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The extension monoid product of preinjective and preprojective Kronecker modules

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Abstract. Let P, P' be preprojective and I, I' preinjective Kronecker modules. Working with the extension monoid product we give conditions for the existence of short exact sequences of the form $0 \to P \to I \to I' \to 0$ (and dually for $0 \to P' \to P \to I \to 0$). We show that the existence of these short exact sequences is equivalent with the existence of certain short exact sequences of preinjective (respectively preprojective) Kronecker modules, hence they obey the combinatorial rule described in [11].

1. Introduction and motivation

We begin by putting together a short compilation of definitions and well-known facts about the category of Kronecker modules. The calculations, justifications and proofs leading to these results can be found in many standard textbooks on representation theory of algebras, see for example [1], [7], [2], [6].

Let K be the *Kronecker quiver*, i.e. the quiver having two vertices and two parallel arrows: b,

$$K: 1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$$

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and κ an arbitrary field. The path algebra of the Kronecker quiver is the *Kronecker algebra* and we will denote it by κK. A finite dimensional right module over the Kronecker algebra is called a *Kronecker module*. We denote by mod-κK the category of finite dimensional right modules over the Kronecker algebra.

A (finite dimensional) κ-linear representation of the quiver K is a quadruple $M = (V_1, V_2; \varphi_\alpha, \varphi_\beta)$ where V_1, V_2 are finite dimensional κ-vector spaces (corresponding to the vertices) and $\varphi_\alpha, \varphi_\beta : V_2 \to V_1$ are κ-linear maps (corresponding to the arrows). Thus a κ-linear representation of K associates vector spaces to the vertices and compatible κ-linear functions (or equivalently, matrices) to the arrows. Let us denote by rep-κK the category of finite dimensional κ-representations of the Kronecker quiver. There is a well-known equivalence of categories between mod-κK and rep-κK, so that every Kronecker module can be identified with a representation of K.

The *simple Kronecker modules* (up to isomorphism) are

$$S_1 : \kappa \leftleftarrows 0 \quad \text{and} \quad S_2 : 0 \leftleftarrows \kappa.$$

For a Kronecker module M we denote by $\underline{\dim}M$ its *dimension*. The dimension of M is a vector $\underline{\dim}M = ((\dim M)_1, (\dim M)_2) = (m_{S_1}(M), m_{S_2}(M))$, where $m_{S_i}(M)$ is the number of factors isomorphic with the simple module S_i in a composition series of M, $i = \overline{1,2}$. Regarded as a representation, $M : V_1 \underset{\varphi_\beta}{\overset{\varphi_\alpha}{\leftleftarrows}} V_2$, we have that $\underline{\dim}M = (\dim_\kappa V_1, \dim_\kappa V_2)$.

The *defect* of $M \in$ mod-κK with $\underline{\dim}M = (a, b)$ is defined in the Kronecker case as $\partial M = b - a$.

An indecomposable module $M \in$ mod-κK is a member in one of the following three families: preprojectives, regulars and preinjectives. In what follows we give some details on these families.

The *preprojective indecomposable Kronecker modules* are determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by P_n the indecomposable preprojective module of dimension $(n+1, n)$. So P_0 and P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple). It is known that (up to isomorphism) $P_n = (\kappa^{n+1}, \kappa^n; f, g)$, where choosing the canonical basis in κ^n and κ^{n+1}, the matrix of $f : \kappa^n \to \kappa^{n+1}$ (respectively of $g : \kappa^n \to \kappa^{n+1}$) is $\binom{E_n}{0}$ (respectively $\binom{0}{E_n}$). Thus in this case

$$P_n : \kappa^{n+1} \underset{\binom{E_n}{0}}{\overset{\binom{0}{E_n}}{\leftleftarrows}} \kappa^n,$$

where E_n is the $n \times n$ identity matrix. We have for the defect $\partial P_n = -1$.

We define a *preprojective Kronecker module* P as being a direct sum of indecomposable preprojective modules: $P = P_{a_1} \oplus P_{a_2} \oplus \cdots \oplus P_{a_l}$, where we use the convention that $a_1 \leq a_2 \leq \cdots \leq a_l$.

The *preinjective indecomposable Kronecker modules* are also determined up to isomorphism by their dimension vector. For $n \in \mathbb{N}$ we will denote by I_n the indecomposable preinjective module of dimension $(n, n+1)$. So I_0 and I_1 are the injective indecomposable modules ($P_0 = S_2$ being simple). It is known that (up to isomorphism) $I_n = (\kappa^n, \kappa^{n+1}; f, g)$, where choosing the canonical basis in κ^{n+1} and κ^n, the matrix of $f : \kappa^{n+1} \to \kappa^n$ (respectively of $g : \kappa^{n+1} \to \kappa^n$) is $(\, E_n \quad 0\,)$ (respectively $(\, 0 \quad E_n\,)$). Thus in this case

$$I_n \colon \kappa^n \overset{(E_n\,0)}{\underset{(0\,E_n)}{\leftleftarrows}} \kappa^{n+1},$$

where E_n is the $n \times n$ identity matrix. We have for the defect $\partial I_n = 1$.

We define a *preinjective Kronecker module* I as being a direct sum of indecomposable preinjective modules: $I = I_{a_1} \oplus I_{a_2} \oplus \cdots \oplus I_{a_l}$, where we use the convention that $a_1 \geq a_2 \geq \cdots \geq a_l$.

The *regular indecomposable Kronecker modules* are those indecomposable modules $M \in$ mod-κK which are neither preprojective nor preinjective. A regular indecomposable is isomorphic as representation with one of the following (f_X denotes the multiplication by X, id is the identity function and $n \geq 1$):

- $R_\infty(n) \colon \kappa[X]/(X^n) \overset{f_X}{\underset{id}{\leftleftarrows}} \kappa[X]/(X^n)$;
- $R_\phi(n) \colon \kappa[X]/(\phi(X)^n) \overset{id}{\underset{f_X}{\leftleftarrows}} \kappa[X]/(\phi(X)^n)$, where ϕ is a monic polynomial with $\deg \phi \geq 2$, irreducible in κ;
- $R_k(n) \colon \kappa[X]/((X-k)^n) \overset{id}{\underset{f_X}{\leftleftarrows}} \kappa[X]/((X-a)^n)$, where $k \in \kappa$ (hence $R_k(n)$ is just a notation for $R_{X-k}(n)$).

This is consistent with everything claimed about Kronecker modules so far, since we have the following isomorphism g of representations.

$$\begin{array}{ccc}
\kappa^{nd} & \overset{id}{\underset{C_{\phi(X)^n}}{\leftleftarrows}} & \kappa^{nd} \\
{\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
\kappa[X]/(\phi(X)^n) & \overset{id}{\underset{X}{\leftleftarrows}} & \kappa[X]/(\phi(X)^n)
\end{array}$$

Here ϕ is an arbitrary monic irreducible polynomial, with $\deg \phi = d \geq 1$, where $g \colon \kappa^{nd} \to \kappa[X]/(\phi(X)^n)$, $g(k_0, k_1, \ldots, k_{nd-1}) = k_0 + k_1 X + \cdots + k_{nd-1} X^{nd-1} +$

$(\phi(X)^n)$ for any $(k_0, k_1, \dots, k_{nd-1}) \in \kappa^{nd}$ and $C_{\phi(X)^n}$ is the companion matrix of the polynomial $\phi(X)^n$. We also have the isomorphism

$$\begin{tikzcd} \kappa^n \arrow[d, "g"'] & \kappa^n \arrow[l, shift right, "J_0^{(n)}"'] \arrow[l, shift left, "id"] \arrow[d, "g"] \\ \kappa[X]/(X^n) & \kappa[X]/(X^n) \arrow[l, shift right, "f_X"'] \arrow[l, shift left, "id"] \end{tikzcd}$$

where $g\colon \kappa^n \to \kappa[X]/(X^n)$ is defined as before and $J_0^{(n)} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ is the nilpotent Jordan block of degree n.

To simplify notations and terminology, let us introduce the following set:

$$\mathcal{P} = \{\infty\} \cup \kappa \cup \{\phi \mid \phi \text{ is a monic irreducible polynomial of degree } \deg\phi \geq 2 \text{ over } \kappa\}$$

and call an element $p \in \mathcal{P}$ of this set simply a "point". We will denote by d_p the degree of the point p, where $d_p = \begin{cases} 1, & p \in \{\infty\} \cup \kappa, \\ \deg\phi, & p \in \mathcal{P} \setminus (\{\infty\} \cup \kappa). \end{cases}$ We also use the convention $R_p(0) = 0$, for any $p \in \mathcal{P}$.

Hence the dimension of a regular indecomposable will be $\underline{\dim} R_p(n) = (nd_p, nd_p)$ and we have for the defect $\partial R_p(n) = 0$.

It is known that $R_p(n)$ is uniserial, i.e. there is only one chain of submodules $0 \subset M_1 \subset \cdots \subset M_n = R_p(n)$ with $M_i/M_{i-1} \cong R_p(1)$ and M_i a direct sum of regular indecomposables. Regular modules form an extension closed abelian subcategory of mod-κK.

If $\kappa = \bar{\kappa}$ is algebraically closed, then all irreducible polynomials are of the form $\phi(X) = X - k$ and the companion matrix $C_{(X-k)^n}$ is similar to $J_k^{(n)}$, where $J_k^{(n)}$ is the $n \times n$ Jordan block $J_k^{(n)} = kE_n + J_0^{(n)}$. In this case $\mathcal{P} = \{\infty\} \cup \kappa$ and the regular indecomposables are

$$R_k(n)\colon \kappa^n \overset{J_k^{(n)}}{\underset{E_n}{\leftleftarrows}} \kappa^n \text{ for } k \in \kappa \text{ and } R_\infty(n)\colon \kappa^n \overset{E_n}{\underset{J_0^{(n)}}{\leftleftarrows}} \kappa^n.$$

A module $R \in$ mod-κK will be called a *regular Kronecker module* if it is a direct sum of regular indecomposables. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a partition, then we use the notation $R_p(\lambda) = R_p(\lambda_1) \oplus R_p(\lambda_1) \oplus \cdots \oplus R_p(\lambda_m)$.

The category mod-κK is a is a *Krull-Schmidt category*, meaning that every module $M \in$ mod-κK has a unique decomposition

$$M = (P_{c_1} \oplus \cdots \oplus P_{c_n}) \oplus (\oplus_{p \in \mathcal{P}} R_p(\lambda^{(p)})) \oplus (I_{d_1} \oplus \cdots \oplus I_{d_m}),$$

where

- $(c_1, \ldots, c_n)$ is a finite increasing sequence of nonnegative integers;
- $\lambda^{(p)} = (\lambda_1, \ldots, \lambda_t)$ is a nonzero partition for finitely many $p \in \mathcal{P}$;
- $(d_1, \ldots, d_m)$ is a finite decreasing sequence of nonnegative integers.

The integer sequences $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_m)$ together with the partitions $\lambda^{(p)}$ corresponding to every $p \in \mathcal{P}$ are called the *Kronecker invariants* of the module M. Hence Kronecker invariants determine a module $M \in$ mod-κK up to isomorphism.

The following well-known lemmas (to be found for example in [9]) summarize some facts on morphisms, extensions and short exact sequences in mod-κK:

Lemma 1. *Denoting by R, P and I a preprojective, a regular, respectively a preinjective Kronecker module, we have (where $m, n, t, t_1, t_2 \in \mathbb{N}$, $p \in \mathcal{P}$ and d_p is the degree of the point p):*

(a) $\mathrm{Hom}(R, P) = \mathrm{Hom}(I, P) = \mathrm{Hom}(I, R) = \mathrm{Ext}^1(P, R) = \mathrm{Ext}^1(P, I) = \mathrm{Ext}^1(R, I) = 0$.

(b) *For $n \leq m$, we have* $\dim_\kappa \mathrm{Hom}(P_n, P_m) = m - n + 1$ *and* $\mathrm{Ext}^1(P_n, P_m) = 0$; *otherwise* $\mathrm{Hom}(P_n, P_m) = 0$ *and* $\dim_\kappa \mathrm{Ext}^1(P_n, P_m) = n - m - 1$. *In particular* $\mathrm{End}(P_n) \cong \kappa$ *and* $\mathrm{Ext}^1(P_n, P_n) = 0$.

(c) *For $n \geq m$, we have* $\dim_\kappa \mathrm{Hom}(I_n, I_m) = n - m + 1$ *and* $\mathrm{Ext}^1(I_n, I_m) = 0$; *otherwise* $\mathrm{Hom}(I_n, I_m) = 0$ *and* $\dim_\kappa \mathrm{Ext}^1(I_n, I_m) = m - n - 1$. *In particular* $\mathrm{End}(I_n) \cong \kappa$ *and* $\mathrm{Ext}^1(I_n, I_n) = 0$.

(d) *If $p \neq p'$, then* $\mathrm{Hom}(R_p(t_1), R_{p'}(t_2)) = \mathrm{Ext}^1(R_p(t_1), R_{p'}(t_2)) = 0$.

(e) $\dim_\kappa \mathrm{Hom}(P_n, I_m) = n + m$ *and* $\dim_\kappa \mathrm{Ext}^1(I_m, P_n) = m + n + 2$.

(f) $\dim_\kappa \mathrm{Hom}(P_n, R_p(t)) = \dim_\kappa \mathrm{Hom}(R_p(t), I_n) = d_p t$ *and* $\dim_\kappa \mathrm{Ext}^1(R_p(t), P_n) = \dim_\kappa \mathrm{Ext}^1(I_n, R_p(t)) = d_p t$.

(g) $\dim_\kappa \mathrm{Hom}(R_p(t_1), R_p(t_2)) = \dim_\kappa \mathrm{Ext}^1(R_p(t_1), R_p(t_2)) = d_p \min(t_1, t_2)$.

Lemma 2. *If there is a short exact sequence $0 \to M' \to M \to M'' \to 0$ of Kronecker modules, then $\underline{\dim} M = \underline{\dim} M' + \underline{\dim} M''$ and $\partial M = \partial M' + \partial M''$.*

The category of Kronecker modules has been extensively studied because the Kronecker algebra is a very important example of a tame hereditary algebra. Moreover, the category has also a geometric interpretation, since it is derived equivalent with the category $\mathrm{Coh}(\mathbb{P}^1(\kappa))$ of coherent sheaves on the projective line (see [3]).

Moreover, Kronecker modules correspond to matrix pencils in linear algebra, so the Kronecker algebra relates representation theory with numerical linear algebra and matrix theory. Recall that a *matrix pencil* over a field κ is a matrix $A + \lambda B$ where A, B are matrices over κ of the same size and λ is an indeterminate. This correspondence and the connection to an important open problem in the theory of matrix pencils (the matrix subpencil problem) is made clear in [4] and [10].

In this paper we continue the investigation of the short exact sequences of Kronecker modules, started in [11] and [10].

2. Extensions of Kronecker modules over arbitrary fields

For $d \in \mathbb{N}^2$ let $M_d = \{[M] | M \in \text{mod-}\kappa K, \underline{\dim} M = d\}$ be the set of isomorphism classes of Kronecker modules of dimension d. Following Reineke in [5] for subsets $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$ we define

$$\mathcal{A}*\mathcal{B} = \{[X] \in M_{d+e} \,|\, \exists\, 0 \to N \to X \to M \to 0 \text{ exact for some } [M] \in \mathcal{A}, [N] \in \mathcal{B}\}.$$

So the product $\mathcal{A} * \mathcal{B}$ is the set of isoclasses of all extensions of modules M with $[M] \in \mathcal{A}$ by modules N with $[N] \in \mathcal{B}$. This is in fact Reineke's extension monoid product using isomorphism classes of modules instead of modules. It is important to know (see [5]) that the product above is associative, i.e. for $\mathcal{A} \subset M_d$, $\mathcal{B} \subset M_e$, $\mathcal{C} \subset M_f$, we have $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$. Also $\{[0]\} * \mathcal{A} = \mathcal{A} * \{[0]\} = \mathcal{A}$. We will call the operation "$*$" simply the *extension monoid product*.

Remark 3. For $M, N \in \text{mod-}\kappa K$ and κ finite, the product $\{[M]\} * \{[N]\}$ coincides with the set $\{[M][N]\}$ of terms in the Ringel Hall product $[M][N]$ (see Section 4 from [11]).

The aim of this section is to present recent results on some products of the form $\{[M]\} * \{[N]\}$, i.e. the description of all extensions of N by M. It is important to note that by saying "an extension of N by M" we mean a module X, which is a middle term in $\mathrm{Ext}^1(M, N)$. We emphasize that all the results are valid over an arbitrary field κ.

Theorem 4. *We have the following rules for the monoid product of various Kronecker modules:*

(a) $\{[P]\} * \{[R]\} * \{[I]\} = \{[P \oplus R \oplus I]\}$, *where* $P, R, I \in$ mod-κK *are arbitrary preprojective, regular respectively preinjective modules;*

(b) $\{[R]\} * \{[R']\} = \{[R']\} * \{[R]\}$, *moreover this set contains only regulars (for* $R, R' \in$ mod-κK *arbitrary regulars),*

(c) $\{[I_i]\} * \{[I_j]\} =$
$$= \begin{cases} \{[I_i \oplus I_j]\}, & i-j \geq -1, \\ \{[I_j \oplus I_i], [I_{j-1} \oplus I_{i+1}], \dots, [I_{j-[\frac{j-i}{2}]} \oplus I_{i+[\frac{j-i}{2}]}]\}, & i-j < -1. \end{cases}$$

(d) $\{[P_i]\} * \{[P_j]\} =$
$$= \begin{cases} \{[P_i \oplus P_j]\}, & i-j \leq -1, \\ \{[P_j \oplus P_i], [P_{j+1} \oplus P_{i-1}], \dots, [P_{j+[\frac{i-j}{2}]} \oplus P_{i-[\frac{i-j}{2}]}]\}, & i-j > -1. \end{cases}$$

(e) $\{[I_{n-1-i}]\} * \{[P_i]\} = \mathcal{R}_n \cup \{[P_i \oplus I_{n-1-i}]\}$, *where*

$$\mathcal{R}_n = \{[R_{p_1}(t_1) \oplus \cdots \oplus R_{p_s}(t_s)] \mid s \in \mathbb{N}^*, p_i \neq p_j \text{ if } i \neq j, t_1 d_{p_1} + \cdots + t_s d_{p_s} = n\}.$$

(f) $\{[I_m]\} * \mathcal{R}_n = \mathcal{R}_n * \{[I_m]\} \cup \mathcal{R}_{n-1} * \{[I_{m+1}]\} \cup \cdots \cup \{[I_{m+n}]\}$.

(g) $\mathcal{R}_n * \{[P_m]\} = \{[P_m]\} * \mathcal{R}_n \cup \{[P_{m+1}]\} * \mathcal{R}_{n-1} \cup \cdots \cup \{[P_{m+n}]\}$.

The claims (a), (b), (c) and (d) are direct consequences of results presented in Section 4 of [9] (formulas for the corresponding Ringel Hall products) and the field independence of short exact sequences of preinjective and preprojective modules, as shown in Theorem 3.3 from [10]. The remaining claims are proved in [8].

Taking into account that middle terms in short exact sequences of preinjective and preprojective Kronecker modules do not depend on the base field κ, we can restate Theorem 8 from [11] in the following way:

Theorem 5. *If* $a_1 \geq \cdots a_p \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$ *and* $c_1 \geq \cdots \geq c_r \geq 0$ *are nonnegative integers, then* $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$ *if and only if* $r = n+p$, $\exists \beta \colon \{1, \dots, n\} \to \{1, \dots, n+p\}$, $\exists \alpha : \{1, \dots, p\} \to \{1, \dots, n+p\}$ *both functions strictly increasing with* $\mathrm{Im}\alpha \cap \mathrm{Im}\beta = \emptyset$ *and* $\exists m_j^i \geq 0$, $1 \leq i \leq n$, $1 \leq j \leq p$, *such that* $\forall \ell \in \{1, \dots, n+p\}$

$$c_\ell = \begin{cases} b_i - \sum_{\substack{\beta(i) < \alpha(j) \\ 1 \leq j \leq p}} m_j^i, & \text{where } i = \beta^{-1}(\ell),\ \ell \in \mathrm{Im}\beta, \\ a_j + \sum_{\substack{\beta(i) < \alpha(j) \\ 1 \leq i \leq n}} m_j^i, & \text{where } j = \alpha^{-1}(\ell),\ \ell \in \mathrm{Im}\alpha. \end{cases} \tag{2.1}$$

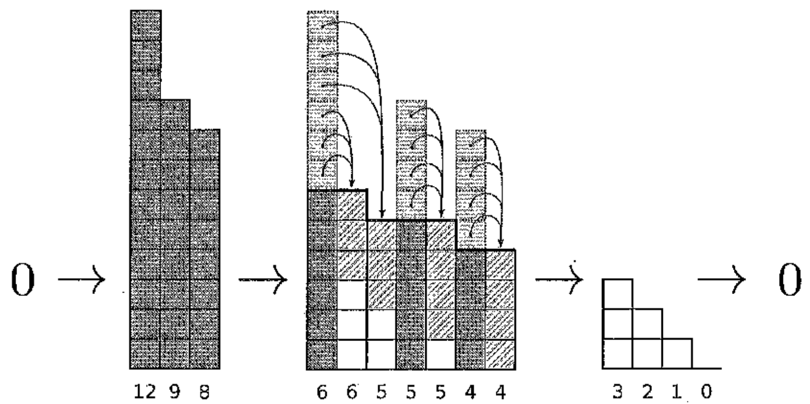
Example 6. In mod-κK we have that

$$[I_6 \oplus I_6 \oplus I_5 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4] \in \{[I_3 \oplus I_2 \oplus I_1 \oplus I_0]\} * \{[I_{12} \oplus I_9 \oplus I_8]\}.$$

Using the notations from Theorem 5, we have $p = 4$, $n = 3$, $r = 7$ and the two strictly increasing functions are $\beta : \{1, 2, 3\} \to \{1, \ldots, 7\}$ with $\beta(1) = 1$, $\beta(2) = 4$, $\beta(3) = 6$ and $\alpha : \{1, \ldots, 4\} \to \{1, \ldots, 7\}$ with $\alpha(1) = 2$, $\alpha(2) = 3$, $\alpha(3) = 5$, $\alpha(4) = 7$. For the values m^i_j, $1 \le i \le n$, $1 \le j \le p$, we have $m^1_1 = m^1_2 = 3$, $m^2_3 = m^3_4 = 4$ and $m^i_j = 0$ in all other cases. Hence there exists a short exact sequence

$$0 \to I_{12} \oplus I_9 \oplus I_8 \to I_6 \oplus I_6 \oplus I_5 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4 \to I_3 \oplus I_2 \oplus I_1 \oplus I_0 \to 0,$$

illustrated as follows:



So, less formally, Theorem 5 claims that $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_p}]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$ if and only if the sequence $c_1 \ge \cdots \ge c_r \ge 0$ is obtained by merging the sequences $a_1 \ge \cdots \ge a_p \ge 0$ and $b_1 \ge \cdots \ge b_n \ge 0$ and by applying the "box dropping rule" illustrated in the picture above. This rule says that in the middle term boxes can be dropped only to the right and only from columns corresponding to elements of the sequence $b_1 \ge \cdots \ge b_n \ge 0$ on top of columns corresponding to elements of the sequence $a_1 \ge \cdots \ge a_p \ge 0$. The values m^i_j from the theorem denote the number of boxes dropped from the column corresponding to the element b_i on top of the column corresponding to the element a_j.

As immediate consequences of Theorem 5, we get the following two corollaries:

Corollary 7. *Let $I, I', I'' \in$ mod-κK be preinjective Kronecker modules, where $I = I_{c_1} \oplus \cdots \oplus I_{c_r}$, $I' = I_{a_1} \oplus \cdots \oplus I_{a_p}$ and $I'' = I_{b_1} \oplus \cdots \oplus I_{b_n}$. Then there is a short exact sequence*

$$0 \to I_{b_1} \oplus \cdots \oplus I_{b_n} \to I_{c_1} \oplus \cdots \oplus I_{c_r} \to I_{a_1} \oplus \cdots \oplus I_{a_p} \to 0$$

if and only if there is a short exact sequence

$$0 \to I_{b_1+m} \oplus \cdots \oplus I_{b_n+m} \to I_{c_1+m} \oplus \cdots \oplus I_{c_r+m} \to I_{a_1+m} \oplus \cdots \oplus I_{a_p+m} \to 0$$

for some $m \in \mathbb{N}$.

follow

Proof. Results immediately from (2.1) in Theorem 5. ■

Corollary 8. *For $b_1 \geq \cdots \geq b_n \geq 0$, $c_1 \geq \cdots \geq c_p \geq 0$ and $a \geq 0$ nonnegative integers, we have that*

$$[I_{c_1} \oplus \cdots \oplus I_{c_p}] \in \{[I_a]\} * \{[I_{b_1} \oplus \cdots \oplus I_{b_n}]\}$$

if and only if $p = n+1$, $c_1 = b_1 - m_1$, ..., $c_{l-1} = b_{l-1} - m_{l-1}$, $c_l = a + \sum_{i=1}^{l-1} m_i$, $c_{l+1} = b_l$, ..., $c_{n+1} = b_n$ for some $l \in \{1, \ldots, n+1\}$ and $m_i \geq 0$, $i = \overline{l, n}$.

Proof. Just take $\beta\colon \{1, \ldots, n\} \to \{1, \ldots, n+1\}$, $\beta(i) = \begin{cases} i, & i < l, \\ i+1, & i \geq l, \end{cases}$ and $\alpha\colon \{1\} \to \{1, \ldots, n+1\}$, $\alpha(1) = l$ for the functions and $m_1^i = \begin{cases} m_i, & i < l, \\ 0, & i \geq l, \end{cases}$ for the nonnegative integer values from Theorem 5. ■

We will also need Lemma 5 from [11] rewritten using the extension monoid product:

Lemma 9. *Let $a_1 \geq \cdots \geq a_n$ be a decreasing and $(b_1, \ldots, b_n)$ an arbitrary sequence of nonnegative integers such that $(b_1, \ldots, b_n) \geq (a_1, \ldots, a_n)$. If $[I_{c_1} \oplus \cdots \oplus I_{c_n}] \in \{[I_{b_1}]\} * \cdots * \{[I_{b_n}]\}$ then $(c_1, \ldots, c_n) \geq (a_1, \ldots, a_n)$.*

Remark 10. For two arbitrary sequences $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ we say that $(b_1,\dots,b_n) \geq (a_1,\dots,a_n)$ if $b_i \geq a_i$ for $i=\overline{1,n}$.

3. The extension monoid product of a preinjective and a preprojective Kronecker module

In the sequel we prove that the preinjective elements of the extension monoid product of a preprojective and a preinjective Kronecker module can be described by the extension monoid product of two preinjectives. Or, equivalently, we show that the existence of short exact sequences of the form $0 \to P \to I \to I' \to 0$ depends on the existence of certain short exact sequences of the form $0 \to \overline{I} \to \overline{I}' \to I'' \to 0$, where P is preprojective and $I, I', I'', \overline{I}, \overline{I}'$ are preinjective Kronecker modules.

Lemma 11. *Let $d_1 \geq \dots \geq d_q \geq 0$ and $c_1 \geq \dots \geq c_r \geq 0$ be nonnegative integers. Then $[I_{c_1} \oplus \dots \oplus I_{c_r}] \in \{[I_{d_1} \oplus \dots \oplus I_{d_q}]\} * \{[P_n]\}$ if and only if $r = q-1$ and we have $c_1 = d_1 + m_1, \dots, c_l = d_l + m_l$, $c_{l+1} = d_{l+2}, \dots, c_{q-1} = d_q$ for some $l \in \{1,\dots,q-1\}$ with $m_i \geq 0$, $i = \overline{1,l}$ and $\sum_{i=1}^{l} m_i = d_{l+1} + n + 1$.*

Proof. By applying first the rule (c), and repeatedly rule (e) then (f) from Theorem 4 we can write the following sequence of inclusions:

$$
\begin{aligned}
&\{[I_{d_1} \oplus \dots \oplus I_{d_q}]\} * \{[P_n]\} \\
&\overset{(c)}{=} \{[I_{d_1}]\} * \dots * \{[I_{d_q}]\} * \{[P_n]\} \\
&\overset{(e)}{\supseteq} \{[I_{d_1}]\} * \dots * \{[I_{d_{q-1}}]\} * \{[P_n]\} * \{[I_{d_q}]\} \\
&\vdots \\
&\overset{(e)}{\supseteq} \{[I_{d_1}]\} * \dots * \{[I_{d_l}]\} * \Big(\{[I_{d_{l+1}}]\} * \{[P_n]\}\Big) * \{[I_{d_{l+2}}]\} * \dots * \{[I_{d_q}]\} \\
&\overset{(e)}{\supseteq} \{[I_{d_1}]\} * \dots * \Big(\{[I_{d_l}]\} * \mathcal{R}_{d_{l+1}+n+1}\Big) * \{[I_{d_{l+2}}]\} * \dots * \{[I_{d_q}]\} \\
&\overset{(f)}{\supseteq} \{[I_{d_1}]\} * \dots \Big(\{[I_{d_{l-1}}]\} * \mathcal{R}_{d_{l+1}+n+1-m_l'}\Big) * \{[I_{d_l+m_l'}]\} * \{[I_{d_{l+2}}]\} * \dots * \{[I_{d_q}]\} \\
&\vdots \\
&\overset{(f)}{\supseteq} \{[I_{d_1+m_1'}]\} * \dots * \{[I_{d_l+m_l'}]\} * \{[I_{d_{l+2}}]\} * \dots * \{[I_{d_q}]\}
\end{aligned}
$$

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$$= \{[I_{c'_1}]\} * \cdots * \{[I_{c'_l}]\} * \{[I_{c'_{l+1}}]\} * \cdots * \{[I_{c'_{q-1}}]\}$$
$$\overset{(c)}{=} \{[I_{c'_1}]\} * \cdots * \{[I_{c'_l}]\} * \{[I_{c'_{l+1}} \oplus \cdots \oplus I_{c'_{q-1}}]\},$$

where $c'_1 = d_1 + m'_1, \ldots, c'_l = d_l + m'_l$ and $c'_{l+1} = d_{l+2}, \ldots, c'_{q-1} = d_q$ for some $l \in \{1, \ldots, q-1\}$ with $m'_i \geq 0$, $i = \overline{1,l}$ and $\sum_{i=1}^{l} m'_i = d_{l+1}+n+1$. By examining the rules for the extension monoid product described in Theorem 4 we can conclude that $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_n]\}$ if and only if $r = q-1$ and $[I_{c_1} \oplus \cdots \oplus I_{c_r}] \in \{[I_{c'_1}]\} * \cdots * \{[I_{c'_l}]\} * \{[I_{c'_{l+1}} \oplus \cdots \oplus I_{c'_{q-1}}]\}$ for some $c'_1, \ldots, c'_{q-1}$ defined as before (any other choice for applying the rules would lead to a set of isoclasses of Kronecker modules that are not preinjective). But since $(c'_1, c'_2, \ldots, c'_l) \geq (d_1, d_2, \ldots, d_l)$, by using Lemma 9 we get that any $[I_{c_1} \oplus \cdots \oplus I_{c_l}] \in \{[I_{c'_1}]\} * \cdots * \{[I_{c'_l}]\}$ with $c_1 \geq \cdots \geq c_l$ can be written in the form $c_1 = d_1 + m_1, \ldots, c_l = d_l + m_l$ and since $c_l \geq d_{l+2}$, the statement of the lemma follows. ∎

Lemma 12. *Let $d_1 \geq \cdots \geq d_q \geq 0$ and $c_1 \geq \cdots \geq c_{q-1} \geq 0$ be nonnegative integers. Then $[I_{c_1} \oplus \cdots \oplus I_{c_{q-1}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_n]\}$ if and only if $[I_{d_1+n+1} \oplus \cdots \oplus I_{d_q+n+1}] \in \{[I_0]\} * \{[I_{c_1+n+1} \oplus \cdots \oplus I_{c_{q-1}+n+1}]\}$.*

Proof. ? Follows easily from Corollary 8 and Lemma 11. ∎

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In what follows we are going to need two lemmas from [10]:

Lemma 13. ([10]). *Let $N_1, N_2, M_1, M_2 \in$ mod-κK be Kronecker modules (where κ is an arbitrary field) such that $\mathrm{Ext}^1(N_1, N_2) = 0$ and $\mathrm{Hom}(N_2, M_1) = 0$. Then there exists an exact sequence of the form*

$$0 \to N_1 \oplus N_2 \to M_1 \oplus M_2 \to Y \to 0$$

if and only if there is a module X with exact sequences

$$0 \to N_2 \to M_2 \to X \to 0,$$
$$0 \to N_1 \to M_1 \oplus X \to Y \to 0.$$

Dually, we have:

Lemma 14. *Let N_1, N_2, M_1, M_2 be finite dimensional right modules over the Kronecker algebra κK (where κ is a field) such that* $\mathrm{Ext}^1(N_1, N_2) = 0$ *and* $\mathrm{Hom}(M_2, N_1) = 0$. *Then there exists an exact sequence of the form*

$$0 \to Y \to M_1 \oplus M_2 \to N_1 \oplus N_2 \to 0$$

if and only if there is a module X with exact sequences

$$0 \to X \to M_1 \to N_1 \to 0$$
$$0 \to Y \to X \oplus M_2 \to N_2 \to 0.$$

We are now ready to prove our main theorem, which gives a characterization (via Theorem 5) of the short exact sequences of the form $0 \to P \to I' \to I \to 0$.

We will use the following notation:

- for $I = I_{c_1} \oplus \cdots \oplus I_{c_n} \in$ mod-κK preinjective and $d \in \mathbb{N}$, $I^{(+d)} = I_{c_1+d} \oplus \cdots \oplus I_{c_n+d}$;
- for $P = P_{a_1} \oplus \cdots \oplus P_{a_n} \in$ mod-κK preprojective and $d \in \mathbb{N}$, $P^{(+d)} = P_{a_1+d} \oplus \cdots \oplus P_{a_n+d}$.

Theorem 15. *Let $q > n > 0$, $d_1 \geq \cdots \geq d_q \geq 0$, $c_1 \geq \cdots \geq c_{q-n} \geq 0$ and $0 \leq a_1 \leq \cdots \leq a_n$ be nonnegative integers, $I = I_{c_1} \oplus \cdots \oplus I_{c_{q-n}}$ and $I' = I_{d_1} \oplus \cdots \oplus I_{d_q}$. Then $[I_{c_1} \oplus \cdots \oplus I_{c_{q-n}}] \in \{[I_{d_1} \oplus \cdots \oplus I_{d_q}]\} * \{[P_{a_1} \oplus \cdots \oplus P_{a_n}]\}$ if and only if $[I_{d_1+a_n+1} \oplus \cdots \oplus I_{d_q+a_n+1}] \in \{[I_{a_n-a_1} \oplus \cdots \oplus I_{a_n-a_{n-1}} \oplus I_0]\} * \{[I_{c_1+a_n+1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1}]\}$, or equivalently there is a short exact sequence*

$$0 \to P_{a_1} \oplus \cdots \oplus P_{a_n} \to I \to I' \to 0$$

if and only if there is a short exact sequence

$$0 \to I^{(+a_n+1)} \to I'^{(+a_n+1)} \to I_{a_n-a_1} \oplus \cdots \oplus I_{a_n-a_{n-1}} \oplus I_0 \to 0.$$

Proof. We use induction on n. For $n = 1$ the theorem is true by Lemma 12. Let $n > 1$ and suppose the theorem holds for $n - 1$. Due to Lemma 13 we know that there is a short exact sequence

$$0 \to P_{a_1} \oplus \cdots \oplus P_{a_n} \to I_{c_1} \oplus \cdots \oplus I_{c_{q-n}} \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0$$

if and only if we have the short exact sequences

$$0 \to P_{a_1} \to X \to I_{d_1} \oplus \cdots \oplus I_{d_q} \to 0 \tag{3.1}$$

and

$$0 \to P_{a_2} \oplus \cdots \oplus P_{a_n} \to I_{c_1} \oplus \cdots \oplus I_{c_{q-n}} \to X \to 0, \tag{3.2}$$

where $X = I_{e_1} \oplus \cdots \oplus I_{e_{q-1}}$ in our case, because a preinjective module project onto X and by Lemma 2.

On one hand by the induction hypothesis, the existence of the short exact sequence (3.1) is equivalent with the existence of the short exact sequence

$$0 \to I_{e_1+a_1+1} \oplus \cdots \oplus I_{e_{q-1}+a_1+1} \to I_{d_1+a_1+1} \oplus \cdots \oplus I_{d_q+a_1+1} \to I_0 \to 0,$$

but in turn by Corollary 7 this is equivalent with the existence of the short exact sequence

$$0 \to I_{e_1+a_n+1} \oplus \cdots \oplus I_{e_{q-1}+a_n+1} \to I_{d_1+a_n+1} \oplus \cdots \oplus I_{d_q+a_n+1} \to I_{a_n-a_1} \to 0.$$

On the other hand, again by the induction hypothesis, the existence of the short exact sequence (3.2) is equivalent with the existence of the short exact sequence

$$\begin{aligned} 0 \to I_{c_1+a_n+1} \oplus \cdots \oplus I_{c_{q-n}+a_n+1} \to I_{e_1+a_n+1} \oplus \cdots \\ \cdots \oplus I_{e_q+a_n+1} \to I_{a_n-a_2} \oplus \cdots \oplus I_{a_n-a_{n-1}} \oplus I_0 \to 0. \end{aligned}$$

Using Lemma 14, we are done. ∎

Remark 16. If we are looking to the derived category of the initial Kronecker module category, then one can see that the preprojectives are in fact the shifted versions of the preinjectives. In this sense the short exact sequence $0 \to P_{a_1} \oplus \cdots \oplus P_{a_n} \to I \to I' \to 0$ may be regarded as a "shifted version" of the short exact sequence $0 \to I^{(+a_n+1)} \to I'^{(+a_n+1)} \to I_{a_n-a_1} \oplus \cdots \oplus I_{a_n-a_{n-1}} \oplus I_0 \to 0$.

Example 17. We have a short exact sequence

$$0 \to P_0 \oplus P_1 \oplus P_2 \oplus P_3 \to I_8 \oplus I_5 \oplus I_4 \to I_2 \oplus I_2 \oplus I_1 \oplus I_1 \oplus I_1 \oplus I_0 \oplus I_0 \to 0$$

because there exists a short exact sequence

$$0 \to I_{12} \oplus I_9 \oplus I_8 \to I_6 \oplus I_6 \oplus I_5 \oplus I_5 \oplus I_5 \oplus I_4 \oplus I_4 \to I_3 \oplus I_2 \oplus I_1 \oplus I_0 \to 0,$$

as shown in Example 6.

Remark 18. We have developed in [12] methods for checking (in linear time) the existence of a preinjective only short exact sequence $0 \to I \to I' \to I'' \to 0$ and to efficiently generate all middle terms, factors or kernels if two of the preinjectives I, I', I'' are given. So these results will immediately apply to preinjective-preprojective mixed short exact sequences as well.

As it can be seen from the introductory part, preinjective modules are categorical duals of preprojectives, so Theorem 15 can be stated dually, in the following way:

Theorem 19. *Let $q > n > 0$, $0 \le d_1 \le \cdots \le d_q$, $0 \le c_1 \le \cdots \le c_{q-n}$ and $a_1 \ge \cdots \ge a_n \ge 0$ be nonnegative integers, $P = P_{c_1} \oplus \cdots \oplus P_{c_{q-n}}$ and $P' = P_{d_1} \oplus \cdots \oplus P_{d_q}$. Then $[P_{c_1} \oplus \cdots \oplus P_{c_{q-n}}] \in \{[I_{a_1} \oplus \cdots \oplus I_{a_n}]\} * \{[P_{d_1} \oplus \cdots \oplus P_{d_q}]\}$ if and only if $[P_{d_1+a_n+1} \oplus \cdots \oplus P_{d_q+a_n+1}] \in \{[P_{a_n-a_1} \oplus \cdots \oplus P_{a_n-a_{n-1}} \oplus P_0]\} * \{[P_{c_1+a_n+1} \oplus \cdots \oplus P_{c_{q-n}+a_n+1}]\}$, or equivalently there is a short exact sequence*

$$0 \to P' \to P \to I_{a_1} \oplus \cdots \oplus I_{a_n} \to 0$$

if and only if there is a short exact sequence

$$0 \to P_0 \oplus P_{a_1-a_2} \oplus \cdots \oplus P_{a_1-a_n} \to P'^{(+a_1+1)} \to P^{(+a_1+1)} \to 0.$$

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