{a}_{X|Y} = \overline{\rho}\,\dfrac{\overline{\sigma}_X}{\overline{\sigma}_Y}$ is the coefficient of regression of X on Y and

$$\overline{\rho}^2 = \overline{a}_{Y|X}\,\overline{a}_{X|Y}.$$

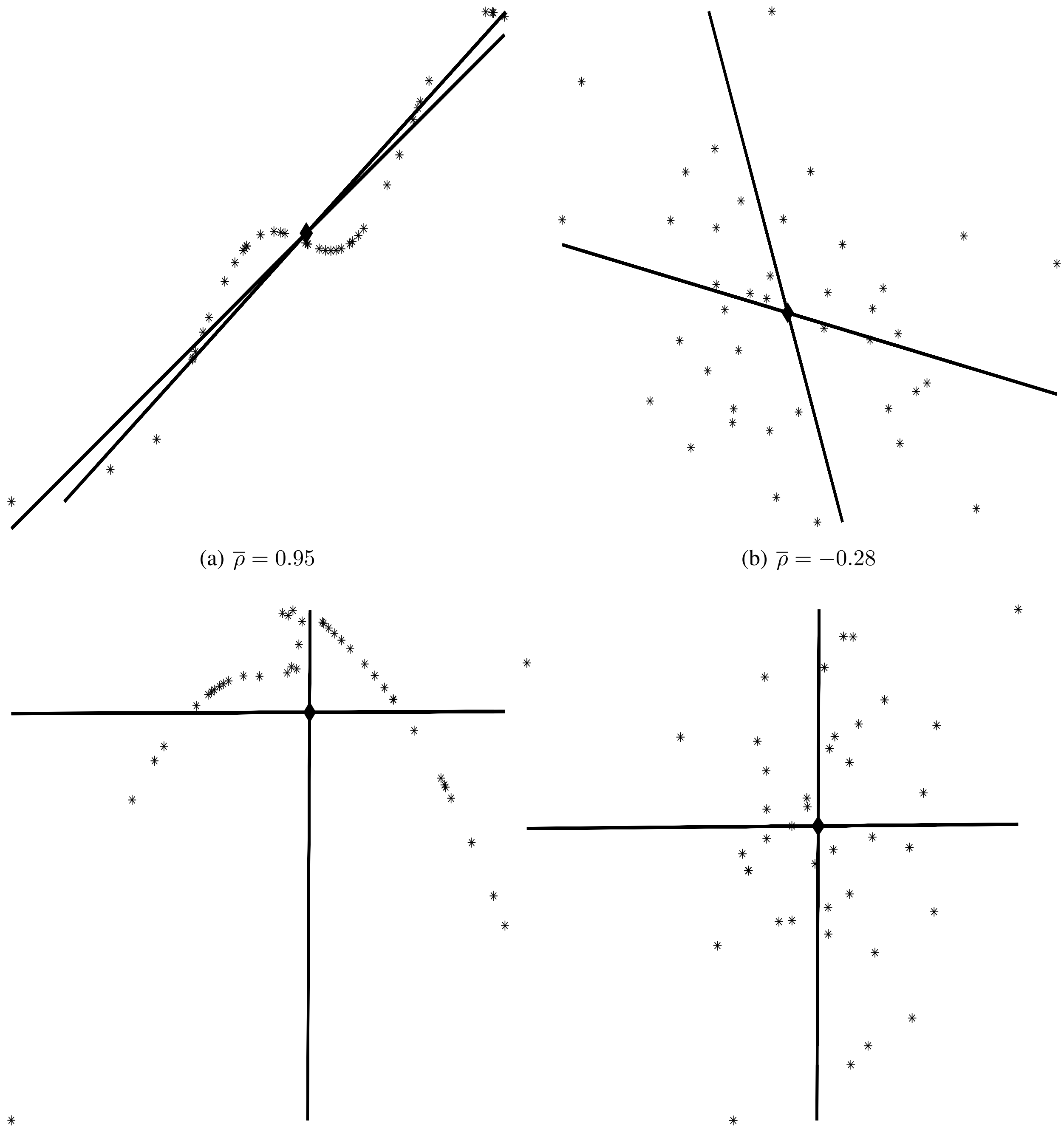
3. For the angle α between the two lines of regression, we have

$$\tan\alpha \;=\; \frac{1-\overline{\rho}^2}{\overline{\rho}^2}\cdot\frac{\overline{\sigma}_X\overline{\sigma}_Y}{\overline{\sigma}_X^2 + \overline{\sigma}_Y^2}.$$

So, if $|\,\overline{\rho}\,| = 1$, then $\alpha = 0$, i.e. the two lines coincide. If $|\,\overline{\rho}\,| = 0$ (for instance, if X and Y are independent), then $\alpha = \dfrac{\pi}{2}$, i.e. the two lines are perpendicular.

Example 3.8. Let us examine the situations graphed in Figure 1.

- In Figure 1(a) $\overline{\rho} = 0.95$, positive and very close to 1, suggesting a strong positive linear trend. Indeed, most of the points are on or very close to the line of regression of Y on X. The positivity indicates that large values of X are associated with large values of Y. Also, since the correlation coefficient is so close to 1, the two lines of regression almost coincide.



(a) $\overline{\rho} = 0.95$

(b) $\overline{\rho} = -0.28$

(c) $\overline{\rho} = 0$

(d) $\overline{\rho} = 0$

Fig. 1: Scattergram, Lines of Regression and Centroid

- In Figure 1(b) $\overline{\rho} = -0.28$, negative and fairly small, close to 0. If a relationship exists between X and Y, it does not seem to be linear. In fact, they are very close to being independent, since the points are scattered around the plane, no pattern being visible. The two lines of regression are very distinct and both have negative slopes, suggesting that large values of X are associated with small values of Y.
- In Figure 1(c) $\overline{\rho} = 0$, so the two characteristics are uncorrelated, no linear relationship exists between them. However they are not independent, they were chosen so that $Y = -X^2 + \sin\left(\frac{1}{X}\right)$. Notice also, that the two lines of regression are perpendicular.
- Finally, in Figure 1(d) $\overline{\rho} = 0$, again, so no linear relationship exists. In fact the two characteristics are independent, which is suggested by their random scatter inside the plane.

Remark 3.9. Other types of curves of regression that are fairly frequently used are
– *exponential* regression $y = ab^x$,
– *logarithmic* regression $y = a \log x + b$,
– *logistic* regression $y = \frac{1}{ae^{-x} + b}$,
– *hyperbolic* regression $y = \frac{a}{x} + b$.