An algorithm to compute the Wedderburn decomposition of semisimple group algebras implemented in the GAP package `wedderga`

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Abstract

We present an algorithm to compute the Wedderburn decomposition of semisimple group algebras based on a computational approach of the Brauer-Witt theorem. The algorithm was implemented in the GAP package `wedderga`.

Key words: Group Algebras, Wedderburn decomposition.

The Wedderburn decomposition of a semisimple group algebra FG is the decomposition of FG as a direct sum of simple algebras and it has applications to the study of units and automorphisms of group rings (Her; Jes-Lea; Oli-Río-Sim-2; Rit-Seh). The computation of the Wedderburn decomposition of group algebras and, in particular, of the primitive central idempotents, has attracted the attention of several authors (Bro-Pol; Bro-Río; Jes-Lea-Paq; Oli-Río; Oli-Río-Sim-1).

The Brauer-Witt Theorem states that the Wedderburn components of FG (i.e the factors of its Wedderburn decomposition) are Brauer equivalent to cyclotomic algebras (see (Yam) or the original papers of Brauer and Witt (Bra; Wit)). By the computation of the Wedderburn decomposition of FG we mean the description of its Wedderburn components as Brauer equivalent to cyclotomic algebras. The identities of the Wedderburn

⋆ Research supported by M.E.C. of Romania (CEEX-ET 47/2006), D.G.I. of Spain and Fundación Séneca of Murcia.

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components of FG are the primitive central idempotents of FG and can be computed from the character table of the group G. A character-free method to compute the primitive central idempotents of $\mathbb{Q}G$ for G nilpotent has been introduced in (Jes-Lea-Paq). In (Oli-Río-Sim-1), it was shown how to extend the methods of (Jes-Lea-Paq) to compute not only the primitive central idempotents of $\mathbb{Q}G$, if G is a strongly monomial group, but also the Wedderburn decomposition of $\mathbb{Q}G$. (See below for the definition of strongly monomial groups.) This approach was used to produce a GAP package (GAP), called `wedderga`, which was able to compute the Wedderburn decomposition of $\mathbb{Q}G$ provided G is strongly monomial. (See (Oli-Río) where the main algorithm of the first version of `wedderga` is explained.) The GAP package `LAGUNA` provides other useful functions for computation with group rings (LAGUNA).

Recently, a computational approach of the Brauer-Witt Theorem was given in (Olt). Using this approach, the functionality of `wedderga` was extended and the new version of the package is now able to compute the Wedderburn decomposition of FG for any finite group G, provided F is an abelian extension of the rationals or a finite field (Wedderga). In this paper we report on the main algorithm of this implementation.

1. The theoretical background

Throughout F is a field of zero characteristic [1], G is a finite group and FG is the group algebra of G over F.

We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G. (All characters are assumed to be complex characters.) Let $\chi \in \mathrm{Irr}(G)$. Following (Yam), $A(\chi, F)$ denotes the unique Wedderburn component I of FG such that $\chi(I) \neq 0$. The identity of $A(\chi, F)$ is denoted by $e_F(\chi)$. The degree of $A(\chi, F)$ is $\chi(1)$, the degree of the character χ. The center of $A(\chi, F)$ is isomorphic to $F(\chi) = F(\chi(g) : g \in G)$, the field of character values of χ over F (Yam). If $\chi, \chi' \in \mathrm{Irr}(G)$, then $A(\chi, F) = A(\chi', F)$ if and only if $e_F(\chi) = e_F(\chi')$ if and only if $\chi' = \sigma \circ \chi$, for some $\sigma \in \mathrm{Gal}(F(\chi)/F)$. In that case we say that χ and χ' are *F-equivalent*. If θ is a character of a subgroup of G, then θ^G denotes the character of G induced by θ.

A *crossed product* of G over a field E is an associative algebra over E having a basis $\{u_g : g \in G\}$ of invertible elements such that there are two maps $\sigma : G \to \mathrm{Aut}(E)$ and $\tau : G \times G \to E^*$ satisfying

$$au_g = u_g a^{\sigma_g} \quad \text{and} \quad u_g u_h = u_{gh}\tau(g, h)$$

for each $g, h \in G$ and $a \in E$. (Here $\sigma_g = \sigma(g)$ and we are using exponential notation for the action of automorphisms.) The maps σ and τ are called the *action* and *twisting* of the crossed product and we denote the crossed product with action σ and twisting τ by $E *_\tau^\sigma G$ or simply $E * G$. For example, $EG = E *_\tau^\sigma G$, with $\sigma_g = 1_E$, the identity map of E, and $\tau(g, h) = 1$, the identity of the field E, for every $g, h \in G$.

A *cyclotomic algebra* over a field $\mathbb{F}$ is a crossed product of the form $E *_\tau^\sigma G$, where E is a finite cyclotomic extension of $\mathbb{F}$, $G = \mathrm{Gal}(E/\mathbb{F})$, the action σ is the inclusion $\mathrm{Gal}(E/\mathbb{F}) \hookrightarrow \mathrm{Aut}(E)$ and the values taking by the twisting τ are roots of unity. Such a cyclotomic algebra is usually denoted as $(E/\mathbb{F}, \tau)$ and it is always a central simple $\mathbb{F}$-algebra. The twisting τ is a 2-cocycle of $\mathrm{Gal}(E/\mathbb{F})$ with coefficients in E^* and if τ' is another 2-cocycle which is

[1] See Final Remarks.

cohomologically equivalent to τ (i.e. $\tau'\tau^{-1}$ is a 2-coboundary) then $(E/\mathbb{F},\tau)\simeq(E/\mathbb{F},\tau')$. So one may consider τ as an element in $H^2(\mathrm{Gal}(E/\mathbb{F}),E^*)$, the second cohomology group. If L is subextension of $E/\mathbb{F}$, then $\mathrm{Cor}_{L\to\mathbb{F}}:H^2(\mathrm{Gal}(E/L),E^*)\to H^2(\mathrm{Gal}(E/\mathbb{F}),E^*)$ denotes the corestriction and $\mathrm{Inf}_{L\to E}:H^2(\mathrm{Gal}(L/\mathbb{F}),L^*)\to H^2(\mathrm{Gal}(E/\mathbb{F}),E^*)$ denotes the inflation (Pie).

Let H be a subgroup of G and K a normal subgroup of H. If $H=K$ then let $\varepsilon(H,K)=\widehat{K}=\frac{1}{K}\sum_{k\in K}k\in\mathbb{Q}G$. If $H\neq K$ then let

$$\varepsilon(H,K)=\prod_{M/K}(\widehat{K}-\widehat{M})$$

where M/K runs through the minimal normal non-trivial subgroups of H/K. Furthermore, $e(G,H,K)$ denotes the sum of the different G-conjugates of $\varepsilon(H,K)$. Clearly $\varepsilon(H,K)$ is a central idempotent of $\mathbb{Q}H$ and $e(G,H,K)$ is a central element of $\mathbb{Q}G$. Moreover, if the G-conjugates of $\varepsilon(H,K)$ are orthogonal then $e(G,H,K)$ is a central idempotent of $\mathbb{Q}G$.

A *strong Shoda pair* of G is a pair (H,K) of subgroups of G satisfying the following properties: K is a normal subgroup of H, H is normal in the normalizer $N_G(K)$ of K in G, H/K is cyclic and maximal abelian in $N_G(K)/H$ and the G-conjugates of $\varepsilon(H,K)$ are orthogonal.

Now we define a central simple algebra $A_F(G,H,K)$ as a matrix algebra of a cyclotomic algebra associated to a strong Shoda pair (H,K) of G. Set $m=[H:K]$, $H/K=\langle h\rangle$ and $N=N_G(K)$. Consider a linear character ψ of H with kernel K and let $\psi(h)=\xi_m$, a primitive m-th root of unity. Denote by θ the induced character ψ^G and set $\mathbb{F}=F(\theta)$. Then $A_F(G,H,K)=M_{nd}((\mathbb{F}(\xi_m)/\mathbb{F},\tau_F))$, where $n=[G:N]$, $d=\frac{[\mathbb{Q}(\xi_m):\mathbb{Q}(\theta)]}{[\mathbb{F}(\xi_m):\mathbb{F}]}$ and the twisting τ_F is obtained as follows: First consider the 2-cocycle α of N/H with coefficients in H/K associated to the natural exact sequence

$$1\to H/K\hookrightarrow N/K\xrightarrow{\pi}N/H\to 1$$

that is, select $\varphi:N/H\to N/K$, a right inverse of π, and let

$$\alpha(g,h)=\varphi(gh)^{-1}\varphi(g)\varphi(h).$$

Second, consider the map $f:N/H\to\mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$ given by $\xi_m^{f(n)}=\xi_m^t$, provided $\varphi(n)^{-1}h\varphi(n)=h^t$. Then, f is a group isomorphism and one defines a 2-cocycle β of $\mathrm{Gal}(\mathbb{Q}(\xi_m)/\mathbb{Q})$ with coefficients in $\mathbb{Q}(\xi_m)$ by setting

$$\beta(g,h)=\psi\circ\alpha(f^{-1}(g),f^{-1}(h)).$$

Finally, τ_F is the restriction of β to $\mathrm{Gal}(\mathbb{F}(\xi_m)/\mathbb{F})$.

The algebra $A_F(G,H,K)$ is isomorphic to a Wedderburn component of FG. (See (Oli-Río-Sim-1) and (Olt) for details.)

Proposition 1. *Let (H,K) be a strong Shoda pair of G and θ defined as above. Then θ is irreducible, $e_{\mathbb{Q}}(\theta)=e(G,H,K)$ and $A(\theta,F)\simeq A_F(G,H,K)$.* $\square$

The independence of the description of $A(\theta,F)$ from the election of φ follows because the cocycles obtained from the different right inverses of π are cohomologically equivalent.

In the sequel, for each positive integer m, we denote an m-th primitive root of unity by ξ_m. Notice that the character θ of Proposition 1 depends not only on the strong Shoda

pair (H, K), but also on the choice of ξ_m. We refer to any of the possible characters $\theta = \psi^G$ (with ψ a linear character of H with kernel K) as a *character induced by* the strong Shoda pair (H, K). By Proposition 1, if θ and θ' are two characters of G induced by (H, K) (with different choice of m-th roots of unity) then $e_{\mathbb{Q}}(\theta) = e(G, H, K) = e_{\mathbb{Q}}(\theta')$, i.e. θ and θ' are $\mathbb{Q}$-equivalent. Two strong Shoda pairs of G are said to be *equivalent* if they induce $\mathbb{Q}$-equivalent characters. An irreducible character χ of G is said to be *strongly monomial* if it is the character induced by a strong Shoda pair of G. We say that G is *strongly monomial* if every irreducible character of G is strongly monomial.

Proposition 1 allows one to compute the Wedderburn decomposition of FG, provided G is strongly monomial. This is the theoretical basis of the first version of `wedderga` to compute the Wedderburn decomposition of rational group algebras of strongly monomial groups (see (Oli-Río)). The theoretical background that has allowed to extend the functionality of `wedderga` to arbitrary groups is based on a computationally oriented proof of the Brauer-Witt Theorem given in (Olt). We explain now the ingredients of this approach which are relevant to us.

If A is a central simple $\mathbb{F}$-algebra, then $[A]$ denotes the class of A in $\mathrm{Br}(\mathbb{F})$. If we write $\langle [A] \rangle = P \times Q$, where P is the Sylow p-subgroup of $\langle [A] \rangle$, then the projection $[A]_p$ of $[A]$ in P is called the p-part of $[A]$. If $p_1, \ldots, p_n$ are the different primes dividing the exponent of A and, for each i, A_{p_i} is a central simple $\mathbb{F}$-algebra such that $[A_{p_i}] = [A]_{p_i}$, then A is Brauer equivalent to $A_{p_1} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} A_{p_n}$. So, if χ is an irreducible character of G, in order to describe $A(\chi, \mathbb{F})$ up to Brauer equivalence, it is enough to describe representatives of its p-parts.

If E/F is an abelian finite field extension and p is a prime, then the p'-part of E/F is the maximal subextension L of E/F such that $[L : F]$ is coprime to p. The keystone for the computational approach to the Brauer-Witt Theorem relies on the following proposition (see (Olt)).

Proposition 2. *Let n be the exponent of G and χ an irreducible character of G.*

(1) For every prime p, there is a strongly monomial character θ of a subgroup M of G satisfying:

$(*)$ *(χ_M, θ) is coprime to p and θ takes values in L_p, the p'-part of $F(\xi_n)/F(\chi)$.*

(2) If θ satisfies condition $(*)$, then the p-part of $A(\chi, F)$ is Brauer equivalent to $\mathrm{Cor}_{L_p \to F(\chi)}(A(\theta, L_p))^{\otimes r}$, where r is an inverse of $[L_p : F(\chi)]$ modulo the maximum p-th power dividing $\chi(1)$. $\square$

Proposition 2 shows that one may describe $A(\chi, F)$ by making use of Proposition 1 to compute its p-parts up to Brauer equivalence. In other words, each p-part of $A(\chi, F)$ can be described in terms of $A_{L_p}(M, H, K)$, where (H, K) is a suitable strong Shoda pair of a subgroup M of G. A *strong Shoda triple* of G is by definition a triple (M, K, H), where M is a subgroup of G and (H, K) is a strong Shoda pair of G. This suggests the following algorithm that was proposed in (Olt).

Algorithm 1. Theoretical algorithm for the computation of the Wedderburn decomposition of FG.

INPUT**:** A group algebra FG of a finite group G over a field F of zero characteristic.

PRECOMPUTATION: Compute n, the exponent of G and E, a set of representatives of the F-equivalence classes of the irreducible characters of G.

COMPUTATION: For every $\chi \in E$:

(1) Compute $\mathbb{F} := F(\chi)$, the field of character values of χ over F.
(2) Compute $p_1, \ldots, p_r$, the common prime divisors of $\chi(1)$ and $[\mathbb{F}(\xi_n) : \mathbb{F}]$.
(3) For each $p \in [p_1, \ldots, p_r]$:
  (a) Compute L_p, the p'-part L_p of $\mathbb{F}(\xi_n)/\mathbb{F}$.
  (b) Search for a strong Shoda triple (M_p, H_p, K_p) of G such that the character θ_p of M_p induced by (H_p, K_p) satisfies:
    $(*)$ (χ_{M_p}, θ_p) is coprime to p and θ_p takes values in L_p.
  (c) Compute $A_p := (L_p(\xi_{m_p})/L_p, \tau_p = \tau_{L_p})$, as in Proposition 1.
  (d) Compute $\tau'_p := \mathrm{Cor}_{L_p \to \mathbb{F}}(\tau_p)$.
  (e) Compute a_p, an inverse of $[L_p : \mathbb{F}]$ modulo the maximum p-th power dividing $\chi(1)$.
(4) Compute m, the least common multiple of $m_{p_1}, \ldots, m_{p_r}$.
(5) Compute $\widehat{\tau_{p_i}} := \mathrm{Inf}_{\mathbb{F}(\xi_{m_{p_i}}) \to \mathbb{F}(\xi_m)}(\tau'_{p_i})$, for each $i = 1, \ldots, r$.
(6) Compute $B := (\mathbb{F}(\xi_m)/\mathbb{F}, \tau)$, where $\tau = \widehat{\tau_{p_1}}^{a_{p_1}} \cdots \widehat{\tau_{p_r}}^{a_{p_r}}$.
(7) Compute $A_\chi := M_{d_1/d_2}(B)$, where d_1, d_2 are the degrees of χ and B respectively.

OUTPUT: $\{A_\chi : \chi \in E\}$, the Wedderburn components of FG.

In some cases, the algebra A_χ obtained in (7) is not a genuine matrix algebra because d_2 does not divide d_1 necessarily. This undesired phenomenon can not be avoided because it is not true, in general, that every Wedderburn component of FG is a matrix algebra of a cyclotomic algebra (see (Olt) for an example). Luckily, this is a rare phenomenon and even when it is encountered, the information $\frac{d_1}{d_2}$ and B is still useful to describe A_χ (for example, it can be used to compute the index of A_χ).

2. A working algorithm

Algorithm 1 is not the most efficient way to compute the Wedderburn decomposition of FG for several reasons.

Firstly, it is easy to compute the Wedderburn decomposition of FG from the Wedderburn decomposition of $\mathbb{Q}G$. More precisely, if χ is an irreducible character of G, $\Bbbk = \mathbb{Q}(\chi)$ and $\mathbb{F} = F(\chi)$, then $A(\chi, F) \simeq \mathbb{F} \otimes_{\Bbbk} A(\chi, \mathbb{Q})$. In particular, if $A(\chi, \mathbb{Q})$ is equivalent to the cyclotomic algebra $(\Bbbk(\xi)/\Bbbk, \tau)$, then $A(\chi, F)$ is Brauer equivalent to $(\mathbb{F}(\xi)/\mathbb{F}, \tau')$, where τ' is the restriction of τ via the inclusion $\mathrm{Gal}(\mathbb{F}(\xi)/\mathbb{F}) \subseteq \mathrm{Gal}(\Bbbk(\xi)/\Bbbk)$. Moreover, the degrees of $A(\chi, \mathbb{Q})$ and $A(\chi, F)$ are equal (the degree of χ). This suggests to use the description of the Wedderburn decomposition of $\mathbb{Q}G$ as information to be stored as an attribute of G. (Recall that an attribute of a GAP object is information about the object saved when computed, to be quickly accessed in subsequent computations). The algorithm implemented computes some data which can be easily used to compute the Wedderburn decomposition of $\mathbb{Q}G$. A small modification will be enough to use this data to produce the Wedderburn decomposition of FG.

Secondly, if χ is a strongly monomial character of G, then $A(\chi, F)$ can be computed at once by using Proposition 1. That is, there is no need to compute the p-parts separately and merging them together.

Example 1. Let p be a prime and consider $\mathbb{Z}_p^*$ acting on $\mathbb{Z}_p$ by multiplication. Let $G = \mathbb{Z}_p \rtimes Z_p^*$ be the corresponding semidirect product. Then $(\mathbb{Z}_p, 1)$ is a strong Shoda pair of G and if χ is the strongly monomial character induced, then $A = A(\chi, F)$ has degree $p-1$. For example, if $p = 31$, then A has degree 30. So according to Algorithm 1, one should describe the p-parts for $p = 2, 3$ and 5. This is not needed using Proposition 1. $\square$

In particular, if G is strongly monomial (as so is the group of Example 1), then instead of running through the irreducible characters χ of G and looking for some strong Shoda pairs (H, K) of G such that χ is the character of G induced by (H, K), it is more efficient to produce a list of strong Shoda pairs of G and at the same time produce the primitive central idempotents $e(G, H, K)$ of $\mathbb{Q}G$, which helps to control if the list is complete. This was the approach in (Oli-Río).

Thirdly, even if χ is not strongly monomial and the number r of primes appearing in step (2) of Algorithm 1 is greater than 1, it may happen that just one strongly monomial character θ of a subgroup M of G satisfies condition $(*)$ of Proposition 2 for more than one prime p.

Example 2. Consider the permutation group $G = \langle (3,4)(5,6), (1,2,3)(4,5,7) \rangle$ and its subgroup $M = \langle (1,3,5)(4,6,7), (1,6)(5,7) \rangle$. Then G has an irreducible character χ of degree 6, such that $\mathbb{Q}(\chi) = \mathbb{Q}$ and $(\chi_M, 1_M) = 1$. Clearly 1_M, the trivial character of M, is strongly monomial and satisfies condition $(*)$ for the two possible primes 2 and 3. Using this, it follows at once that $A(\chi, F) = M_6(F)$ for each field F, and so there is no need to consider the two primes separately. $\square$

Fourthly, one strongly monomial character θ of a subgroup of G may satisfy condition $(*)$ for more than one irreducible character χ of G.

Example 3. Consider the group $G = \mathrm{SL}(2,3) = \langle a, b \rangle \rtimes \langle c \rangle$ (where $\langle a, b \rangle$ is the quaternion group of order 8 and c has order 3). The group G has one non-strongly monomial character χ_1 of degree 2 with $\mathbb{Q}(\chi_1) = \mathbb{Q}$ and two non-strongly monomial $\mathbb{Q}$-equivalent characters χ_2 and χ_2', also of degree 2, with $\mathbb{Q}(\chi_2) = \mathbb{Q}(\chi_2') = \mathbb{Q}(\xi_3)$. Then $(M = \langle a, b \rangle, H = \langle a \rangle, 1)$ is a strong Shoda triple. If θ is the strongly monomial character of M induced by $(H, 1)$, then θ satisfies condition $(*)$ for both χ_1 and χ_2 and $p = 2$, the unique prime involved. $\square$

Finally, the weakest part of Algorithm 1 is step (3)(b), where a blind search of a strong Shoda triple of G satisfying condition $(*)$ for each irreducible character of G and each prime $p_1, \ldots, p_r$ may be too costly.

Taking all these into account, it is more efficient to run through the strong Shoda triples of G and for each such triple evaluate its contribution to the p-parts of $A(\chi, F)$ for the different irreducible characters χ of G and the different primes p. This leads to the question on what is the most efficient way to systematically compute strong Shoda triples of G. The first version of `wedderga` included a function `StrongShodaPairs` which computes a list of representatives of the equivalence classes of the strong Shoda pairs of the group given as input. So one can use this function to compute the strong Shoda pairs for each subgroup of G. However, most of the strong Shoda triples of G are not necessary. For example, if G is strongly monomial, we only need to compute the strong Shoda triples of the form (G, H, K), i.e. in this case one needs to compute only the strong Shoda pairs (H, K) of G. Again, this is the original approach in (Oli-Río). This suggests

to start computing the strong Shoda pairs of G and the associated simple components as in Proposition 1. If the group is strongly monomial, we are done.

Which are the next natural candidates of subgroups M of G for which we should compute the strong Shoda pairs of M? That is, what are the strong Shoda triples (M, H, K) most likely to actually contribute in the computation? Take any strong Shoda triple (M, H, K) of G. If M_1 is a subgroup of M containing H, then (M_1, H, K) is also a strong Shoda triple of G. Now let ψ be a linear character of H with kernel K and set $\theta = \psi^M$ and $\theta_1 = \psi^{M_1}$. Then, for every irreducible character χ of G, $(\chi_{M_1}, \theta_1) = (\chi, \theta_1^G) = (\chi, \theta^G) = (\chi_M, \theta)$, by Frobenius Reciprocity. So θ satisfies the first part of condition $(*)$ if and only if so does θ_1. However, $F(\theta) \subseteq F(\theta_1)$ and so, the bigger M, the more likely θ to satisfy the second condition of $(*)$ and, in fact, all the contributions of θ_1 are already realized by θ.

Example 3. (Continuation). Notice that $(H, 1)$ is a strong Shoda pair of M, but it is not strong Shoda pair of G. In some sense, $(H, 1)$ is very close to be a strong Shoda pair of G because it is a strong Shoda pair in a subgroup of prime index in G. On the other hand, $(H, H, 1)$ is also a strong Shoda triple of G. However, the strongly monomial character θ of H (in fact linear) induced by $(H, 1)$ does not satisfy condition $(*)$ with respect to either χ_1 or χ_2 because the field of character values of θ contains $i = \sqrt{-1}$. So, G is too big for $(G, H, 1)$ to be a strong Shoda triple of G, while H is too small for $(H, H, 1)$ to contribute in terms of satisfying condition $(*)$. $\square$

Notice also that if M is a subgroup of G and $g \in G$, then the strong Shoda pairs of M and M^g are going to contribute equally in terms of satisfying condition $(*)$ for a given irreducible character χ. This is because if (H, K) is a strong Shoda pair of M then (H^g, K^g) is a strong Shoda pair of M^g and if θ is the character of M induced by (H, K), then θ^g is the character induced by (H^g, K^g). Then $(\chi_M, \theta) = (\chi_M, \theta^g)$ and θ and θ^g take the same values. So, we only have to compute strong Shoda pairs for one representative of each conjugacy class of subgroups of G.

Summarizing, we chose the algorithm to run through conjugacy classes of subgroups of G in decreasing order and evaluate the contribution on as many p-parts of as many irreducible characters as possible. In fact, we consider the group $M = G$ separately because Proposition 1 tells us how to compute the corresponding simple algebras without having to consider the p-parts separately. This is called the Strongly Monomial Part of the algorithm and takes care of the Wedderburn components of the form $A(\chi, F)$ for $\chi \in \mathrm{Irr}(G)$ strongly monomial. The remaining components are computed in the Non-Strongly Monomial Part, where we consider proper subgroups M (actually representatives of conjugacy classes). For such an M we use `StrongShodaPairs` to compute a set of representatives of strong Shoda pairs (H, K) of M and for each (H, K) we check to which p-parts of the non-strongly monomial characters of G the character θ induced by (H, K) contributes (i.e. condition $(*)$ is satisfied). The algorithm stops when all the p-parts of all the irreducible characters are covered. In most cases, only a few subgroups M of G have to be used.

We are ready to present the algorithm.

Algorithm 2. Computes data for the Wedderburn decomposition of $\mathbb{Q}G$.

Input**:** A finite group G (of exponent n).

Strongly Monomial Part:

(1) Compute S, a list of representatives of strong Shoda pairs of G.

(2) Compute $Data := [[n_x, \Bbbk_x, m_x, \mathrm{Gal}_x, \tau_x] : x \in S]$, where for each $x = (H, K) \in S$:
- $n_x := [G : N]$, with $N = N_G(K)$;
- $\Bbbk_x := \mathbb{Q}(\theta_x)$, for θ_x a strongly monomial character of G induced by (H, K);
- $m_x := [H : K]$;
- $\mathrm{Gal}_x := \mathrm{Gal}(\Bbbk_x(\xi_{m_x})/\Bbbk_x)$;
- $\tau_x := \tau_{\mathbb{Q}}$, the 2-cocycle of Gal_x with coefficients in $\mathbb{Q}(\xi_{m_x})$ given as in Proposition 1.

Non-Strongly Monomial Part: If G is not strongly monomial

(1) Compute E, a set of representatives of the $\mathbb{Q}$-equivalence classes of the non-strongly monomial irreducible characters of G.

(2) Compute $PrimesLps := [PrimesLp_\chi : \chi \in E]$, where $PrimesLP_\chi$ is the list of pairs $[p, L_p]$, with p a prime dividing $\gcd(\chi(1), [\mathbb{Q}(\xi_n) : \mathbb{Q}(\chi)])$ and L_p is the p'-part of the extension $\mathbb{Q}(\xi_n)/\mathbb{Q}(\chi)$.

(3) Initialize $E' := E$, a copy of E, and
$Parts := [Parts_\chi := [\,] : \chi \in E]$, a list of length $|E|$ formed by empty lists.

(4) For M running in decreasing order through a set of representatives of conjugacy classes of proper subgroups of G (while $E' \neq \emptyset$):
Compute S_M, the strong Shoda pairs of M and for each $(H, K) \in S_M$:
- Compute θ, a strongly monomial character of M induced by (H, K).
- Compute $Drop := [Drop_\chi : \chi \in E]$, where $Drop_\chi$ is the set of $[p, L_p]$ in $PrimesLps_\chi$, for which $(*)$ holds.
- For each $[p, L_p]$ in $Drop_\chi$, compute m_p, τ'_p and a_p as in Step (3) of Algorithm 1 and add this information to $Parts_\chi$.
- $PrimesLps_\chi := PrimesLps_\chi \setminus Drop_\chi$.
- $E' := E' \setminus \{\chi \in E : PrimesLps = \emptyset\}$.

(5) Compute $Data' := [[n_\chi, \Bbbk_\chi, m_\chi, \mathrm{Gal}_\chi, \tau_\chi] : \chi \in E]$, where
- $\Bbbk_\chi := \mathbb{Q}(\chi)$;
- $m_\chi :=$ Least common multiple of the m_p's appearing in $Parts_\chi$;
- $n_\chi := \frac{\chi(1)}{[\Bbbk_\chi(\xi_{m_\chi}):\Bbbk_\chi]}$;
- $\mathrm{Gal}_\chi := \mathrm{Gal}(\Bbbk_\chi(\xi_{m_\chi})/\Bbbk_\chi)$;
- τ_χ is computed from $m = m_\chi$ and the τ'_p's and a_p's in $Parts_\chi$, as in Steps (3)-(6) of Algorithm 1.

Output: The list obtained merging $Data$ and $Data'$.

The output of Algorithm 2 can be used right away to produce the Wedderburn decomposition of $\mathbb{Q}G$. Each entry $[n, \Bbbk, m, \mathrm{Gal}, \tau]$ parametrizes one Wedderburn component of $\mathbb{Q}G$ which is isomorphic to $M_n((\Bbbk(\xi_m)/\Bbbk, \tau))$.

For an arbitrary field of zero characteristic F, some modifications are needed. The number of 5-tuples, say r, of the output of Algorithm 2 is the number of $\mathbb{Q}$-equivalence classes of irreducible characters of G. Let $\chi_1, \ldots, \chi_r$ be a set of representatives of $\mathbb{Q}$-equivalence classes of irreducible characters of G. Then $\mathbb{Q}G = \oplus_{i=1}^r A(\chi_i, \mathbb{Q})$ and so $FG = F \otimes_{\mathbb{Q}} \mathbb{Q}G = \oplus_{i=1}^r F \otimes_{\mathbb{Q}} A(\chi_i, \mathbb{Q})$. Moreover, if $A = A(\chi, \mathbb{Q})$, then

$$F \otimes_{\mathbb{Q}} A = F \otimes_{\mathbb{Q}} \mathbb{Q}(\chi) \otimes_{\mathbb{Q}(\chi)} A \simeq [F \cap \mathbb{Q}(\chi) : \mathbb{Q}]F(\chi) \otimes_{\mathbb{Q}(\chi)} A = [F \cap \mathbb{Q}(\chi) : \mathbb{Q}]A(\chi, F).$$

Thus, an entry $[n, \Bbbk, m, \mathrm{Gal}, \tau]$ of the output parametrizes $[F \cap \Bbbk : \mathbb{Q}]$ Wedderburn components of FG, each one isomorphic to $\mathbb{F} \otimes_{\Bbbk} M_n((\Bbbk(\xi_m)/\Bbbk, \tau)) \simeq M_{nd}((\mathbb{F}(\xi_m)/\mathbb{F}, \tau'))$, where $\mathbb{F}$ is the compositum of $\Bbbk$ and F, $d = \frac{[\Bbbk(\xi_m):\Bbbk]}{[\mathbb{F}(\xi_m):\mathbb{F}]} = \frac{|\mathrm{Gal}|}{|\mathrm{Gal}'|}$, $\mathrm{Gal}' = \mathrm{Gal}(\mathbb{F}(\xi_m)/\mathbb{F})$ and τ' is the restriction of $\tau \in H^2(\mathrm{Gal}, \mathbb{F}(\xi_m))$ to a 2-cocycle $\tau' \in H^2(\mathrm{Gal}', \mathbb{F}(\xi_m))$.

If $\xi_m \in \Bbbk$ then $\mathrm{Gal} = 1$ and, in fact, Algorithm 2 only loads the information $[n, \Bbbk]$, which parametrizes the simple component $M_n(\Bbbk)$ of $\mathbb{Q}G$ and $[F \cap \Bbbk : \mathbb{Q}]$ simple components of FG isomorphic to $M_n(\mathbb{F})$. If $\xi_m \notin \Bbbk$, then the simple component of $\mathbb{Q}G$ is a matrix algebra of size n of a non-commutative cyclotomic algebra. However, if $\xi_m \in \mathbb{F}$, (equivalently if $\mathrm{Gal}' = 1$), then the simple components of FG given by this entry of the output are isomorphic to $M_{nd}(\mathbb{F})$.

3. Examples

In this section we show how Algorithm 2 works in two examples.

Example 4. Consider the group $G = \langle (3,4)(5,6), (1,2,3)(4,5,7) \rangle$ of Example 2. This group is the group $(168, 42)$ from the GAP library of small groups and it is isomorphic to $\mathrm{SL}(3,2)$. The Wedderburn decomposition of $\mathbb{Q}G$ can be computed using the function `WedderburnDecomposition` of `wedderga`.

```
gap> G:=SmallGroup(168,42);;
gap>QG:=GroupRing(Rationals,G);;
gap> WedderburnDecomposition(QG);
[ Rationals, ( Rationals^[ 7, 7 ] ), ( NF(7,[ 1, 2, 4 ])^[ 3, 3 ] ),
( Rationals^[ 6, 6 ] ), ( Rationals^[ 8, 8 ] ) ]
```

Thus

$$\mathbb{Q}G \simeq \mathbb{Q} \oplus M_7(\mathbb{Q}) \oplus M_3(\mathbb{Q}(\sqrt{-7})) \oplus M_6(\mathbb{Q}) \oplus M_8(\mathbb{Q}).$$

Notice that the center in the third component is $\mathbb{Q}(\sqrt{-7})$, the subfield of $\mathbb{Q}(\xi_7)$ consisting of the elements fixed by the automorphism $\xi_7 \mapsto \xi_7^2$.

Now we explain how the package obtains this information. As it is explained above, the first part of the algorithm computes a list of representatives of the strong Shoda pairs of G using the function `StrongShodaPairs`. This part of the algorithm provides two strong Shoda pairs and the first two Wedderburn components of $\mathbb{Q}G$.

```
gap> StrongShodaPairs(G);
[ [ Group([ (3,4)(5,6), (1,2,3)(4,5,7) ]),
      Group([ (3,4)(5,6), (1,2,3)(4,5,7) ]) ],
  [ Group([ (3,4)(5,6), (1,7)(5,6), (1,3,5)(4,6,7), (3,6)(4,5) ]),
      Group([ (3,4)(5,6), (1,7)(5,6), (1,3,5)(4,6,7) ]) ] ]
```

The other part of the calculation provides another three pairs. They correspond to three $\mathbb{Q}$-equivalence classes of non-strongly monomial characters represented by the fol-

lowing characters, where $\alpha = \xi_7 + \xi_7^2 + \xi_7^4 = \frac{-1+\sqrt{-7}}{2}$:

	1	$(3,4)(5,6)$	$(2,3,4)(5,6,7)$	$(2,3,7,5)(4,6)$	$(1,2,3,5,6,7,4)$	$(1,2,3,7,4,6,5)$
χ_1	3	-1	0	1	$-1-\alpha$	α
χ_2	6	2	0	0	-1	-1
χ_3	8	0	-1	0	1	1

So the center of $A_1 := A(\chi_1, \mathbb{Q})$ is $\mathbb{Q}(\chi_1) = \mathbb{Q}(\alpha)$ and the center of $A_2 := A(\chi_2, \mathbb{Q})$ and $A_3 := A(\chi_3, \mathbb{Q})$ is $\mathbb{Q}(\chi_2) = \mathbb{Q}(\chi_3) = \mathbb{Q}$. Now the program has to compute cyclotomic algebras equivalent to A_1, A_2 and A_3. The degrees of these algebras are 3, 6 and 8 respectively. Since the index of a central simple algebra divides its degree, one has to describe the 3-part of A_1, the 2 and 3-parts of A_2 and the 2-part of A_3. By Proposition 2, the 2 and 3-parts of A_2 can be obtained by using two strong Shoda triples of G. However, as we have seen in Example 2, $((\chi_2)_M, 1_M) = 1$ for $M = \langle (1,3,5)(4,6,7), (1,6)(5,7) \rangle$. So, there is a unique strong Shoda triple of G, namely (M, M, M), which provides the strongly monomial character 1_M satisfying condition $(*)$ for the two primes involved. It was already explained that $A(\chi_2, \mathbb{Q}) \simeq M_6(\mathbb{Q})$ and this takes care of the fourth entry given as output by `WedderburnDecomposition`.

For the other two characters the algorithm obtains the strong Shoda triple $(M, H = \langle (3,4)(5,6), (1,6,7,5)(3,4) \rangle, K = \langle (1,6,7,5)(3,4) \rangle)$ for both of them. Since $H = N_M(K)$ and $[H : K] = 2$, the algebra $A(M, H, K)$ is Brauer equivalent to $\mathbb{Q}(\xi_2) = \mathbb{Q}$ (Proposition 1). Since $A(\chi_1, \mathbb{Q})$ and $A(\chi_3, \mathbb{Q})$ are Brauer equivalent to $A(M, H, K)$ (Proposition 2), we obtain that $A(\chi_1, \mathbb{Q}) \simeq M_3(\mathbb{Q})$ and $A(\chi_3, \mathbb{Q}) \simeq M_8(\mathbb{Q})$.

Notice that for all the strong Shoda triples (L, H, K) of G used, the subgroup L is either G (for the Strongly Monomial Part) or M (for the Non-Strongly Monomial Part). The group G has 15 conjugacy classes of subgroups, one formed by G, two classes formed by subgroups of order 24 and the other classes formed by subgroups of smaller order. The advantage of running through subgroups in decreasing order becomes apparent in this computation for only the groups M and G have been considered in the search of "useful" strong Shoda triples. This has avoided many unnecessary computations. □

The Wedderburn components of $\mathbb{Q}G$ for the group G of Example 4 are matrix algebras over fields. Of course this does not occur always. In general, the Wedderburn components are equivalent to cyclotomic algebras, which `WedderburnDecomposition` presents as matrix algebras over crossed products. In this case it is difficult to use the output `WedderburnDecomposition` to describe the corresponding factors. The function `WedderburnDecompositionInfo` provides a numerical alternative, giving as output a list of tuples of length 2, 4 or 5, with numerical information describing the Wedderburn decomposition of the group algebra given as input. The tuples of length 5 are of the form

$$[n, \Bbbk, m, [o_i, \alpha_i, \beta_i]_{1 \le i \le l}, [\gamma_{ij}]_{1 \le i < j \le l}], \tag{1}$$

where $\Bbbk$ is a field and $n, k, m, o_i, \alpha_i > 0$ and $\beta_i, \gamma_{ij} \ge 0$ are integers. The data of (1) represents the matrix algebra $M_n(A)$ with A the cyclotomic algebra given by the following presentation:

$$A = \Bbbk(\xi_m)(g_1, \ldots, g_l | \xi_m^{g_i} = \xi_m^{\alpha_i}, g_i^{o_i} = \xi_m^{\beta_i}, g_j g_i = g_i g_j \xi_m^{\gamma_{ij}}, 1 \le i < j \le l). \tag{2}$$

The tuples of length 2 and 4 are simplified forms of the 5-tuples and take the forms $[n, \Bbbk]$ and $[n, \Bbbk, m, [o, \alpha, \beta]]$ respectively. They represent the matrix algebra $M_n(\Bbbk)$ and $M_n(A)$, where A has an interpretation as in (2) for $l = 1$.

In Example 4 each Wedderburn component is described using a unique strong Shoda triple. The next example shows one Wedderburn component which cannot be given by a unique strong Shoda triple.

Example 5. Consider the group $G = \langle x, y\rangle \rtimes \langle a, b\rangle$, where $\langle x, y\rangle = Q_8$, the quaternion group of order 8 and $\langle a, b\rangle$ is the group of order 27, with $a^9 = 1$, $a^3 = b^3$ and $ab = ba^4$. The action of a, b on $\langle x, y\rangle$ is given by $(x, a) = (y, a) = 1$, $x^b = y$ and $y^b = xy$. This is the small group $(216, 39)$ from the GAP library.

```
gap> G:=SmallGroup(216,39);;
gap>QG:=GroupRing(Rationals,G);;
gap> WedderburnDecompositionInfo(QG);
[ [ 1, Rationals ], [ 1, CF(3) ], [ 1, CF(3) ], [ 1, CF(3) ],
  [ 1, CF(3) ], [ 3, Rationals ], [ 3, CF(3) ], [ 3, CF(3) ],
  [ 3, CF(9) ], [ 1, Rationals, 4, [ 2, 3, 2 ] ],
  [ 1, CF(3), 12, [ 2, 7, 6 ] ], [ 1, CF(3), 12, [ 2, 7, 6 ] ],
  [ 1, CF(3), 12, [ 2, 7, 6 ] ], [ 1, CF(3), 4, [ 2, 3, 2 ] ],
  [ 1, CF(3), 36, [ 6, 31, 18 ] ] ]
```

Using (2) one obtains

$$\mathbb{Q}G = \mathbb{Q} \oplus 4\mathbb{Q}(\xi_3) \oplus M_3(\mathbb{Q}) \oplus 2M_3(\mathbb{Q}(\xi_3)) \oplus M_3(\mathbb{Q}(\xi_9)) \oplus A_1 \oplus 3A_2 \oplus A_3 \oplus A_4$$

where

$$\begin{aligned} A_1 &= \mathbb{Q}(\xi_4)[u : \xi_4^u = \xi_4^3, u^2 = \xi_4^2 = -1] \\ A_2 &= \mathbb{Q}(\xi_{12})[u : \xi_{12}^u = \xi_{12}^7, u^2 = \xi_{12}^6 = -1] \\ A_3 &= \mathbb{Q}(\xi_3)(\xi_4)[u : \xi_4^u = \xi_4^3, u^2 = \xi_4^2 = -1] \\ A_4 &= \mathbb{Q}(\xi_{36})[u : \xi_{36}^u = \xi_{36}^{31}, u^6 = \xi_{36}^{18} = -1] \end{aligned}$$

Let $\mathbb{H}(\Bbbk) = \Bbbk[i, j | i^2 = j^2 = -1, ji = -ij]$, the Hamiltonian quaternion algebra with center $\Bbbk$. Then $A_1 = \mathbb{H}(\mathbb{Q})$ and $A_2 = A_3 = \mathbb{H}(\mathbb{Q}(\xi_3))$. Moreover, using that -1 belongs to the image of the norm map $N_{\mathbb{Q}(\xi_3)/\mathbb{Q}}$ and known properties of cyclic algebras (see e.g. (Rei)) one has that $A_2 = A_3 \simeq M_2(\mathbb{Q}(\xi_3))$ and $A_4 = M_6(\mathbb{Q}(\xi_3))$.

Now we explain which are the strong Shoda triples that the program discovers and uses to describe the last Wedderburn component A_4. The simple algebra A_4 is $A(\chi, \mathbb{Q})$ where χ is one of the two ($\mathbb{Q}$-equivalent) characters of degree 6 of G. The field $\Bbbk = \mathbb{Q}(\chi)$ of character values of χ is $\mathbb{Q}(\xi_3)$. It turns out that, unlike in Example 4, the factor A_4 of $\mathbb{Q}G$ cannot be given by a unique strong Shoda triple able to cover both primes 2 and 3 in terms of satisfying condition $(*)$. Indeed, if such a strong Shoda triple (M, H, K) exists and θ is a character of M induced by (H, K), then (χ_M, θ) is coprime with 6 and $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\xi_3)$ because the exponent of G is 36 and $[\mathbb{Q}(\xi_{36}) : \Bbbk = \mathbb{Q}(\xi_3)] = 6$. The following computation shows that such a strong Shoda triple does not exist.

```
gap> chi:=Irr(G)[30];;
gap> ForAny(List(ConjugacyClassesSubgroups(G),Representative),
>     M->ForAny(StrongShodaPairs(M),
>         x->
>         Gcd(6,ScalarProduct( Restricted(chi,M) ,
>           LinCharByKernel(x[1],x[2])^M )) = 1 and
>         ForAll(List(ConjugacyClasses(M),Representative),
>           c -> c^(LinCharByKernel(x[1],x[2])^M) in CF(3) )
>        )
>     );
false
```

The function `LinCharByKernel` is a two argument function which applied to a pair (H, K) of groups with $K \trianglelefteq H$ and H/K cyclic, returns a linear character of H with kernel K.

The two strong Shoda triples of G obtained by the function `WedderburnDecomposition` to describe the 2 and 3-parts of A_4 are

$$(M_2 = \langle a, x, y\rangle, H_2 = \langle a, x\rangle, K_2 = \langle 1\rangle),$$
$$(M_3 = \langle a, x^2, a^2bxy\rangle, H_3 = \langle a^3, x^2, a^2bxy\rangle, K_3 = \langle a^2bxy\rangle).$$

The $2'$ and $3'$-parts of $\mathbb{Q}(\xi_{36})/\Bbbk$ are $L_2 = \mathbb{Q}(\xi_9)$ and $L_3 = \mathbb{Q}(\xi_{12})$, respectively. Following Propositions 1 the algorithm computes $A_{L_2}(M_2, H_2, K_2) = (\mathbb{Q}(\xi_{36})/\mathbb{Q}(\xi_9), \tau_2)$ and $A_{L_3}(M_3, H_3, K_3) = M_2(\mathbb{Q}(\xi_{12}))$ (the latter is equivalent to $(\mathbb{Q}(\xi_{12})/\mathbb{Q}(\xi_{12}), \tau_3 = 1)$). Then the algorithm inflates τ_2 and τ_3 to $\mathbb{Q}(\xi_{36})$, corestricts to $\mathbb{Q}(\xi_3)$ and computes the cocycle τ_χ as in steps $(3) - (6)$ of Algorithm 2. This gives rise to the numerical information `[ 1, CF(3), 36, [ 6, 31, 18 ] ] ]` obtained above. We have seen that the interpretation of this data is that A_4 is isomorphic to $M_6(\mathbb{Q}(\xi_3))$. This may have been obtained also noticing that $A_{L_2}(M_2, H_2, K_2) = \mathbb{H}(\mathbb{Q}(\xi_9)) \simeq M_2(\mathbb{Q}(\xi_9))$. Then the 2 and 3-parts of A_4 are trivial in the Brauer group, and so $A_4 \simeq M_6(\mathbb{Q}(\xi_3))$. $\square$

Final Remarks: The basic approach presented in this paper is still valid if F has positive characteristic provided FG is semisimple (i.e. the characteristic of F is coprime with the order of G) (see (Bro-Río) for the strongly monomial part). On the one hand we have only considered the zero characteristic case for simplicity. On the other hand the problem in positive characteristic is somehow simpler because the Wedderburn components of FG are split, that is they are matrices over fields.

The functionality of the package `wedderga` depends on the capacity of constructing fields in the GAP system. In practice `wedderga` can compute the Wedderburn decomposition of semisimple group algebras over finite abelian extensions of the rationals and finite fields.

Acknowledgements

The authors of this article would like to thank all the authors of `wedderga` for their contribution to the construction of the package.

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