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Baer-Galois connections and applications

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ABSTRACT. We define Baer-Galois connections between bounded modular lattices. We relate them to lifting lattices and we show that they unify the theories of (relatively) Baer and dual Baer modules.

1. INTRODUCTION

Galois connections have been useful tools for transferring properties between partially ordered sets and have found applications in various fields of mathematics: classical Galois theory, group theory, algebraic topology etc. In particular, their theory has been successfully applied to some classical module-theoretic Galois connections in order to relate properties of a module with properties of its endomorphism ring. In a long series of articles on these topics we mention the recent papers [4, 5], which developped a theory of some special Galois connections that offer a more efficient and transparent way to deal with the above problem from module theory. This paper has a similar motivation for introducing Baer-Galois connections. They are showed to be a suitable setting for establishing lattice-theoretic analogues of some module-theoretic results connected to the theories of Baer and dual Baer modules. These have been introduced by Rizvi and Roman [12] and Keskin Tütüncü and Tribak [11] respectively, and are important concepts related to the intensively studied extending and lifting modules [3, 8].

In this article we prove that Baer-Galois connections are related to lifting lattices. The proofs of our results are different and much easier as those for modules, using only properties of a lattice-theoretic nature. Our general theory has a wide range of applications, and we illustrate its strength on two particular Galois connections between submodule lattices, observed by Albu and Năstăsescu [1, pp. 25-26]. Then Baer-Galois connections reduce to (relatively) Baer and dual Baer modules, and one easily obtains some corresponding results from module theory.

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2. BAER-GALOIS CONNECTIONS

Throughout the paper all lattices will be modular and bounded. Then they have a least element, denoted by 0, and a greatest element, denoted by 1, and we assume that $0 \neq 1$. For elements a, a' of the lattice $(A, \leq)$, we denote $[a, a'] = \{x \in A | a \leq x \leq a'\}$.

We recall the definition of (monotone) Galois connection (e.g., see [9]), which in fact holds more generally for partially ordered sets.

Definition 2.1. Let $(A, \leq)$ and $(B, \leq)$ be lattices. A *Galois connection* between them consists of a pair (α, β) of two order-preserving functions $\alpha : A \to B$ and $\beta : B \to A$ such that for all $a \in A$ and $b \in B$, we have $\alpha(a) \leq b \iff a \leq \beta(b)$. Equivalently, (α, β) is a Galois connection if and only if for all $a \in A$, $a \leq \beta\alpha(a)$ and for all $b \in B$, $\alpha\beta(b) \leq b$.

The following results on Galois connections are well-known (e.g., see [1, Proposition 3.3], [9]), and will be freely used.

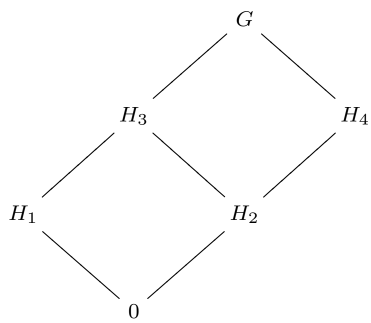
Lemma 2.1. *Let (α, β) a Galois connection between lattices A and B. Then $\alpha\beta\alpha = \alpha$, $\beta\alpha\beta = \beta$, α preserves all suprema in A, β preserves all infima in B, $\alpha(0) = 0$ and $\beta(1) = 1$.*

Now we introduce the main notion of the paper.

Definition 2.2. Let (α, β) be a Galois connection between lattices A and B. We say that (α, β) is *Baer* (or *Baer-Galois*) if $\alpha(a)$ is a complement in B for every $a \in A$, and $\beta(b)$ is a complement in A for every $b \in B$.

In general, for a Galois connection (α, β), the two conditions from Definition 2.2 are independent, as we may see in the following example. One may see the algorithmic approach from [6, 7] to have a clearer insight.

Example 2.1. Consider the abelian group $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for some primes p and q with $p \neq q$, where $\mathbb{Z}_n$ denotes the cyclic group of order $n \in \mathbb{N}$. The subgroup lattice $L(G)$ of G is given by a diagram of the following form



where $H_1 \simeq \mathbb{Z}_q$, $H_2 \simeq \mathbb{Z}_p$, $H_3 \simeq \mathbb{Z}_{pq}$ and $H_4 \simeq \mathbb{Z}_{p^2}$.

Consider the functions $\alpha : L(G) \to L(G)$ defined by $\alpha(0) = 0$, $\alpha(H_1) = H_1$, $\alpha(H_2) = 0$, $\alpha(H_3) = H_1$, $\alpha(H_4) = H_4$, $\alpha(G) = G$, and $\beta : L(G) \to L(G)$

defined by $\beta(0) = H_2$, $\beta(H_1) = H_3$, $\beta(H_2) = H_2$, $\beta(H_3) = H_3$, $\beta(H_4) = H_4$, $\beta(G) = G$. Then (α, β) is a Galois connection from the lattice $(L(G), \subseteq)$ to itself. Then $\alpha(a)$ is a complement in $B = L(G)$ for every $a \in A = L(G)$, but $\beta(0) = H_2$ is not a complement in $A = L(G)$.

In certain cases we may simplify the definition of a Baer-Galois connection. Inspired by the behaviour of additive functors in additive categories, we say that $\beta : B \to A$ is *additive* if $\beta(b \vee b') = \beta(b) \vee \beta(b')$ for every $b, b' \in B$ with $b \wedge b' = 0$. Note that if β is additive, then β preserves complements. Dually, one has the terminology that $\alpha : A \to B$ is *coadditive*.

Lemma 2.2. *Let (α, β) be a Galois connection between lattices A and B such that $\alpha : A \to B$ is coadditive, $\beta : B \to A$ is additive, $\alpha(1) = 1$ and $\beta(0) = 0$. Then the following are equivalent*
(i) (α, β) is Baer.
(ii) $\alpha(a)$ is a complement in B for every $a \in A$.
(iii) $\beta(b)$ is a complement in A for every $b \in B$.

Proof. It is enough to prove the equivalence (ii)⇔(iii). Assume (ii). If $b \in B$, then $\alpha\beta(b)$ is a complement in B, and so, $\beta(b) = \beta\alpha\beta(b)$ is a complement in A by the additivity of β and (iii) holds. The converse is dual. □

The following concepts will be useful, which generalize some corresponding module-theoretic notions (e.g., see [3, 8, 10]).

Definition 2.3. Let B be a lattice.
(1) Let $b, b' \in B$ be such that $b' \leq b$. Then b is called *coessential* (or *cosmall*) in $[b', 1]$ if for any $x \in B$, $1 = b \vee x$ implies $1 = b' \vee x$. Also, b is called *superfluous* in B if b is coessential in $[0, 1]$.
(2) B is called *lifting* if for every $b \in B$, there exists a complement $b' \in B$ such that b is coessential in $[b', 1]$.
Let (α, β) be a Galois connection between lattices A and B. Then B is called
(3) α*-nonsingular* if for $a \in A$, $\alpha(a)$ coessential in $[0, 1]$ implies $a = 0$.
(4) β*-cononsingular* if for $b \in B$, $\beta(b) = 0$ implies b coessential in $[0, 1]$.

Considering the concepts of coessential element and lifting lattice for the dual lattice B^{op} of a lattice B, one obtains the notions of *essential* element and *extending* lattice respectively.

Now we may give our main result, which relates Baer-Galois connections with the lifting property for lattices.

Theorem 2.1. *Let (α, β) be a Galois connection between lattices A and B, where $\alpha : A \to B$ is coadditive, $\beta : B \to A$ is additive, $\alpha(1) = 1$ and $\beta(0) = 0$. The following are equivalent*
(i) (α, β) is Baer and B is β-cononsingular.
(ii) B is α-nonsingular lifting.

Proof. $(i) \Rightarrow (ii)$ Assume that (α, β) is Baer and B is β-cononsingular.

Let $a \in A$ be such that $\alpha(a)$ is coessential in $[0, 1]$. Since $\alpha(a)$ is a complement in B, we must have $\alpha(a) = 0$. Then $a \leq \beta\alpha(a) = \beta(0) = 0$, hence $a = 0$. Thus B is α-nonsingular.

Let $b \in B$. We claim that b is coessential in $[\alpha\beta(b), 1]$. To this end, let $b_0 \in B$ be such that $b \vee b_0 = 1$. Since $\alpha\beta(b)$ is a complement in B, there is $b' \in B$ such that $\alpha\beta(b) \vee b' = 1$ and $\alpha\beta(b) \wedge b' = 0$. Then we have $\beta(b \wedge b') = \beta(b) \wedge \beta(b') = \beta(\alpha\beta(b) \wedge b') = \beta(0) = 0$. But B is β-cononsingular, whence $b \wedge b'$ is coessential in $[0, 1]$. Since $\alpha\beta(b) \leq b$, it follows by modularity that

$$1 = b \vee b_0 = (b \wedge (\alpha\beta(b) \vee b')) \vee b_0 = \alpha\beta(b) \vee (b \wedge b') \vee b_0.$$

Then we must have $\alpha\beta(b) \vee b_0 = 1$, which shows that b is coessential in $[\alpha\beta(b), 1]$. Hence B is lifting.

$(ii) \Rightarrow (i)$ Assume that B is α-nonsingular lifting.

Let $b \in B$. Then there is a complement $b' \in B$ such that b is coessential in $[b', 1]$. Hence there is $b'' \in B$ such that $b' \vee b'' = 1$ and $b' \wedge b'' = 0$. Then it follows that $b \wedge b''$ is coessential in $[b' \wedge b'', 1] = [0, 1]$, that is, $b \wedge b''$ is superfluous in B. Since $\alpha\beta(b \wedge b'') \leq b \wedge b''$, $\alpha(\beta(b) \wedge \beta(b''))$ is superfluous in B. Since B is α-nonsingular, we have $\beta(b) \wedge \beta(b'') = 0$. By modularity, it follows that

$$\beta(b) = \beta(b) \wedge (\beta(b') \vee \beta(b'')) = \beta(b') \vee (\beta(b) \wedge \beta(b'')) = \beta(b').$$

Hence $\beta(b)$ is a complement in A. Then (α, β) is Baer by Lemma 2.2.

Let $b \in B$ be such that $\beta(b) = 0$. Then there is a complement $b' \in B$ such that b is coessential in $[b', 1]$. Hence there is $b'' \in B$ such that $b' \vee b'' = 1$ and $b' \wedge b'' = 0$. Since β is additive, $\beta(b') \vee \beta(b'') = \beta(1) = 1$. But $\beta(b') \leq \beta(b) = 0$, hence $\beta(b'') = 1$. Then $1 = \alpha(1) = \alpha\beta(b'') \leq b''$, and so $b'' = 1$. Now $b' = 0$, hence b is coessential in $[0, 1]$. Thus B is β-cononsingular. $\square$

Remark 2.1. Note that the coadditivity of $\alpha : A \to B$ in Theorem 2.1 is only used to ensure the equivalence of the two conditions from the definition of a Baer-Galois connection (α, β).

3. Applications

Now we recall some relevant Galois connections between submodule lattices, previously pointed out in the literature (e.g., see [1]).

Let R be an associative ring with (non-zero) identity. Let M and N be two right R-modules, and denote $U = \mathrm{Hom}_R(M, N)$, $S = \mathrm{End}_R(M)$ and $T = \mathrm{End}_R(N)$. Then ${}_TU_S$ is a bimodule.

For every submodule X of M_R and every submodule Z of ${}_TU$ denote

$$l_U(X) = \{f \in U \mid X \subseteq \mathrm{Ker}(f)\}, \quad r_M(Z) = \bigcap_{f \in Z} \mathrm{Ker}(f).$$

For every submodule Y of N_R and every submodule Z of U_S denote

$$l'_U(Y) = \{f \in U \mid \mathrm{Im}(f) \subseteq Y\}, \quad r'_N(Z) = \sum_{f \in Z} \mathrm{Im}(f).$$

Theorem 3.2. [1, Proposition 3.4] *(r_M, l_U) is a Galois connection between the submodule lattices $L({}_TU)$ and $L(M_R)^{\mathrm{op}}$, and (r'_N, l'_U) is a Galois connection between $L(U_S)$ and $L(N_R)$.*

The notions of Baer module [12, Definition 2.2] and dual Baer module [11, p. 262] may be generalized as follows.

Definition 3.4. Let M and N be two right R-modules. Then
(1) M is called *N-Baer* if the Galois connection (r_M, l_U) is Baer.
(2) N is called *M-dual Baer* if the Galois connection (r'_N, l'_U) is Baer.

The notions of Baer module and dual Baer module are related to that of Baer-Galois connection in the following way.

Theorem 3.3. *Let M be a right R-module with $S = \mathrm{End}_R(M)$. Then*
(i) M is Baer if and only if M is M-Baer if and only if the Galois connection (r_M, l_S) is Baer.
(ii) M is dual Baer if and only if M is M-dual Baer if and only if the Galois connection (r'_M, l'_S) is Baer.

Proof. (i) Note that M is Baer if and only if $l_S(X)$ is a direct summand of ${}_SS$ for every submodule X of M_R if and only if $r_M(Z)$ is a direct summand of M_R for every submodule Z of ${}_SS$ [12, Definition 2.2].

(ii) Note that M is dual Baer if and only if $l'_S(Y)$ is a direct summand of S_S for every submodule Y of M_R if and only if $r'_M(Z)$ is a direct summand of M_R for every submodule Z of S_S [11, p. 262]. □

For the above module-theoretic Galois connections we have the following two corollaries, which are [12, Theorem 2.12] and [11, Theorem 2.14].

Corollary 3.1. *Let M be a right R-module. The following are equivalent*
(i) M is a Baer $\mathcal{K}$-cononsingular module.
(ii) M is a $\mathcal{K}$-nonsingular extending module.

Proof. Consider the Galois connection (r_M, l_S) between the submodule lattices $L({}_SS)$ and $L(M_R)^{\mathrm{op}}$. Then l_S is additive by [2, Lemma 4.9]. The module M_R is r_M-nonsingular if for every submodule Z of ${}_SS$, $r'_M(Z)$ essential in M implies $Z = 0$. Hence M_R is r_M-nonsingular if and only if M_R is $\mathcal{K}$-nonsingular in the sense of [12, Definition 2.5]. The module M_R is l_S-cononsingular if for every submodule X of M_R, $l_S(X) = 0$ implies X essential in M. Hence M_R is l_S-cononsingular if and only if M_R is $\mathcal{K}$-cononsingular in the sense of [12, Definition 2.7]. The lattice $L(M_R)^{\mathrm{op}}$ is lifting if and only if M_R is extending. Now use Theorem 2.1 for the Galois connection (r_M, l_S), together with Remark 2.1 and Theorem 3.3. □

Corollary 3.2. *Let M be a right R-module. The following are equivalent*
(i) M is a dual Baer $\mathcal{K}$-module.
(ii) M is a $\mathcal{T}$-non-cosingular lifting module.

Proof. Consider the Galois connection (r'_M, l'_S) between the submodule lattices $L(S_S)$ and $L(M_R)$. Then l'_S is clearly additive. The module M_R is r'_M-nonsingular if for every submodule Z of S_S, $r'_M(Z)$ coessential in M implies $Z = 0$. Hence M_R is r'_M-nonsingular if and only if M_R is $\mathcal{T}$-non-cosingular in the sense of [11, p. 261]. The module M_R is l'_S-cononsingular if for every submodule Y of M_R, $l'_S(Y) = 0$ implies Y coessential in M. Hence M_R is l'_S-cononsingular if and only if M_R is a $\mathcal{K}$-module in the sense of [11, p. 264]. The lattice $L(M_R)$ is lifting if and only if M_R is lifting. Now use Theorem 2.1 for the Galois connection (r'_M, l'_S), together with Remark 2.1 and Theorem 3.3. □

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