1.3 Hermite Interpolation

Consider the following situation: For a moving object, we know the distances traveled $d_0, d_1, \ldots,$ d_m, at times $t_0, t_1, \ldots, t_m$, and we want a polynomial approximation of the distance function $d = d(t)$ on the entire interval containing the points $t_0, \ldots, t_m$. Obviously, this is a Lagrange interpolation problem and we already know how to find the interpolation polynomial.

Now, assume that, in addition, we also know the values of the *velocities* v_i of the object at times $t_i, i = \overline{0, m}$. We would expect that this additional information helps us find an *even better* approximation of the function d. However, from what we know about Lagrange interpolation, there is *no way* to include this data into our approximation. Since the velocity is the derivative with respect to time of the distance traveled, this means that we also have information about the *derivatives* of the function we want to interpolate. This is a **Hermite interpolation** problem. The ideas and computational formulas are similar to the ones we used to determine the Lagrange interpolation polynomial.

1.3.1 Interpolation with double nodes

For a variety of applications, as the one described above, it is convenient to consider polynomials $P(x)$ that interpolate a function $f(x)$ and in addition have the derivative polynomial $P'(x)$ also interpolate the derivative function $f'(x)$.

Hermite interpolation problem with double nodes. Given $m+1$ distinct nodes $x_i, i = \overline{0, m}$ and the values $f(x_i), f'(x_i)$ of an unknown function f and its derivative, find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$\begin{aligned} P(x_i) &= f(x_i), \\ P'(x_i) &= f'(x_i),\ i = \overline{0, m}. \end{aligned} \tag{1.1}$$

Since for each node there are two values (of the function and of its derivative) given, we call them *double* nodes.

There are $2m+2$ conditions in (1.1), so we seek a polynomial of degree (at most) $n = 2m+1$. We determine this polynomial in a similar way to the construction of the Lagrange polynomial. Recall the notations:

$$\begin{aligned} \psi_m(x) &= (x - x_0)\ldots(x - x_{m-1})(x - x_m), \\ l_i(x) &= \frac{(x - x_0)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_m)}{(x_i - x_0)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_m)} = \frac{\psi_m(x)}{(x - x_i)\psi'_m(x_i)}, \end{aligned} \tag{1.2}$$

for $i = 0, 1, \ldots, m$.

Theorem 1.1. *There is a unique polynomial $H_n f$ of degree at most n, satisfying the interpolation conditions* (1.1). *This polynomial can be written as*

$$H_n f(x) \;=\; \sum_{i=0}^{m} \Big[h_{i0}(x) f(x_i) + h_{i1}(x) f'(x_i)\Big], \tag{1.3}$$

where

$$\begin{aligned} h_{i0}(x) &= \big[1 - 2l_i'(x_i)(x - x_i)\big]\big[l_i(x)\big]^2, \\ h_{i1}(x) &= (x - x_i)\big[l_i(x)\big]^2,\; i = 0, \ldots, m. \end{aligned} \tag{1.4}$$

$H_n f$ is called the ***Hermite interpolation polynomial*** *of f at the double nodes $x_0, x_1, \ldots, x_m$. The functions $h_{i0}(x), h_{i1}(x),\; i = \overline{0, m}$ are called* ***Hermite fundamental (basis) polynomials*** *associated with these points.*

Proof. First we will prove that the polynomial in (1.3) *does* satisfy all interpolation conditions (i.e., existence), and then we will show that it is *the only one* to do so (i.e., uniqueness).
The degree of polynomials l_i from (1.2) is m, so the degree of h_{i0}, h_{i1} and $H_n f$ is $2m + 1 = n$.
The derivatives of the Hermite fundamental polynomials are

$$\begin{aligned} h_{i0}'(x) &= -2l_i'(x_i)\big(l_i(x)\big)^2 + 2\big[1 - 2l_i'(x_i)(x - x_i)\big] l_i'(x) l_i(x), \\ h_{i1}'(x) &= \big(l_i(x)\big)^2 + 2(x - x_i) l_i'(x) l_i(x). \end{aligned}$$

Notice that $l_i(x),\; i = \overline{0, m}$ are the Lagrange fundamental polynomials, thus,

$$l_i(x_j) \;=\; \delta_{ij} \;=\; \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Then,

$$\begin{aligned} h_{i0}(x_j) &= 0,\; j \neq i, \\ h_{i0}(x_i) &= 1 \cdot \big(l_i(x_i)\big)^2 \;=\; 1, \\ h_{i1}(x_j) &= 0,\; j \neq i, \\ h_{i1}(x_i) &= 0. \end{aligned}$$

The values of the derivatives at the nodes are

$$\begin{aligned} h'_{i0}(x_j) &= 0,\ j \neq i, \\ h'_{i0}(x_i) &= -2l'_i(x_i) + 2l'_i(x_i) \ = \ 0, \\ h'_{i1}(x_j) &= 0,\ j \neq i, \\ h'_{i1}(x_i) &= 1 + 0 \ = \ 1. \end{aligned}$$

It follows that

$$\begin{aligned} (H_n f)(x_k) &= \sum_{i=0}^{m} \Big[h_{i0}(x_k) f(x_i) + h_{i1}(x_k) f'(x_i) \Big] \ = \ f(x_k), \\ (H_n f)'(x_k) &= \sum_{i=0}^{m} \Big[h'_{i0}(x_k) f(x_i) + h'_{i1}(x_k) f'(x_i) \Big] \ = \ f'(x_k),\ k = \overline{0, m}, \end{aligned}$$

hence, the polynomial $H_n f$ given in (1.3) satisfies the interpolation conditions (1.1).

To prove uniqueness, assume there exists another polynomial G_n (of degree at most $n = 2m+1$) satisfying relations (1.1) and consider

$$Q_n \ = \ H_n - G_n.$$

Then Q_n is also a polynomial of degree at most $n = 2m + 1$. From the interpolation conditions, it follows that

$$\begin{aligned} Q_n(x_i) &= H_n(x_i) - G_n(x_i) \ = \ f(x_i) - f(x_i) \ = \ 0,\ i = 0, \ldots, m, \\ Q'_n(x_i) &= H'_n(x_i) - G'_n(x_i) \ = \ f'(x_i) - f'(x_i) \ = \ 0,\ i = 0, \ldots, m. \end{aligned}$$

So, Q_n, a polynomial of degree at most $2m + 1$, has $m + 1$ *double* roots. By the Fundamental Theorem of Algebra, Q_n must be identically zero, thus proving the uniqueness of H_n.

□

Example 1.2. One of the most widely used form of Hermite interpolation is the cubic Hermite polynomial, which solves the interpolation problem with two double nodes $a < b$,

$$\begin{aligned} P(a) &= f(a),\ P(b) \ = \ f(b), \\ P'(a) &= f'(a),\ P'(b) \ = \ f'(b). \end{aligned} \tag{1.5}$$

Solution. First of all, let us compute the degree. The degree of the polynomial is $\big[2*$(number of nodes) $-1\big]$, so, in this case,

$$n = 2 \cdot 2 - 1 = 3.$$

Letting $x_0 = a,\ x_1 = b$, with our previous notations and formulas, we have

$$\begin{aligned}
\psi_1(x) &= (x-a)(x-b), \\
l_0(x) &= \frac{x-b}{a-b},\ l_0'(x) \ = \ \frac{1}{a-b}, \\
l_1(x) &= \frac{x-a}{b-a},\ l_1'(x) \ = \ \frac{1}{b-a}.
\end{aligned}$$

The Hermite fundamental polynomials are given by

$$\begin{aligned}
h_{00}(x) &= \big(1 - 2l_0'(a)(x-a)\big)\big(l_0(x)\big)^2 \ = \ \left[1 + 2\frac{x-a}{b-a}\right]\left[\frac{b-x}{b-a}\right]^2, \\
h_{10}(x) &= \big(1 - 2l_1'(b)(x-b)\big)\big(l_1(x)\big)^2 \ = \ \left[1 + 2\frac{b-x}{b-a}\right]\left[\frac{x-a}{b-a}\right]^2, \\
h_{01}(x) &= (x-a)\big(l_0(x)\big)^2 \ = \ \frac{(x-a)(b-x)^2}{(b-a)^2}, \\
h_{11}(x) &= (x-b)\big(l_1(x)\big)^2 \ = \ -\frac{(x-a)^2(b-x)}{(b-a)^2}.
\end{aligned}$$

So the cubic Hermite polynomial is

$$\begin{aligned}
H_3 f(x) &= \left[1 + 2\frac{x-a}{b-a}\right]\left[\frac{b-x}{b-a}\right]^2 \cdot f(a) \ + \ \left[1 + 2\frac{b-x}{b-a}\right]\left[\frac{x-a}{b-a}\right]^2 \cdot f(b) \\
&\quad + \ \frac{(x-a)(b-x)^2}{(b-a)^2} \cdot f'(a) \ - \ \frac{(x-a)^2(b-x)}{(b-a)^2} \cdot f'(b).
\end{aligned}$$

■

1.3.2 Newton's divided differences form

Just as in the case of Lagrange interpolation, Newton's divided differences provide a more easily computable form of the Hermite interpolation polynomial.

Consider $2m+2$ distinct nodes $z_0, z_1, \ldots, z_{2m}, z_{2m+1}$ and the Newton polynomial interpolating a function f at these nodes.

$$N_{2m+1}(x) \ = \ f(z_0) + f[z_0, z_1](x - z_0) + \cdots + f[z_0, \ldots, z_{2m+1}](x - z_0) \ldots (x - z_{2m}),$$

with the error given by

$$R_{2m+1}(x) \;=\; f(x) - N_{2m+1}(x) \;=\; f[x, z_0, \ldots, z_{2m+1}](x - z_0)\ldots(x - z_{2m+1}).$$

We take the limits in the two relations above

$$z_0, z_1 \to x_0, \quad z_2, z_3 \to x_1, \quad \ldots, \quad z_{2i}, z_{2i+1} \to x_i, \quad \ldots \quad z_{2m}, z_{2m+1} \to x_m.$$

Denoting by $n = 2m + 1$, we get

$$\begin{aligned} N_n(x) &= f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 \\ &+ f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) + \ldots \\ &+ f[x_0, x_0, \ldots, x_m, x_m](x - x_0)^2 \ldots (x - x_{m-1})^2(x - x_m) \end{aligned} \tag{1.6}$$

and for the remainder,

$$f(x) - N_n(x) \;=\; f[x, x_0, x_0, \ldots, x_m, x_m](x - x_0)^2 \ldots (x - x_m)^2. \tag{1.7}$$

Proposition 1.3. *Let $[a, b] \subset \mathbb{R}$ be the smallest interval containing the distinct nodes $x_0, \ldots, x_m$ and $f : [a, b] \to \mathbb{R}$ be a function of class $C^{2m+2}[a, b]$. Then, for the two polynomials in* (1.3) *and* (1.6)*, we have*

$$H_n f(x) \;=\; N_n(x), \forall x \in [a, b], \tag{1.8}$$

with the interpolation error

$$R_n(x) \;=\; f(x) - H_n f(x) \;=\; \big[\psi_m(x)\big]^2 \frac{f^{(n+1)}(\xi_x)}{(n+1)!},\; \xi_x \in (a, b). \tag{1.9}$$

Proof. By the way it was constructed (in (1.6)), obviously the polynomial N_n has degree at most n. Then, by the uniqueness of the Hermite interpolation polynomial, it suffices to show that N_n satisfies the interpolation conditions (1.1).
From (1.7), it follows that

$$f(x_i) - N_n(x_i) \;=\; 0,\; i = 0, \ldots, m.$$

Also, by the same relation, we have for the derivatives,

$$\begin{aligned} f'(x) - N_n'(x) &= (x-x_0)^2 \dots (x-x_m)^2 \frac{\partial}{\partial x} f[x, x_0, x_0, \dots, x_m, x_m] \\ &+ 2f[x, x_0, x_0, \dots, x_m, x_m] \sum_{i=0}^{m} \left[(x-x_i) \prod_{\substack{j=0 \\ j \neq i}}^{m} (x-x_j)^2 \right], \end{aligned}$$

hence,

$$f'(x_i) - N_n'(x_i) = 0, \; i = 0, \dots, m.$$

Thus,

$$H_n f(x) = N_n(x), \forall x \in [a, b]$$

and the error formula (1.9) follows directly from (1.7) and the mean-value formula for divided differences.

□

Example 1.4. Let us find the polynomial and the remainder for the Hermite interpolation problem with two double nodes $a < b$, from Example 1.2.

Solution. We have

$$\begin{aligned} H_3 f(x) &= f(a) + f[a, a](x-a) + f[a, a, b](x-a)^2 \\ &+ f[a, a, b, b](x-a)^2(x-b). \end{aligned}$$

The divided differences table for two double nodes is

$$\begin{array}{l|lclclcl} z_0 = a & f(a) & \longrightarrow & f[a,a] = f'(a) & \longrightarrow & f[a,a,b] & \longrightarrow & f[a,a,b,b] \\ & & \nearrow & & \nearrow & & \nearrow & \\ z_1 = a & f(a) & \longrightarrow & f[a,b] = \dfrac{f(b)-f(a)}{b-a} & \longrightarrow & f[a,b,b] & & \\ & & \nearrow & & \nearrow & & & \\ z_2 = b & f(b) & \longrightarrow & f[b,b] = f'(b) & & & & \\ & & \nearrow & & & & & \\ z_3 = b & f(b), & & & & & & \end{array}$$

where

$$\begin{aligned}
f[a,a,b] &= \frac{f[a,b]-f'(a)}{b-a}, \\
f[a,b,b] &= \frac{f'(b)-f[a,b]}{b-a}, \\
f[a,a,b,b] &= \frac{f[a,b,b]-f[a,a,b]}{b-a} = \frac{f'(b)-2f[a,b]+f'(a)}{(b-a)^2}.
\end{aligned}$$

The interpolation error is given by

$$\begin{aligned}
f(x)-H_3f(x) &= (x-a)^2(x-b)^2 f[x,a,a,b,b] \\
&= \frac{(x-a)^2(x-b)^2}{24} f^{(4)}(\xi_x),
\end{aligned}$$

with ξ_x belonging to the smallest interval that contains the points a, b and x.
We can find a bound for the error. Considering that on $[a,b]$, the maximum of the function $|(x-a)(x-b)|$ occurs at the midpoint of the interval, $\frac{a+b}{2}$, and that the maximum value is $\frac{(b-a)^2}{4}$, we have

$$\max_{x\in[a,b]} \left|f(x)-H_3f(x)\right| \leq \frac{(b-a)^4}{384} \max_{t\in[a,b]} \left|f^{(4)}(t)\right|.$$

■

Example 1.5 (Continuation of Example 1.1 in Lecture 4)**.** Consider the function $f:[0.5,5]\to\mathbb{R},\ f(x)=\sqrt{x}$ and the nodes $a=1, b=4$. Let us compare Lagrange and Hermite approximations.

Solution. For the *simple* nodes $a=1, b=4$, we have the interpolation conditions

$$\begin{aligned}
L_1f(a) &= f(a) = 1, \\
L_1f(b) &= f(b) = 2,
\end{aligned}$$

satisfied by the Lagrange polynomial of degree 1

$$L_1f(x) = \frac{1}{3}x + \frac{2}{3}.$$

If the nodes are *double*, the interpolation conditions are

$$\begin{aligned}
H_3 f(a) &= f(a) \;=\; 1,\\
H_3 f(b) &= f(b) \;=\; 2,\\
\big(H_3 f\big)'(a) &= f'(a) \;=\; 1/(2\sqrt{1}) \;=\; 1/2,\\
\big(H_3 f\big)'(b) &= f'(b) \;=\; 1/(2\sqrt{4}) \;=\; 1/4.
\end{aligned}$$

The divided differences table is

$$\begin{array}{r|lclclcl}
z_0 = 1 & \boldsymbol{f(1) = 1} & \longrightarrow & \boldsymbol{f'(1) = 1/2} & \longrightarrow & \boldsymbol{f[1,1,4] = -1/18} & \longrightarrow & \boldsymbol{f[1,1,4,4] = 1/108}\\
 & & \nearrow & & \nearrow & & \nearrow & \\
z_1 = 1 & f(1) = 1 & \longrightarrow & f[1,4] = 1/3 & \longrightarrow & f[1,4,4] = -1/36 & & \\
 & & \nearrow & & \nearrow & & & \\
z_2 = 4 & f(4) = 2 & \longrightarrow & f'(4) = 1/4 & & & & \\
 & & \nearrow & & & & & \\
z_3 = 4 & f(4) = 2, & & & & & &
\end{array}$$

The corresponding cubic Hermite interpolation polynomial is given by

$$H_3 f(x) \;=\; 1 + \frac{1}{2}(x-1) - \frac{1}{18}(x-1)^2 + \frac{1}{108}(x-1)^2(x-4),$$

with derivative

$$\big(H_3 f\big)'(x) \;=\; \frac{1}{2} - \frac{1}{9}(x-1) + \frac{1}{108}(x-1)\big[2(x-4) + (x-1)\big].$$

Check that $H_3 f$ found above satisfies the interpolation conditions:

$$\begin{aligned}
H_3 f(1) &= 1 \;=\; f(1),\\
H_3 f(4) &= 1 + \frac{3}{2} - \frac{1}{18}\cdot 9 \;=\; 2 \;=\; f(4),\\
\big(H_3 f\big)'(1) &= \frac{1}{2} \;=\; f'(1),\\
\big(H_3 f\big)'(4) &= \frac{1}{2} - \frac{1}{3} + \frac{1}{108}\cdot 9 \;=\; \frac{1}{4} \;=\; f'(4).
\end{aligned}$$

The graphs of f and the two interpolation polynomials, L_1, H_3, on the interval $[0.5, 5]$, are shown in Figure 1. The interpolation errors are plotted in Figure 2.

■

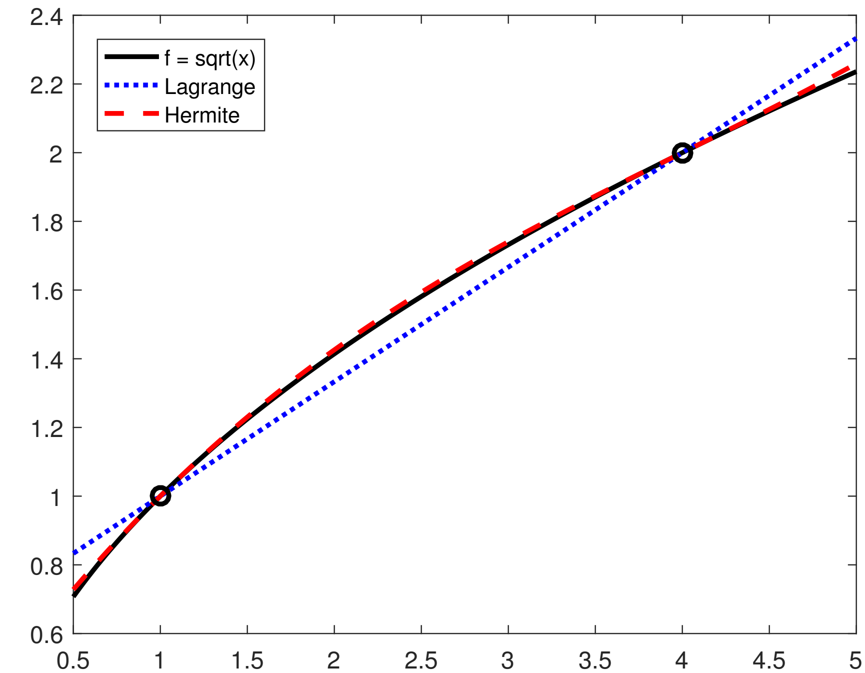

Fig. 1: Lagrange and Hermite interpolation with 2 nodes of the function $\sqrt{x}$

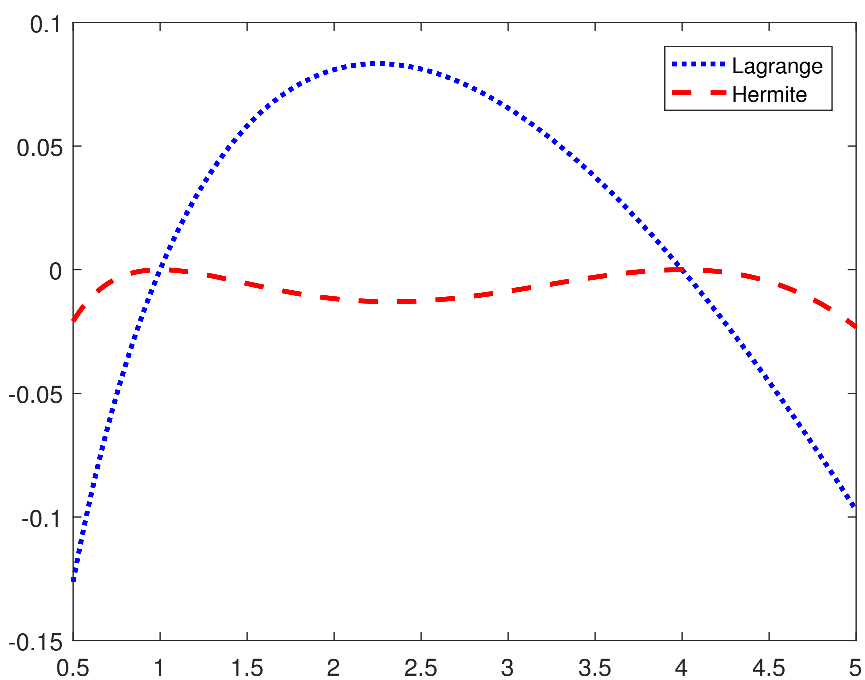

Fig. 2: Error of Lagrange and Hermite interpolation with 2 nodes of the function $\sqrt{x}$

1.3.3 General case

Hermite interpolation problem. Given $m+1$ distinct nodes $x_i \in [a,b], i=\overline{0,m}$,

$$\begin{aligned}&x_0, \text{ of multiplicity } r_0+1,\\&x_1, \text{ of multiplicity } r_1+1,\\&\quad\dots\\&x_i, \text{ of multiplicity } r_i+1,\\&\quad\dots\\&x_m \text{ of multiplicity } r_m+1,\end{aligned}$$

and the values $f^{(j)}(x_i),\ i=0,1,\dots,m,\ j=0,\dots,r_i$, of an unknown function $f:[a,b]\to\mathbb{R}$ whose derivatives of order up to r_i exist at $x_i, i=\overline{0,m}$, find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$P^{(j)}(x_i) = f^{(j)}(x_i),\ i=\overline{0,m},\ j=\overline{0,r_i}. \tag{1.10}$$

Above, there are

$$n+1 \overset{\text{not}}{=} \sum_{i=0}^{m}(r_i+1)$$

conditions, so the polynomial satisfying these relations will have degree at most n.

Theorem 1.6. *There is a unique polynomial $H_n f$ of degree at most n, satisfying the interpolation conditions* (1.10)*. This polynomial is called the **Hermite interpolation polynomial** of the function f, relative to the nodes $x_0, x_1, \dots, x_m$ and the integers $r_0, r_1, \dots, r_m$, and it can be written as*

$$H_n f(x) = \sum_{i=0}^{m}\sum_{j=0}^{r_i} h_{ij}(x) f^{(j)}(x_i). \tag{1.11}$$

Remark 1.7.

1. The functions $h_{ij}(x),\ i=\overline{0,m}, j=\overline{0,r_i}$, are called **Hermite fundamental (basis) polynomials** and they satisfy the relations

$$\begin{aligned} h_{ij}^{(k)}(x_l) &= 0, \quad l\neq i,\ k=\overline{0,r_l},\\ h_{ij}^{(k)}(x_i) &= \delta_{jk}, \quad k=\overline{0,r_i}.\end{aligned} \tag{1.12}$$

2. If we denote by

$$\begin{aligned} u(x) &= \prod_{i=0}^{m}(x-x_i)^{r_i+1}, \\ u_i(x) &= \prod_{\substack{j=0\\ j\neq i}}^{m}(x-x_j)^{r_j+1} = \frac{u(x)}{(x-x_i)^{r_i+1}}, \end{aligned} \tag{1.13}$$

then the fundamental polynomials $h_{ij}(x)$ din (1.11) can be written as

$$h_{ij}(x) = \frac{(x-x_i)^j}{j!}\left[\sum_{k=0}^{r_i-j}\frac{(x-x_i)^k}{k!}\left[\frac{1}{u_i(x)}\right]^{(k)}_{x=x_i}\right]u_i(x). \tag{1.14}$$

2. A more computable form can be found using Newton divided differences. Re-indexing the nodes according to their multiplicity,

$$\begin{aligned} z_0 &= x_0,\ \dots,\ z_{r_0} = x_0, \\ z_{r_0+1} &= x_1,\ \dots,\ z_{(r_0+1)+r_1} = x_1, \\ z_{(r_0+1)+(r_1+1)} &= x_2,\ \dots,\ z_{(r_0+1)+(r_1+1)+r_2} = x_2, \\ &\dots \\ z_{n-r_m} &= x_m,\ \dots,\ z_n = x_m, \end{aligned}$$

the Hermite polynomial can be written in Newton's form as

$$N_n f(x) = f(z_0) + f[z_0, z_1](x-z_0) + \cdots + f[z_0, \dots, z_n](x-z_0)\dots(x-z_{n-1}), \tag{1.15}$$

with interpolation error

$$\begin{aligned} R_n(x) &= f(x) - N_n(x) = f[x, z_0, \dots, z_n](x-z_0)\dots(x-z_n) \\ &= \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi_x),\ \xi_x \in (a,b). \end{aligned} \tag{1.16}$$

Example 1.8. Consider the case of a simple node x_0 and a double node x_1. Find the interpolant for this data and an expression for the remainder.

Solution. We have the nodes

$$\begin{aligned} &x_0, \quad \text{of multiplicity } r_0+1 = 1, \\ &x_1, \quad \text{of multiplicity } r_1+1 = 2. \end{aligned}$$

so $n+1 = 1+2$ and the polynomial has degree $n = 2$.

The divided differences table:

$$\begin{array}{c|ccccc}
x_0 & f(x_0) & \longrightarrow & f[x_0,x_1] & \longrightarrow & \dfrac{f'(x_1) - f[x_0,x_1]}{x_1 - x_0} \\
 & & \nearrow & & \nearrow & \\
x_1 & f(x_1) & \longrightarrow & f'(x_1) & & \\
 & & \nearrow & & & \\
x_1 & f(x_1) & & & &
\end{array}$$

Then,

$$\begin{aligned}
H_2 f(x) &= f(x_0) + f[x_0,x_1](x-x_0) + \frac{f'(x_1) - f[x_0,x_1]}{x_1 - x_0}(x-x_0)(x-x_1) \\
&= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x-x_0) + \frac{f'(x_1)}{x_1 - x_0}(x-x_0)(x-x_1) \\
&- \frac{f(x_1) - f(x_0)}{(x_1 - x_0)^2}(x-x_0)(x-x_1) \\
&= h_{00} f(x_0) + h_{10} f(x_1) + h_{11} f'(x_1)
\end{aligned}$$

and the remainder is given by

$$R_2 f(x) = \frac{(x-x_0)(x-x_1)^2}{3!} f'''(\xi),$$

with ξ belonging to the smallest interval containing x_0 and x_1.

Now, since $H_2 f$ has degree 2 (small), we can find it directly: we seek it of the form

$$H_2 f(x) = ax^2 + bx + c$$

and determine coefficients a, b and c from the interpolation conditions:

$$\left\{ \begin{array}{rcl}
H_2 f(x_0) &=& f(x_0) \\
H_2 f(x_1) &=& f(x_1) \\
(H_2 f)'(x_1) &=& f'(x_1)
\end{array} \right. ,$$

i.e., from the linear system

$$\left\{ \begin{array}{rcrcrcl} x_0^2 a & + & x_0 b & + & c & = & f(x_0) \\ x_1^2 a & + & x_1 b & + & c & = & f(x_1) \\ 2x_1 a & + & b & & & = & f'(x_1) \end{array} \right. . \tag{1.17}$$

The matrix of this system,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 0 & 1 & 2x_1 \end{bmatrix},$$

is called a *generalized Vandermonde matrix*. It is invertible and the elements of its inverse are the coefficients of the fundamental polynomials h_{00}, h_{10} and h_{11}.

If the node x_0 is double and x_1 is simple, the corresponding Hermite polynomial and its error are given by

$$\begin{aligned} H_2 f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}(x - x_0)^2, \\ R_2 f(x) &= \frac{(x - x_0)^2 (x - x_1)}{3!} f'''(\xi). \end{aligned}$$

■

Example 1.9. Find a polynomial of minimum degree that interpolates the data $f(0), f(1), f'(1)$ and $f''(1)$ (so, a *simple* node and a *triple* one). Evaluate the error.

Solution. We have the nodes

$$\begin{aligned} x_0 &= 0, \text{ of multiplicity } r_0 + 1 = 1, \\ x_1 &= 1, \text{ of multiplicity } r_1 + 1 = 3. \end{aligned}$$

Hence, we seek the Hermite polynomial of degree at most

$$n = 1 + 3 - 1 = 3.$$

This will be of the form

$$H_3 f(x) = h_{00}(x) f(0) + h_{10}(x) f(1) + h_{11}(x) f'(1) + h_{12}(x) f''(1).$$

We compute the divided differences

$$
\begin{array}{c|lllllll}
0 & \boldsymbol{f(0)} & \longrightarrow & \boldsymbol{f(1)-f(0)} & \longrightarrow & \boldsymbol{f'(1)-f(1)+f(0)} & \longrightarrow & \boldsymbol{\frac{f''(1)}{2}-f'(1)+f(1)-f(0)} \\
 & & \nearrow & & \nearrow & & \nearrow & \\
1 & f(1) & \longrightarrow & f'(1) & \longrightarrow & \frac{f''(1)}{2} & & \\
 & & \nearrow & & \nearrow & & & \\
1 & f(1) & \longrightarrow & f'(1) & & & & \\
 & & \nearrow & & & & & \\
1 & f(1) & & & & & &
\end{array}
$$

Then the interpolant is

$$
\begin{aligned}
H_3 f(x) &= f(0) + \Big(f(1)-f(0)\Big)x + \Big(f'(1)-f(1)+f(0)\Big)x(x-1) \\
&+ \Big(\frac{f''(1)}{2} - f'(1) + f(1) - f(0)\Big)x(x-1)^2 \\
&= -(x-1)^3 f(0) + x(x^2-3x+3)f(1) - x(x-1)(x-2)f'(1) + \frac{1}{2}x(x-1)^2 f''(1).
\end{aligned}
$$

So the fundamental polynomials are

$$
\begin{aligned}
h_{00}(x) &= -(x-1)^3, \\
h_{10}(x) &= x(x^2-3x+3), \\
h_{11}(x) &= -x(x-1)(x-2), \\
h_{12}(x) &= \frac{1}{2}x(x-1)^2,
\end{aligned}
$$

with derivatives

$$
\begin{aligned}
h'_{00}(x) &= -3(x-1)^2, & h''_{00}(x) &= -6(x-1), \\
h'_{10}(x) &= 3(x-1)^2, & h''_{10}(x) &= 6(x-1), \\
h'_{11}(x) &= -(3x^2-6x+2), & h''_{11}(x) &= -6(x-1), \\
h'_{12}(x) &= \frac{1}{2}(x-1)(3x-2), & h''_{12}(x) &= 3x-2.
\end{aligned}
$$

Now, we can better understand relations (1.12), as we can easily see that

$$\begin{cases} h_{00}(0) &= 1 \\ h_{00}(1) &= 0 \\ h'_{00}(1) &= 0 \\ h''_{00}(1) &= 0 \end{cases}, \quad \begin{cases} h_{10}(0) &= 0 \\ h_{10}(1) &= 1 \\ h'_{10}(1) &= 0 \\ h''_{10}(1) &= 0 \end{cases}, \quad \begin{cases} h_{11}(0) &= 0 \\ h_{11}(1) &= 0 \\ h'_{11}(1) &= 1 \\ h''_{11}(1) &= 0 \end{cases}, \quad \begin{cases} h_{12}(0) &= 0 \\ h_{12}(1) &= 0 \\ h'_{12}(1) &= 0 \\ h''_{12}(1) &= 1 \end{cases}.$$

Also, it is now very easy to check that $H_3 f$ satisfies the interpolation conditions.

Alternatively, we can write the polynomial in the form

$$\begin{aligned} H_3 f(x) &= \Big(-f(0) + f(1) - f'(1) + \frac{1}{2} f''(1)\Big) x^3 + \Big(3f(0) - 3f(1) + 3f'(1) - f''(1)\Big) x^2 \\ &+ \Big(-3f(0) + 3f(1) - 2f'(1) + \frac{1}{2} f''(1)\Big) x + f(0). \end{aligned}$$

For the remainder, we have

$$R_3 f(x) = \frac{u(x)}{4!} f^{(iv)}(\xi) = \frac{x(x-1)^3}{4!} f^{(iv)}(\xi), \ \xi \in (0,1).$$

Now,

$$\begin{aligned} u(x) &= x(x-1)^3 = x^4 - 3x^3 + 3x^2 - x, \\ u'(x) &= 4x^3 - 9x^2 + 6x - 1 = (x-1)^2(4x-1), \end{aligned}$$

so $u(x) \le 0$ on $[0,1]$ and it has a local minimum at $x = \frac{1}{4}$. Thus,

$$|u(x)| \le |u(1/4)| = \left| \frac{1}{4} \left(-\frac{3}{4}\right)^3 \right| = \frac{27}{256}.$$

Then, we find an error bound as

$$|R_3 f(x)| \le \frac{27}{256 \cdot 4!} \max_{t \in [0,1]} |f^{(iv)}(t)| \approx 0.0044 \cdot ||f^{(iv)}||.$$

■

Special cases

1. If all $r_i = 0, i = \overline{0, m}$, all the nodes are simple and we have the Lagrange interpolation formula.

2. If we consider one single node, x_0, of multiplicity $n+1$, the Hermite interpolation polynomial is reduced to Taylor's polynomial:

$$\begin{aligned} H_n f(x) &= T_n f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \dots \\ &+ \frac{(x - x_0)^n}{n!} f^{(n)}(x), \end{aligned} \tag{1.18}$$

with remainder

$$R_n(f)(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x). \tag{1.19}$$

3. Consider two nodes, $x_0 = a$, of multiplicity $m + 1$ and $x_1 = b$, of multiplicity $n + 1$.

The Hermite polynomial has degree

$$(m + 1) + (n + 1) - 1 = m + n + 1.$$

With the notations from Remark 1.7, we have

$$\begin{aligned} u(x) &= (x - a)^{m+1}(x - b)^{n+1}, \\ u_0(x) &= (x - b)^{n+1}, \\ u_1(x) &= (x - a)^{m+1}. \end{aligned}$$

The Hermite polynomial is of the form

$$H_{m+n+1} f(x) = \sum_{j=0}^{m} h_{0j}(x) f^{(j)}(a) + \sum_{i=0}^{n} h_{1i}(x) f^{(i)}(b) \tag{1.20}$$

and the fundamental polynomials are given by

$$\begin{aligned} h_{0j}(x) &= \frac{(x - a)^j}{j!} \left[\sum_{k=0}^{m-j} \frac{(x - a)^k}{k!} \left[\frac{1}{(x - b)^{n+1}} \right]_{x=a}^{(k)} \right] (x - b)^{n+1}, \\ h_{1i}(x) &= \frac{(x - b)^i}{i!} \left[\sum_{k=0}^{n-i} \frac{(x - b)^k}{k!} \left[\frac{1}{(x - a)^{m+1}} \right]_{x=b}^{(k)} \right] (x - a)^{m+1}. \end{aligned}$$

In Newton's form (1.15),

$$
\begin{aligned}
H_{m+n+1}f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(m)}(a)}{m!}(x-a)^m \\
&+ f[\underbrace{a,\ldots,a}_{m+1},b](x-a)^{m+1} + f[\underbrace{a,\ldots,a}_{m+1},b,b](x-a)^{m+1}(x-b) \\
&+ \cdots + f[\underbrace{a,\ldots,a}_{m+1},\underbrace{b,\ldots,b}_{n+1}](x-a)^{m+1}(x-b)^n,
\end{aligned}
$$

with remainder

$$
\begin{aligned}
R_{m+n+1} &= f[x,\underbrace{a,\ldots,a}_{m+1},\underbrace{b,\ldots,b}_{n+1}](x-a)^{m+1}(x-b)^{n+1} \\
&= \frac{f^{(m+n+2)}(\xi_x)}{(m+n+2)!}(x-a)^{m+1}(x-b)^{n+1},\ \xi_x \in (a,b).
\end{aligned}
$$