Seminar 1

1. Let A and B be sets with n elements ($n \in \mathbb{N}^*$) and let $f : A \to B$. Show that the following are equivalent:
(i) f is injective;
(ii) f is surjective;
(iii) f is bijective.

2. Let $f : A \to B$. Show that f is injective $\Longleftrightarrow f(X \cap Y) = f(X) \cap f(Y)$, $\forall X, Y \subseteq A$.

3. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations (composition laws) on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

4. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

5. Let " $*$ " be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

6. Let " $*$ " be the operation defined on $\mathbb{N}$ by $x * y = g.c.d.(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation " $*$ " on D_6.

7. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

8. Let " $\cdot$ " be an operation on a set A and let $X, Y \subseteq A$. Denote by $\mathcal{P}(A)$ the *power set* of A, that is, the set of all subsets of A. Define an operation " $\cdot$ " on $\mathcal{P}(A)$ by

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
(i) If " $\cdot$ " on A is commutative, then " $\cdot$ " on $\mathcal{P}(A)$ is commutative.
(ii) If " $\cdot$ " on A is associative, then " $\cdot$ " on $\mathcal{P}(A)$ is associative.
(iii) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), \cdot)$ is a monoid.
(iv) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), \cdot)$ is not a group (for $|A| \geq 2$).

9. Let $(G, \cdot)$ and $(G', \cdot)$ be two groups with identity elements e and e' respectively. Define on $G \times G'$ the operation " $\cdot$ " by

$$(g_1, g_1') \cdot (g_2, g_2') = (g_1 \cdot g_2, g_1' \cdot g_2'), \quad \forall (g_1, g_1'), (g_2, g_2') \in G \times G'.$$

Prove that $(G \times G', \cdot)$ is a group, called the *direct product* of the groups G and G'.

10. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2 y^2$.
(ii) If $\forall x \in G$, $x^2 = 1$, then G is abelian.

Seminar 2

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\, r\, y \Longleftrightarrow x < y$$

$$x\, s\, y \Longleftrightarrow x|y$$

$$x\, t\, y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x\, v\, y \Longleftrightarrow x \equiv y \pmod 3 .$$

Write the graphs R, S, T, V of the given relations.

2. Let A and B be sets with n and m elements respectively $(m, n \in \mathbb{N}^*)$. Determine the number of:
(i) relations having the domain A and the codomain B;
(ii) homogeneous relations on A.

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\, r\, z_2 \Longleftrightarrow |z_1| = |z_2|\,; \qquad z_1\, s\, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x\, \rho_n\, y \Longleftrightarrow n|(x - y)\,.$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

Seminar 3

1. Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

2. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\} \quad (n \in \mathbb{N}^*)$ be the *set of n-th roots of unity.* Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

3. Let K be one of the sets $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(i) $GL_n(K) = \{A \in M_n(K) \mid det\, A \neq 0\}$ is a stable subset of the monoid $(M_n(K), \cdot)$;
(ii) $(GL_n(K), \cdot)$ is a group, called the *general linear group of rank n*;
(iii) $SL_n(K) = \{A \in M_n(K) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(K), \cdot)$.

4. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

5. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo n and $(U_n, \cdot)$ of n-th roots of unity are isomorphic.

6. Let $M \neq \emptyset$ and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R\,, \quad (f + g)(x) = f(x) + g(x)\,, \quad \forall x \in M\,,$$

$$f \cdot g : M \to R\,, \quad (f \cdot g)(x) = f(x) \cdot g(x)\,, \quad \forall x \in M\,.$$

Show that $(R^M, +, \cdot)$ is a ring. If R is commutative, with identity or without zero divisors, does R^M have the same property?

7. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ of residue classes modulo n ($n \in \mathbb{N}$, $n \geq 2$) and let $\widehat{a} \in \mathbb{Z}_n^*$.
(i) Prove that $\widehat{a}$ is invertible $\Longleftrightarrow g.c.d.(a, n) = 1$.
(ii) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\Longleftrightarrow n$ is prime.

8. Show that the following sets are subrings of the corresponding rings:
(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

9. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

Seminar 4

1. Let K be a field. Show that $(K[X], K, +, \cdot)$ is a vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\ k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n \,.$$

2. Let K be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $(M_{m,n}(K), K, +, \cdot)$ is a vector space, with the usual addition and scalar multiplication of matrices.

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that $(K^A, K, +, \cdot)$ is a vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f + g)(x) = f(x) + g(x)\,,$$

$$(k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

In particular, $(\mathbb{R}^{\mathbb{R}}, \mathbb{R}, +, \cdot)$ is a vector space.

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations:

$$x \perp y = xy\,,$$

$$k \top x = x^k\,,$$

$\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that $(V, \mathbb{R}, \perp, \top)$ is a vector space.

5. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.
(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.
(ii) Does there exist a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

6. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

7. Which ones of the following sets are subspaces:
(i) $[-1, 1]$ of the real vector space $\mathbb{R}$;
(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
(iii) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(iv) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

8. Show that the set of all solutions of the homogeneous system of equations with real coefficients

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = 0 \\ a_{21}x_1 + a_{22}x_2 = 0 \end{cases}$$

is a subspace of the real vector space $\mathbb{R}^2$.

Seminar 5

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces (with a minimal number of generators).

3. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

4. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

5. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$

$$g(x, y) = (2x - y, 4x - 2y),$$

$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

6. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

7. (i) Determine the kernel and the image for the endomorphisms f, g, h from Exercise **5.**
(ii) Prove that $Ker\, h \oplus Im\, h = \mathbb{R}^3$.

8. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

Seminar 6

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:

(i) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.

(ii) v_1, v_2 are linearly independent.

2. Prove that the following vectors are linearly independent:

(i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.

(ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

5. Let V be a vector space over K and let $X = (v_1, v_2, v_3, v_4)$, $Y = (v_1 + v_4, v_2 + v_4, v_3 + v_4, v_4)$ be lists of vectors in V. Prove that X is linearly independent $\Longleftrightarrow$ Y is linearly independent.

6. Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.

(i) Show that the list (v_1, v_2, v_3) is a basis of the real vector space $\mathbb{R}^3$.

(ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of the vectors v_1, v_2 and v_3.

(iii) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in the each of the two bases.

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

9. (i) Let K be a field. Prove that $\forall x \in K^*$, the set $\{x\}$ is a basis for the canonical vector space K over K.

(ii) Determine a basis of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

10. Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

Seminar 7

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}\,.$$

2. Let K be a field and $S = \{(x_1, \dots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
(i) Prove that S is a subspace of the canonical vector space K^n over K.
(ii) Determine a basis and the dimension of S.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$.
(i) Show that $v = (0, 10, 2) \in Ker\, f$ and $v' = (1, 2, -1) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Determine the coordinates of v and v' in the bases of $Ker\, f$, respectively $Im\, f$.

5. Determine a complement for the following subspaces:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$ in the real vector space $\mathbb{R}^3$;
(iii) $C = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

6. Let V be a vector space over K and let S, T, U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

7. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}\,,$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

8. Consider the subspaces

$$S =< (1, 0, 4), (2, 1, 0), (1, 1, -4) >\,,$$

$$T =< (-3, -2, 4), (5, 2, 4), (-2, 0, -8) >$$

of the real vector space $\mathbb{R}^3$. Determine a basis and the dimension of S, T, $S \cap T$ and $S + T$.

Seminar 8

Compute by applying elementary operations the rank of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$; $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **2.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Compute by applying elementary operations the inverse of the matrices:

3. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **4.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

5. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

6. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

7. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine $dim < X >$ and a basis of $< X >$.

8. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1, 0, 4), (2, 1, 0), (1, 1, -4) >\,,$$

$$T = < (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > \;.$$

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S = < (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) >\,,$$

$$T = < (2, 5, -6, -5), (-1, 2, -7, -3) > \;.$$

10. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$ and a basis for the first three of them, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

Seminar 9

1. Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Show that A is invertible, determine A^{-1} and solve the linear system $AX = B$.

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i) \begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii) \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

4. Decide when the following linear system is compatible determinate and in that case solve it by using Cramer's method:

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad (a, b, c \in \mathbb{R}).$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

$$\textbf{5}.\quad (i) \begin{cases} 2x + 2y + 3z = 3 \\ x - y \quad = 1 \\ -x + 2y + z = 2 \end{cases} \quad (ii) \begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases} \quad (iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

$$\textbf{6}.\quad \begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \quad (\lambda \in \mathbb{R})$$

$$\textbf{7}.\quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \quad (a \in \mathbb{R})$$

8. Determine the positive solutions of the following non-linear system:

$$\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \frac{z}{xy} = 81 \end{cases}$$

Seminar 10

1. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

3. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

4. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1, 2), (1, 3))$ and $B' = (v'_1, v'_2) = ((1, 0), (2, 1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f + g]_B$ and $[f \circ g]_{B'}$.

5. Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x, y) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that f is an automorphism.

6. Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$.

Seminar 11

1. In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v_1', v_2', v_3') = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrix of change of basis $T_{BB'}$ and the coordinates of the vector $u = (2,0,-1)$ in both bases.

2. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

3. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x,y) = (3x+3y, 2x+4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Compute the eigenvalues and the eigenvectors of the matrices:

4. $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$ **5.** $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ **6.** $\begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix}$ $(x, y \in \mathbb{R}^*)$

7. Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A. Prove that:
(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.
(ii) A has all the eigenvalues in $\mathbb{R} \iff (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.

8. Let $A \in M_2(\mathbb{R})$. Show that A is a root of its characteristic polynomial.

Seminar 12

1. Let K be a field and let $f : K^2 \times K^2 \to K$ be defined by

$$f((x_1, x_2), (y_1, y_2)) = \alpha_{11}x_1y_1 + \alpha_{12}x_1y_2 + \alpha_{21}x_2y_1 + \alpha_{22}x_2y_2,$$

where α_{11}, α_{12}, α_{21}, $\alpha_{22} \in K$. Show that f is a bilinear form.

2. Let $n \in \mathbb{N}^*$ and let K be a field. Show that $f : K^n \times K^n \to K$ defined by

$$f((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

is a bilinear form.

3. Consider the bilinear form $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f((x_1, x_2), (y_1, y_2)) = 2x_1y_1 + 3x_1y_2 + 4x_2y_1 - x_2y_2.$$

Determine the matrix $[f]_E$ of f in the canonical basis E of $\mathbb{R}^2$ and the matrix $[f]_B$ of f in the basis $B = ((1, 1), (1, -1))$ of $\mathbb{R}^2$.

4. Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a bilinear form having the matrix $[f]_E = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ in the canonical basis E of $\mathbb{R}^2$. Define f.

5. Determine the matrix in the corresponding canonical basis of the following quadratic forms:
(i) $q : \mathbb{R}^2 \to \mathbb{R}$, $q(x_1, x_2) = 2x_1^2 + 3x_1x_2 + 6x_2^2$.
(ii) $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x_1, x_2, x_3) = x_1^2 + 4x_1x_2 + 4x_2^2 + 2x_1x_3 + x_3^2 + 2x_2x_3$.

Reduce to a canonical form the following quadratic forms:

6. $q : \mathbb{R}^2 \to \mathbb{R}$, $q(x_1, x_2) = 2x_1^2 + 3x_1x_2 + 6x_2^2$.

7. $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x_1, x_2, x_3) = x_1^2 + 4x_1x_2 + 4x_2^2 + 2x_1x_3 + x_3^2 + 2x_2x_3$.

8. $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x_1, x_2, x_3) = 2x_1x_2 - 6x_1x_3 - 6x_2x_3$.

Seminar 13

1. (i) Which of the following received words contain detectable errors when using the (3,2)-parity check code: 110, 010, 001, 111, 101, 000?

(ii) Decode the following words using the (3,1)-repeating code to correct errors: 111, 011, 101, 010, 000, 001. Which of them contain detectable errors?

2. Are $1+X^3+X^4+X^6+X^7$ and $X+X^2+X^3+X^6$ code words in the (8,4) polynomial code generated by $p=1+X^2+X^3+X^4\in\mathbb{Z}_2[X]$?

3. Write down all the words in the (6,3)-code generated by $p=1+X^2+X^3\in\mathbb{Z}_2[X]$.

4. A code is defined by the generator matrix $G=\begin{pmatrix} P \\ I_3\end{pmatrix}\in M_{5,3}(\mathbb{Z}_2)$, where:

$$P=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1\end{pmatrix}.$$

Write down the parity check matrix and all the code words.

5. Determine the minimum Hamming distance between the code words of the code with generator matrix $G=\begin{pmatrix} P \\ I_4\end{pmatrix}\in M_{9,4}(\mathbb{Z}_2)$, where:

$$P=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1\end{pmatrix}.$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

6. Encode the following messages using the generator matrix of the (9,4)-code of Exercise **5.**: 1101, 0111, 0000, 1000.

Determine the generator matrix and the parity check matrix for:

7. The (4,1)-code generated by $p=1+X+X^2+X^3\in\mathbb{Z}_2[X]$.

8. The (7,3)-code generated by $p=(1+X)(1+X+X^3)\in\mathbb{Z}_2[X]$.

Seminar 14

1. Consider a $(63, 56)$-code.
(i) What is the number of digits in the message before coding?
(ii) What is the number of check digits?
(iii) What is the information rate?
(iv) How many different syndromes are there?

2. Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the syndromes and coset leaders

Syndrome	000	001	010	011
Coset leader	000000	001000	010000	000010

Syndrome	100	101	110	111
Coset leader	100000	000110	000100	000001

decode the following words: 101110, 011000, 001011, 111111, 110011.

3. A (7,4)-code is defined by the equations $u_1 = u_4 + u_5 + u_7$, $u_2 = u_4 + u_6 + u_7$, $u_3 = u_4 + u_5 + u_6$, where u_4, u_5, u_6, u_7 are the message digits and u_1, u_2, u_3 are the check digits. Write its generator matrix and parity check matrix. Decode the received words 0000111 and 0001111.

4. Find the syndromes of all the received words in the (3,2)-parity check code and in the (3,1)-repeating code.

5. Construct a table of coset leaders and syndromes for the (7,4)-code with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

6. Determine the parity check matrix and all syndromes and coset leaders of the (5,3)-code with generator matrix $G = \left(\frac{P}{I_3}\right) \in M_{5,3}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

7. Construct a table of coset leaders and syndromes for the (3,1)-code generated by $p = 1 + X + X^2 \in \mathbb{Z}_2[X]$.

8. Construct a table of coset leaders and syndromes for the (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.