Part I

K^2M Operators

Chapter 1

Basic concepts for K^2M operators

Since the K^2M operators technique is an important tool in mathematical economics, we start this section by presenting the concept of K^2M operator.

Let X a vector space over $\mathbb{R}$. A subset A of X is called a linear subspace if for all $x, y \in A$ $x + y \in A$ and for all $x \in X$ and each $\lambda \in \mathbb{R}$ we have that $\lambda \cdot x \in A$. If A is a nonempty subset of X, then $spanA$ is, by definition, the intersection of all subspaces which contains A, i.e., the smallest linear subspace containing A. We have the following characterization of the span.

$$spanA = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \text{ with } x_i \in A, \lambda_i \in \mathbb{R}, n \in \mathbb{N}\}.$$

Similarly, coA is the intersection of all convex subsets of X which contains A, i.e. coA is the smallest convex set which contains A. We have the following characterization of the span.

$$coA = \{x \in X | x = \sum_{i=1}^{n} \lambda_i \cdot x_i, \text{ with } x_i \in A, \lambda_i \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\}.$$

Of course, $coA \subset spanA$.

Definition 10.1. A family $\{A_i|\ i \in I\}$ of sets is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

Definition 10.2. Let X be a vector space and Y a nonempty subset of X. The multivalued operator $G : Y \to P(X)$ is called a Knaster-Kuratowski-Mazurkiewicz operator (briefly K^2M operator) if and only if

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i), \text{ for each finite subset } \{x_1, \ldots, x_n\} \subset Y.$$

For example, let $G : [-2, 2] \to P(\mathbb{R}),\quad G(x) = [-\frac{1+x^3}{5}, \frac{1+x^3}{5}]$.

Then $\bigcup_{x\in[-2,2]} G(x) = [-\dfrac{9}{5}, \dfrac{9}{5}]$.

Since, for every $x \in [-2, -\frac{9}{5}) \cup (\frac{9}{5}, 2]$ we have that $x \notin G(x)$, we immediately get that G is not a KKM operator.

The main property of K^2M operators is given in:

Theorem 10.3. (K^2M principle) *Let X be a vector topological space, Y a nonempty subset of X and $G : Y \to P(X)$ a K^2M operator such that $G(x)$ is closed, for each $x \in Y$. Then the family $\{G(x)|\ x \in Y\}$ of sets has the finite intersection property.*

As an immediate consequence we obtain the following theorem:

Corollary 10.4. (Ky Fan) *Let X be a vector topological space, Y a nonempty subset of X and $G : Y \to P_{cl}(X)$ a K^2M operator. If at least one of the sets $G(x)$, $x \in Y$ is compact, then $\bigcap_{x\in Y} G(x) \neq \emptyset$.*

We observe that same conclusion can be reached in another way, by involving an auxiliary family of sets and a suitable topology on X.

Corollary 10.5. (Ky Fan) *Let X be a vector space, Y a nonempty subset of X and $G : Y \to P(X)$ a K^2M operator. Assume that there is a multivalued operator $T : Y \to P(X)$ such that $G(x) \subset T(x)$ for each $x \in X$ and*

$$\bigcap_{x\in Y} T(x) = \bigcap_{x\in Y} G(x).$$

If there is some topology on X such that each $T(x)$ is compact, then

$$\bigcap_{x\in Y} G(x) \neq \emptyset.$$

Chapter 2

Game theory

If X, Y are two nonempty sets and $A, B : X \to P(Y)$ are two multivalued operators, then by definition $x_0 \in X$ is said to be a coincidence point for A and B if

$$A(x_0) \cap B(x_0) \neq \emptyset.$$

In this case, we denote $C(A, B) := \{x \in X | A(x_0) \cap B(x_0) \neq \emptyset\}$-the set of all coincidence points of A and B.

The following general coincidence result follows from the K^2M principle.

Theorem 12.1. (Ky Fan) *Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. Let $A, B : X \to \mathcal{P}(Y)$ two multivalued operators satisfying the following assumptions:*

i) $A(x) \in \mathcal{P}_{op}(Y)$ and $B(x) \in P_{cv}(Y)$, for each $x \in X$

ii) $A^{-1}(y) \in P_{cv}(X)$ and $B^{-1}(y) \in \mathcal{P}_{op}(X)$, for each $y \in Y$.

Then there exists an element $x_0 \in X$ such that $A(x_0) \bigcap B(x_0) \neq \emptyset$, i.e. $C(A, B) \neq \emptyset$.

Proof. Let $Z = X \times Y$ and $G : X \times Y \to \mathcal{P}(E \times F)$ be given by

$$G(x, y) = Z \setminus (B^{-1}(y) \times A(x)).$$

Because $G(x, y) \in P_{cl}(X \times Y)$ and $X \times Y$ is compact we get that $G(x, y) \in P_{cp}(X \times Y)$.

It is easy to observe that:

$$Z = \bigcup_{(x,y)\in Z} (B^{-1}(y) \times A(x)) \qquad (12.1).$$

Indeed, for the inclusion " $\subseteq$ ", let $(x_0, y_0) \in Z$ be arbitrarily. Choose an $(x, y) \in A^{-1}(y_0) \times B(x_0) \neq \emptyset$, which is equivalent with $(x_0, y_0) \in B^{-1}(y) \times A(x)$. The reverse inclusion is obvious.

From (12.1) we have that

$$\bigcap_{z\in Z} G(z) = \emptyset.$$

From the first Corollary of K^2M principle G cannot be a K^2M operator. Hence there exist $z_1, z_2, \dots, z_n \in Z$ such that

$$co\{z_1, \dots, z_n\} \not\subset \bigcup_{i=1}^{n} G(z_i),$$

which means that there is a $w \in co\{z_1, \dots, z_n\}$,

$$w = \sum_{i=1}^{n} \lambda_i z_i$$

with

$$w \notin \bigcup_{i=1}^{n} G(z_i).$$

Because Z is convex and $z_i \in Z$, for each $i = \overline{1, n}$ we obtain that $w \in Z$. Hence

$$w \in Z \setminus \bigcup_{i=1}^{n} G(z_i) = \bigcap_{i=1}^{n} (B^{-1}(y_i) \times A(x_i)).$$

Since

$$w = \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i \right)$$

it follows that

$$\sum_{i=1}^{n} \lambda_i x_i \in B^{-1}(y_i) \text{ and } \sum_{i=1}^{n} \lambda_i y_i \in A(x_i), \text{ for each } i = \overline{1, n}.$$

Successively we have:

$$y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i\right) \text{ and } x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i\right), \text{ for each } i=\overline{1,n} \Rightarrow$$

$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i\right) \text{ and } \sum_{i=1}^{n} \lambda_i x_i \in A^{-1}\left(\sum_{i=1}^{n} \lambda_i y_i\right) \Rightarrow$$

$$\sum_{i=1}^{n} \lambda_i y_i \in B\left(\sum_{i=1}^{n} \lambda_i x_i\right) \text{ and } \sum_{i=1}^{n} \lambda_i y_i \in A\left(\sum_{i=1}^{n} \lambda_i x_i\right).$$

Writing

$$x_0 := \sum_{i=1}^{n} \lambda_i x_i \text{ and } y_0 := \sum_{i=1}^{n} \lambda_i y_i$$

we get that $y_0 \in A(x_0) \cap B(x_0)$ and hence $C(A,B) \neq \emptyset$. $\square$

We give now an immediate application to game theory, by establishing a general version of the von Neumann min-max principle due to Sion.

Recall that a functional $f; X \to \mathbb{R}$ on a topological space is called:

(a) lower semicontinuous if $\{x \in X | f(x) > r\}$ is open for each $r \in \mathbb{R}$;

(b) upper semicontinuous if $\{x \in X | f(x) < r\}$ is open for each $r \in \mathbb{R}$.

Also, if X is a convex set of a vector space, then f is said to be:

(i) quasi-concave if $\{x \in X | f(x) > r\}$ is convex for each $r \in \mathbb{R}$;

(ii) quasi-convex if $\{x \in X | f(x) < r\}$ is convex for each $r \in \mathbb{R}$.

Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. By definition, a point $(x^*, y^*) \in X \times Y$ is called a saddle point for f if

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \text{ for each } (x,y) \in X \times Y.$$

The above condition is equivalent with

$$\max_{x \in X} f(x, y^*) = f(x^*, y^*) = \min_{y \in Y} f(x^*, y).$$

Moreover, in this case $(x^*, y^*) \in X \times Y$ is a saddle point for f if and only if

$$\min_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \min_{y \in Y} f(x,y).$$

If P and Q are two players having X and respectively Y their the strategies set, then for $x \in X$ and $y \in Y$ the value $f(x, y)$ represents the gain of P and so, the lost of Q. If $(x^*, y^*) \in X \times Y$ is a saddle point for f then

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \text{ for each } (x, y) \in X \times Y.$$

Hence, if Q choose the strategy y^*, then the gain of P is at most $f(x^*, y^*)$ and the maximum will be attained if P has the strategy x^*. Also, if P choose the strategy x^*, the the lost of Q is at least $f(x^*, y^*)$ and the minimum will be obtained if Q has the strategy y^*. In this way, $(x^*, y^*) \in X \times Y$ assures the optimal balance between the interests of the two players.

The following result was proved by John von Neumann in 1927 for the case of $\mathbb{R}^n$. We present here the version based on Sion's proof.

Theorem 12.2. (Min-max principle) *Let E, F vector topological spaces and $X \in P_{cp,cv}(E)$, $Y \in P_{cp,cv}(F)$. Let $f : X \times Y \to \mathbb{R}$ satisfying:*

i) $y \to f(x, y)$ is lower semicontinuous and quasi-convex for each $x \in X$

ii) $x \to f(x, y)$ is upper semicontinuous and quasi-concave for each fixed $y \in Y$.

Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. Because of upper semicontinuity, $\max\limits_{x \in X} f(x, y)$ exists for each $y \in Y$ and it is a lower semicontinuous function of y, so $\min\limits_{y \in Y} \max\limits_{x \in X} f(x, y)$ exists. Similarly, $\max\limits_{x \in X} \min\limits_{y \in Y} f(x, y)$ exists too. Since $f(x, y) \leq \max\limits_{x \in X} f(x, y)$ we have:

$$\min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y),$$

and therefore

$$\max_{x \in X} \min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y).$$

We shall prove now that the strict inequality cannot hold. For, assume it did. Then there exists some real r with:

$$\max_{x \in X} \min_{y \in Y} f(x, y) < r < \min_{y \in Y} \max_{x \in X} f(x, y).$$

Define $A, B : X \to \mathcal{P}(Y)$ by:

$$A(x) = \{y \in Y | f(x,y) > r\} \text{ and } B(x) = \{y \in Y | f(x,y) < r\}.$$

These multivalued operators would satisfy the coincidence result of Ky Fan.

Indeed, $A(x)$ is open by the lower semicontinuity of $y \to f(x,y)$, each $B(x)$ is convex by the quasi-convexity of $y \to f(x,y)$ and it is nonempty because $\max_{x \in X} \min_{y \in Y} f(x,y) < r$. Since

$$A^{-1}(y) = \{x \in X | f(x,y) > r\}$$

and

$$B^{-1}(y) = \{x \in X | f(x,y) < r\},$$

we find in the same way that each $A^{-1}(y)$ is nonempty and convex and each $B^{-1}(y)$ is open. Then, by Ky Fan coincidence result there is $(x_0, y_0) \in X \times Y$ with $y_0 \in A(x_0) \bigcap B(x_0)$, which gives the contradiction $r < f(x_0, y_0) < r$. The proof is complete. $\square$.

Chapter 3

Variational inequalities

An application of the K^2M principle to the theory of variational inequalities will be now presented.

Let $(H, (\cdot,\cdot))$ be a Hilbert space and X be any subset of H. We recall that an operator $f : X \to H$ is monotone decreasing on X if

$$(f(x) - f(y), x - y) \leq 0, \text{ for all } x, y \in X.$$

Theorem 13.1. (Hartman-Stampacchia) *Let H be a Hilbert space, X a closed bounded convex subset of H and $f : X \to H$ monotone decreasing and continuous.*
Then there exists an element $y_0 \in X$ such that

$$(f(y_0), y_0 - x) \geq 0, \textit{ for all } x \in X.$$

Proof. For each $x \in X$, let $G(x) = \{y \in X | (f(y), y - x) \geq 0\}$. We will prove that

$$\bigcap_{x \in X} G(x) \neq \emptyset.$$

We will establish first that G is a K^2M operator. Indeed, let $y_0 \in co\{x_1, \ldots, x_n\}$. Suppose, by contradiction, that $y_0 \notin \bigcup_{i=1}^{n} G(x_i)$. Then we have $(f(y_0), y_0 - x_i) < 0$, for each $i \in \{1, \cdots, n\}$. Since all the x_i would lie in the half-space $\{x \in H|(f(y_0), y_0) < (f(y_0), x)\}$, so also would $co\{x_1, \ldots, x_n\}$ and therefore, since $y_0 \in co\{x_1, \ldots, x_n\}$, we have got the contradiction $(f(y_0), y_0) < (f(y_0), y_0)$. Thus G is a K^2M operator.

Consider now the multivalued operator $T : X \to \mathcal{P}(H)$ given by:

$$T(x) = \{y \in X|(f(x), y - x) \geq 0\}.$$

We show that T satisfies the requirements of the second Corollary of K^2M principle.

(i) $G(x) \subset T(x)$, for all $x \in X$. For, let $y \in G(x)$. Then $(f(y), y-x) \geq 0$. By the monotonicity of f we have that $(f(y) - f(x), y - x) \leq 0$ and so

$$0 \leq (f(y), y - x) \leq (f(x), y - x).$$

It follows $y \in T(x)$.

(ii) $\bigcap\limits_{x \in X} T(x) = \bigcap\limits_{x \in X} G(x)$. For, it is enough to show

$$\bigcap_{x \in X} T(x) \subset \bigcap_{x \in X} G(x).$$

Assume $y_0 \in \bigcap\limits_{x \in X} T(x)$. Choose any $x \in X$ and let $z_t = tx + (1 - t)y_0 = y_0 - t(y_0 - x)$. Because X is convex, we have that $z_t \in X$, for each $0 \leq t \leq 1$. Since $y_0 \in T(z_t)$, for each $t \in [0, 1]$, we find that $(f(z_t), y_0 - z_t) \geq 0$ for all $t \in [0, 1]$. This means that $t(f(z_t), y_0 - x) \geq 0$, for all $t \in [0, 1]$ and in particular, that $(f(z_t), y_0 - x) \geq 0$, for $t \in]0, 1]$. Let $t \to 0$. From the continuity of f, we obtain that $f(z_t) \to f(y_0)$ and therefore we have $(f(y_0, y_0 - x) \geq 0$. Thus $y_0 \in G(x)$, for each $x \in X$ and the second assumption is proved.

(iii) We now equip H with the weak topology. Then X, as a closed bounded convex set in a Hilbert space, is weakly compact. Therefore, each $T(x)$, being the intersection of the closed half-space $\{y \in H|(f(x), y) \geq (f(x), x)\}$ with X is, for the same reason, weakly compact.

All the requirements of the second Corollary of K^2M principle are satisfied and hence $\bigcap\limits_{x \in X} G(x) \neq \emptyset$. The proof is complete. $\square$

Part II

Other Techniques in Mathematical Economics

Chapter 4

Maximal elements

The following theorems give sufficient conditions for a multivalued operator on a compact set to have a maximal element. They also allow us to extend the classical results of equilibrium theory to cover consumers whose preferences may not be representable by utility functions. The problem faced by a consumer is to choose a consumption pattern given his income and prevailing prices. In a market economy, a consumer must purchase his consumption vector at the market prices. The set of all admissible commodity vectors that he can afford at prices p, given an income M (or M_i) is called the budget set and will be denoted by A (or A_i). The budget set can be represented as:

$$A = \{x \in X | p \cdot x \leq M\}.$$

Of course, the budget set can be also empty. An important feature of the budget set is that it is positively homogeneous of degree zero in prices and income. That is, it remains unchanged if the price vector and income are multiplied by the same positive number. If $X = \mathbb{R}^m_+$ and $p > 0$ then the budget set is compact. If some prices are allowed to be zero, then the budget set is no longer compact.

Let us denote by $U(x)$ the set of all consumption vectors which the consumer strictly prefer to x, i. e.

$$U(x) = \{y \in A | y \text{ is strictly preferred to } x\}.$$

Obviously, $U : A \multimap A$ and it is called the preference multifunction or the multivalued operator of preferences. A vector $x^* \in A$ is an optimal preference for a given consumer if and only if $U(x^*) = \emptyset$. Such elements x^* are also called U-maximal or simply maximal. The set of all maximal vectors in the budget set is called the consumer's demand set.

Remark 15.1. Let us remark that if a binary relation U on a set Y is given as follows: it associates to each $x \in Y$ a set $U(x) \subset Y$, which may be interpreted as the set of those elements in Y that are "better" or "larger" than x, then we obtain in fact a multivalued operator $U : Y \multimap Y$, defined by $U(x) = \{y \in Y | y \text{ is better than } x\}$.

Theorem 15.2. (Sommenschein) *Let $Y \subset \mathbb{R}^m_+$ be compact and convex and let $U : Y \multimap Y$ a multivalued operator such that:*

i) $x \notin co\ U(x)$, for all $x \in Y$

ii) If $y \in U^{-1}(x)$ then there exists some $z \in Y$ (possibly $z = y$) such that $y \in int\ U^{-1}(z)$.

Then the U-maximal set is nonempty and compact.

Proof. We have that

$$\{x \in Y | U(x) = \emptyset\} = \bigcap_{x \in Y} (Y - U^{-1}(x)).$$

By hypothesis (ii) we have that

$$\bigcap_{x \in Y} (Y - U^{-1}(x)) = \bigcap_{z \in Y} (Y - \text{int } U^{-1}(z)).$$

This latter intersection is compact. Define a multivalued operator by

$$F(x) = Y - \text{int } U^{-1}(x), \text{ for each } x \in Y.$$

Each $F(x)$ is compact. If $y \in \text{co } \{x_i | i \in \{1, \cdots, n\}\}$ then $y \in \cup_{i=1}^n F(x^i)$. Indeed, if we suppose that $y \notin \cup_{i=1}^n F(x^i)$ then $y \in U^{-1}(x^i)$, for all i, and so $x^i \in U(y)$, for all i. But then $y \in \text{co } \{x_i | i \in \{1, \cdots, n\}\} \subset \text{co } U(y)$, which violates (i). It then follows from the K^2M corollary that $\bigcap_{x \in y} F(x) \neq \emptyset$. $\square$

Remark 15.3. Arrow applied Sonnenschein result to the problem of existence of equilibrium in a political model.

Corollary 15.4. (Ky Fan lemma-Alternate statement) *Let $Y \subset \mathbb{R}^m_+$ be compact and let $U : Y \multimap Y$ a multivalued operator such that:*

i) $x \notin U(x)$, for all $x \in Y$

ii) $U(x)$ is convex, for each $x \in Y$

iii) $Graf U$ is open in $Y \times Y$.

Then the U-maximal set is nonempty and compact.

Chapter 5

Walras type price equilibrium

Recall that a price p is a free disposal equilibrium price if $f(p) \leq 0$, where f denotes the singlevalued excess demand operator.

Theorem 16.1. (Hartman-Stampacchia) *Let Y a compact and convex subset of $\mathbb{R}^m_+$ and let $f : Y \to \mathbb{R}^m_+$ be continuous. Then there exists an element $p^* \in Y$ such that*

$$p^* \cdot f(p^*) \geq p \cdot f(p^*), \text{ for all } p \in Y.$$

Furthermore the set of all such p^* is compact.

Proof. Define a binary relation U on Y by: $q \in U(p)$ if and only if $q \cdot f(p) > p \cdot f(p)$. Obviously we got a multivalued operator

$$U(p) := \{q \in Y | q \cdot f(p) > p \cdot f(p)\}, \text{ for each } p \in Y.$$

Since f is continuous U has open graph. Also $U(p)$ is convex and $p \notin U(p)$, for each $p \in Y$. Thus by Ky Fan lemma (alternative statement) there is a $p^* \in Y$ such that $U(p^*) = \emptyset$., i. e. for each $p \in Y$ it is not true that $p \cdot f(p^*) > p^* \cdot f(p^*)$. Thus for all $p \in Y$ we have $p^* \cdot f(p^*) \geq p \cdot f(p^*)$. Conversely, any such p^* is U-maximal, so the U-maximal set is compact by the same lemma. □

Theorem 16.2. *Let Y be a compact convex set in $\mathbb{R}^{m+1}_+$ and let $f : Y \to \mathbb{R}^{m+1}_+$ be continuous and satisfy $p \cdot f(p) \leq 0$, for all p.*

Then the set $\{p \in Y | f(p) \leq 0\}$ of free disposal equilibrium prices is nonempty and compact.

Proof. Compactness is immediate. From Hartman-Stampacchia theorem and Walras' law there is an element $p^* \in Y$ such that

$$p \cdot f(p^*) \leq p^* \cdot f(p^*) \leq 0, \text{ for all } p \in Y.$$

Thus $f(p^*) \leq 0$. $\square$

Chapter 6

The excess demand multifunction

If we denote by E the excess demand multifunction, then p is an equilibrium price if $0 \in E(p)$ and it is called a free disposal equilibrium price if there exists an element $z \in E(p)$ such that $z \leq 0$.

An auxiliary result is:

Lemma 17.1. Let $C \subset \mathbb{R}^m$ be a closed convex and let $K \subset \mathbb{R}^m$ be compact convex.

Then $K \cap C^* \neq \emptyset$ if and only if for each $p \in C$ there exists $z \in K$ such that $p \cdot z \leq 0$.

The following theorem is fundamental with respect to the existence of a market equilibrium of an economy and generalizes a similar result for a singlevalued excess demand operator.

Theorem 17.2. (Gale-Debreu-Nikaido) *Let $E : \Delta \to P_{cp,cv}(\mathbb{R}^m_+)$ be an u. s. c. multivalued operator such that for each $p \in \Delta$ we have $p \cdot z \leq 0$, for all $z \in E(p)$. Put $N = -\mathbb{R}^{n+1}_+$.*

Then the set $\{p \in \Delta | N \cap E(p) \neq \emptyset\}$ of free disposal equilibrium prices is nonempty and compact.

Proof. For each $p \in \Delta$ set

$$U(p) = \{q | q \cdot z > 0, \text{ for all } z \in E(p)\}.$$

Then $U(p)$ is convex for each p and $p \notin U(p)$. Also $U(p)$ is open for each p. Indeed, if $q \in U^{-1}(p)$, we have $p \cdot z > 0$ for all $z \in E(q)$. Then, since E is upper semicontinuous $E^{+}(\{x|p \cdot x > 0\})$ is a neighborhood of q in $U^{-1}(p)$.

Now p is U-maximal if and only if

$$\text{for each } q \in \Delta \text{ there is a } z \in E(p) \text{ such that } q \cdot z \leq 0.$$

Using an auxiliary result (see lemma below), it follows that p is U-maximal if and only if $E(p) \cap N \neq \emptyset$. Thus by Sonnenschein theorem the set $\{p|E(p) \cap N \neq \emptyset\}$ is nonempty and compact. $\square$