Chapter 1

Game theory

1.1 Introduction

Game theory is a branch of applied mathematics that is used in the social sciences, most notably in economics, as well as in biology (particularly evolutionary biology and ecology), engineering, political science, international relations, computer science, social psychology, philosophy and management. Game theory attempts to mathematically capture behavior in strategic situations, or games, in which an individual's success in making choices depends on the choices of others (Myerson, 1991). While initially developed to analyze competitions in which one individual does better at another's expense (zero sum games), it has been expanded to treat a wide class of interactions, which are classified according to several criteria. Today, "game theory is a sort of umbrella or 'unified field' theory for the rational side of social science, where 'social' is interpreted broadly, to include human as well as non-human players (computers, animals, plants)" (Aumann 1987).

Traditional applications of game theory attempt to find equilibria in these games. In an equilibrium, each player of the game has adopted a strategy that they are unlikely to change. Many equilibrium concepts have been developed (most famously the Nash equilibrium) in an attempt to capture this idea. These equilibrium concepts are motivated differently depending on the field of application, although they often overlap or coincide. This methodology is not without criticism, and debates continue over the appropriateness of particular equilibrium concepts, the appropriateness of equilibria altogether, and the usefulness of mathematical models more generally.

Although some developments occurred before it, the field of game theory came into being with E. Borel's researches in his 1938 book Applications aux Jeux de Hasard, and was followed by the 1944 book Theory of Games and Economic Behavior by John von Neumann and Oskar Morgenstern. This theory was developed extensively in the 1950s by many scholars. Game theory was later explicitly applied to biology in the 1970s, although similar developments go back at least as far as the 1930s. Game theory has been widely recognized as an important tool in many fields. Eight game theorists have won the Nobel Memorial Prize in Economic Sciences, and John Maynard Smith was awarded (1999) the Crafoord Prize for his application of game theory to biology. Note that the Crafoord Prize is an annual science prize established in 1980 by Holger Crafoord, a Swedish industrialist, and his wife Anna-Greta Crafoord. Administered by the Royal Swedish Academy of Sciences, the prize "is intended to promote international basic research in the disciplines: Astronomy and Mathematics, Geosciences, Biosciences, with particular emphasis on Ecology, and Polyarthritis (rheumatoid arthritis)", the disease from which Holger severely suffered in his last years. According to the Academy, "these disciplines are chosen so as to complement those for which the Nobel Prizes are awarded." Only one award is given each year, according to a rotating scheme: astronomy and mathematics, then geosciences, then biosciences. A Crafoord Prize is only awarded for polyarthritis when a special committee decides that substantial progress in the field has been made. The recipient of the Crafoord Prize is announced each year in mid-January; on Crafoord Day in April, the prize is presented by the King of Sweden, who also presents the Nobel Prize Awards at the ceremony in December. The prize sum, which as of 2009 is USD 500,000, is intended to fund further research by the prize winner.

The inaugural winners, Vladimir Arnold and Louis Nirenberg, were cited by the Academy for their work in the field of non-linear differential equations.

1.2 Short history

The first known discussion of game theory occurred in a letter written by James Waldegrave in 1713. In this letter, Waldegrave provides a minimax mixed strategy solution to a two-person version of the card game "le Her".

James Madison made what we now recognize as a game-theoretic analysis of the ways states can be expected to behave under different systems of taxation.

It was not until the publication of Antoine Augustin Cournot's Recherches sur les principes mathematiques de la theorie des richesses (Researches into the Mathematical Principles of the Theory of Wealth) in 1838 that a general game-theoretic analysis was pursued. In this work, Cournot considers a duopoly (A duopoly (from Greek duo(two) + polein(to sell)) is a specific type of economic activity where only two producers exist in one market. More generally, this definition is used where only two firms have dominant control over a market) and presents a solution that is a restricted version of the Nash equilibrium.

Although Cournot's analysis is more general than Waldegrave's, game theory did not really exist as a unique field until John von Neumann published a paper in 1928 (Zur Theorie der Gesellschaftsspiele (On social games theory), Mathematische Annalen, 100(1), pp. 295-320). While the French mathematician Emile Borel did some earlier work on games, von Neumann can rightfully be credited as the inventor of game theory. Von Neumann's work in game theory culminated in the 1944 book Theory of Games and Economic Behavior by J. von Neumann and Oskar Morgenstern. This foundational work contains the method for finding mutually consistent solutions for two-person zero-sum games. During this time period, work on game theory was primarily focused on cooperative game theory, which analyzes optimal strategies for groups of individuals, presuming that they can enforce agreements between them about proper strategies.

In 1950, the first discussion of the prisoner's dilemma appeared, and an experiment was undertaken on this game at the RAND corporation. Around this same time, John Nash developed a criterion for mutual consistency of players' strategies, known as Nash equilibrium, applicable to a wider variety of games than the criterion proposed by von Neumann and Morgenstern. This equilibrium is sufficiently general to allow for the analysis of non-cooperative games in addition to cooperative ones.

Game theory experienced a flurry of activity in the 1950s, during which time the concepts of the core, the extensive form game, fictitious play, repeated games, and the Shapley value were developed. In addition, the first applications of Game Theory to philosophy and political science occurred during this time.

In 1965, Reinhard Selten introduced his solution concept of subgame perfect equilibria, which further refined the Nash equilibrium

In 1967, John Harsanyi developed the concepts of complete information and Bayesian games. Nash, Selten and Harsanyi became Economics Nobel Laureates in 1994 for their contributions to economic game theory.

In the 1970s, game theory was extensively applied in biology, largely as a result of the work of John Maynard Smith and his evolutionarily stable strategy. In addition, the concepts of correlated equilibrium, trembling hand perfection, and common knowledge were introduced and analyzed.

In 2005, game theorists Thomas Schelling and Robert Aumann followed Nash, Selten and Harsanyi as Nobel Laureates. Schelling worked on dynamic models, early examples of evolutionary game theory. Aumann contributed more to the equilibrium school, introducing an equilibrium coarsening, correlated equilibrium, and developing an extensive formal analysis of the assumption of common knowledge and of its consequences.

In 2007, Roger Myerson, together with Leonid Hurwicz and Eric Maskin, was awarded the Nobel Prize in Economics "for having laid the foundations of mechanism design theory". (Mechanism design, sometimes called Reverse game theory is a field in game theory studying solution concepts for a class of private information games. The distinguishing features of these games are: a game "designer" chooses the game structure rather than inheriting one and the designer is interested in the game's outcome. Such a game is called a "game of mechanism design" and is usually solved by motivating agents to disclose their private information.)

Myerson's contributions include the notion of proper equilibrium, and an important graduate text: Game Theory, Analysis of Conflict (Myerson 1997).

In conclusion, game theory studies strategic interaction between individuals in situations called games.

Games can have several features, a few of the most common are listed here:

Number of players: Each person who makes a choice in a game or who receives a payoff from the outcome of those choices is a player.

Strategies per player: In a game each player chooses from a set of possible actions, known as strategies.

Pure strategy and Nash equilibria: A Nash equilibrium is a set of strategies which represents mutual best responses to the other strategies. In other words, if every player is playing their part of a Nash equilibrium, no player has an incentive to unilaterally change his or her strategy. Considering only situations where players play a single strategy without randomizing (a pure strategy) a game can have any number of Nash equilibria.

Sequential game: A game is sequential if one player performs her/his actions after another, otherwise the game is a simultaneous move game.

Perfect information: A game has perfect information if it is a sequential game and every player knows the strategies chosen by the players who preceded them.

Constant sum: A game is constant sum if the sum of the payoffs to every player are the same for every set of strategies. In these games one player gains if and only if another player loses.

Non-cooperative game: is one in which players make decisions independently. Thus, while they may be able to cooperate, any cooperation must be self-enforcing.

Cooperative game: is a game where groups of players (”coalitions”) may enforce cooperative behaviour, hence the game is a competition between coalitions of players, rather than between individual players.

Classes of these games have been given names. This is a list of the most commonly studied games:

Battle of the sexes

Cournot game

Dictator game

Kuhn poker

Prisoner's dilemma

Rock, Paper, Scissors

1.3 Nash equilibrium problem

Roughly speaking, a game is a situation where a number of players, having absolutely independent interests, must each choose a strategy of a certain action and, then, based on these choices, some consequences appears. If we suppose that there are n game participants, with absolutely independent interests, then the game is said to be a noncooperative n-person game.

Let us present now the elements that characterize the noncooperative n-person game. Denote by X_i the set of all strategies of the i player, where $i \in \{1, 2, ..., n\}$.

Then, $X := \prod_{i=1}^{n} X_i$ is the set of all strategy vectors.

Each $x = (x_1, x_2, ..., x_n) \in X$ induces an outcome.

Players preferences are described using the preference multifunction $\widetilde{U}_i : X \multimap X$, defined by $\widetilde{U}_i(x) := \{y \in X | y \text{ is preferred to x }\}$.

We also define, the good reply multifunction.

Denote $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in X_{-i}$, where
$X_{-i} := \prod_{k=1, k\neq i}^{n} X_k$
$x|y_i := (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n) \in X.$

Then, by definition, y_i is a good reply for the player i with respect to the strategy vector x if $x|y_i \in \widetilde{U}_i(x)$.

In this setting, the good reply multifunction for the player i is $U_i : X_{-i} \multimap X_i$ defined by

$$U_i(x_{-i}) := \{y_i \in X_i | x|y_i \in \widetilde{U}_i(x|u), \text{ for every } u \in X_i\}.$$

A game in strategic form or an abstract economy is the pair $(X_i, U_i)_{i\in\{1,2,...,n\}}$.

For example, if we consider $p_i : X \to \mathbb{R}$, for $i \in \{1, 2, ..., n\}$, the pay-off function of the i player, then the good reply multifunction can be expressed by:

$$U_i(x_{-i}) := \{y_i \in X_i | p_i(x|y_i) \geq p_i(x|z), \text{ for each } z \in X_i\}.$$

By definition, $x^* \in X$ is a (noncooperative) Nash equilibrium point for an abstract economy if $x^*_i \in U_i(x^*_{-i})$, for $i \in \{1, 2, ..., n\}$, i.e.,
$x^*|x^*_i \in \widetilde{U}_i(x^*|u), \text{ for every } u \in X_i \text{ and each } i \in \{1, 2, \cdots, n\}.$

Let us observe that the above Nash equilibrium problem is equivalent to the following fixed point problem:

$$x^* \in U(x^*), \text{ where } U(x) := \prod_{i=1}^{n} U_i(x_{-i}).$$

If $x^* = (x_1^*, \cdots, x_n^*) \in X$ is a (noncooperative) Nash equilibrium then each player of the game reckons his choice as acceptable and does't want to change it.

Let us consider now the case of a 2-person game (or an abstract economy with neighborhood effects) given by $(X_1, U_1), (X_2, U_2)$, where X_1, X_2 denote the set of strategies of the player 1, respectively player 2, and $U_1 : X_2 \multimap X_1$, $U_2 : X_1 \multimap X_2$ are the good reply multifunctions for each player.

By definition, (x_1^*, x_2^*) is a Nash equilibrium point if

$$x_1^* \in U_1(x_2^*) \text{ and } x_2^* \in U_2(x_1^*).$$

Another possibility is to define the good reply multifunction $U_i : X \multimap X_i$ as follows:

$$U_i(x) := \{y_i \in X_i | x | y_i \in \widetilde{U}_i(x)\}.$$

Then, by definition, $x^* \in X$ is a Nash equilibrium point if $U_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$, i.e., there is no $y_i \in X_i$ such that $x^* | y_i$ to be preferred to x^*. Thus, x^* is a Nash equilibrium if x^* is a vector strategy such that none of the players has not a better reply.

In what follows we will consider this definition for the good reply multifunction.

Another important concept in game theory is the constraint (feasible) multifunction. It happens frequently that the choices of the players cannot be made independently. Two simple examples are the case of a mineral water exploitation from several springs, by several economic agents or the case of a fish exploitation from a lake by a number of fishers. Each participant has partial control of the price and the strategy x_i of the i player cannot be chosen independently because their sum cannot exceed the total amount of the exploitation. These situations can be, from the mathematical point of view, modelled by introducing the feasibility or constraint multivalued operator $F_i : X \multimap X_i$, which tell us which strategies are actually feasible for the player i, with respect to the strategy vector x.

So, let us denote by $F_i : X \multimap X_i$, the constraint (feasibility) multifunction for the i player, where $i \in \{1, 2, ..., n\}$. Then define

$$F := \prod_{i=1}^{n} F_i : X \multimap X, \text{ by } F(x) := \prod_{i=1}^{n} F_i(x)$$

.

Obviously, the feasible strategy vectors are the fixed points of F, i. e. elements $x \in X$ with $x \in F(x)$.

By definition, a generalized game or a generalized abstract economy is a strategic game (or an abstract economy), which also includes the constraint multifunction F_i, i.e. $(X_i, U_i, F_i)_{i\in\{1,2,...,n\}}$.

A Nash equilibrium point for a generalized abstract economy is a strategy vector $x^* \in X$ such that $x^* \in F(x^*)$ and $U_i(x^*) \cap F_i(x^*) = \emptyset$, for $i \in \{1, 2, ..., n\}$.

1.4 Conclusions

As a conclusion, if $F : X \to \mathcal{P}(X)$ is a multivalued operator, then **fixed points** (i.e. $x \in X$ with $x \in F(x)$), **strict fixed points** (i.e. $x \in X$ with $\{x\} = F(x)$), **maximal elements** (i.e. $x \in X$ with $F(x) = \emptyset$) and **zero points** (i.e. $x \in X$ with $0 \in F(x)$, where $F : X \to \mathcal{P}(\mathbb{E})$, E is a linear space) of the multifunction F have important meanings in the abstract mathematical economics theory.

It is in our intention to report several results in these four directions.