ON PREPROJECTIVE SHORT EXACT SEQUENCES IN THE KRONECKER CASE

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Abstract. Let P, P' preprojective Kronecker modules (i.e. all their indecomposable components are preprojective). We give a numerical criterion in terms of so called Kronecker invariants for the existence of an epimorphism $g : P \to P'$. As an application we describe the possible middle terms in certain preprojective short exact sequences. We also prove that the possible middle terms in preprojective short exact sequences do not depend on the base field k. All the results above can be dualized for preinjective modules.

Key words. Kronecker algebra, preinjective, preprojective modules, matrix pencils, Kronecker invariants.

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1. Introduction

Let K be the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2$ and k a field. We will consider the path algebra kK of K over k (called Kronecker algebra) and the category mod-kK of finite dimensional right modules over kK (called Kronecker modules). The category mod-kK can and will be identified with the category rep-kK of the finite dimensional k-representations of the Kronecker quiver. Recall that such a representation is defined as a quadruple $(M_1, M_2; \alpha, \beta)$ where M_1, M_2 are finite dimensional k-vector spaces (corresponding to the vertices) and $\alpha, \beta : M_2 \to M_1$ are k-linear maps (corresponding to the arrows). The dimension vector of a module (viewed as a representation) $M = (M_1, M_2; \alpha, \beta) \in$ mod-kK is $\underline{dim} M = (\dim_k M_1, \dim_k M_2)$. Up to isomorphism we will have two simple objects in mod-kK corresponding to the two vertices. We shall denote them by S_1 and S_2. For a module $M \in$ mod-kK, $[M]$ will denote the isomorphism class of M and $tM := M \oplus ... \oplus M$ (t-times). For two modules $M, M' \in$ mod-kK we will denote by $M' \hookrightarrow M$ the fact that M' can be embedded in M (i.e. there is a monomorphism $M' \to M$) and by $M \twoheadrightarrow M'$ the fact that M projects on M' (i.e. there is an epimorphism $M \to M'$). For general notions concerning the representation theory of quivers, we refer to [2], [6] or [1].

The indecomposables in mod-kK are divided into three families: the preprojectives, the regulars and the preinjectives.

The preprojective and preinjective indecomposable modules are up to isomorphism uniquely determined by their dimension vectors and they are parametrized by $n \in \mathbb{N}$. We will denote by P_n the indecomposable preprojective module of dimension $(n+1, n)$. So P_0, P_1 are the projective indecomposable modules ($P_0 = S_1$ being simple). It is known that (up to isomorphism) $P_n = (k^{n+1}, k^n; \alpha, \beta)$ with bases $x_1, ..., x_n$ in k^n and $y_0, ..., y_n$ in k^{n+1} such that $\alpha(x_i) = y_{i-1}$ and $\beta(x_i) = y_i$. We will denote by I_n the indecomposable preinjective module of dimension $(n, n+1)$.

A preinjective (regular, preprojective) module is a module with all its indecomposable components preinjective (regular, preprojective). We will usually denote by P a preprojective, by R a regular and by I a preinjective module.

The defect of $M \in$ mod-kK with dimension vector (a, b) is defined in the Kronecker case as $\partial M := b - a$. Observe that if M is a preprojective (preinjective, respectively regular) indecomposable, then $\partial M = -1$

($\partial M = 1$, respectively $\partial M = 0$). Moreover for a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in mod-kK we have $\partial M_2 = \partial M_1 + \partial M_3$.

The following well known lemma summarizes some facts on morphisms and extensions in mod-kK.

Lemma 1.1. *We have:*

a) $\mathrm{Hom}(R,P) = \mathrm{Hom}(I,P) = \mathrm{Hom}(I,R) = \mathrm{Ext}^1(P,R) = \mathrm{Ext}^1(P,I) = \mathrm{Ext}^1(R,I) = 0$.

b) *For* $n \leq m$, *we have* $\dim_k \mathrm{Hom}(P_n, P_m) = m-n+1$ *and* $\mathrm{Ext}^1(P_n,P_m) = 0$; *otherwise* $\mathrm{Hom}(P_n,P_m) = 0$ *and* $\dim_k \mathrm{Ext}^1(P_n,P_m) = n-m-1$. *In particular* $\mathrm{End}(P_n) \cong k$ *and* $\mathrm{Ext}^1(P_n,P_n) = 0$.

An important fact is that Kronecker modules correspond to matrix pencils. Recall that a matrix pencil over a field k is a matrix $A+XB$ where A, B are matrices over k of the same size and X is an indeterminate. Two pencils $A+XB$, $A'+XB'$ are strictly equivalent, denoted by $A+XB \sim A'+XB'$, if and only if there exists invertible, constant (X independent) matrices P, Q such that $P(A'+XB')Q = A+XB$. A pencil $A'+XB'$ is called subpencil of $A+XB$ if and only if there are pencils $A_{12}+XB_{12}$, $A_{21}+XB_{21}$, $A_{22}+XB_{22}$ such that

$$A + XB \sim \begin{pmatrix} A' + XB' & A_{12} + XB_{12} \\ A_{21} + XB_{21} & A_{22} + XB_{22} \end{pmatrix}$$

Kronecker proved that pencils are uniquely determined up to strict equivalence by their classical Kronecker invariants, which are the minimal indices for columns, the minimal indices for rows, the finite elementary divisors, the infinite elementary divisors (see [3] for all the details).

There is an unsolved **Challenge** in pencil theory with lots of applications even outside of algebra (ex. Control Theory, see [5]):

Challenge: If $A+XB$, $A'+XB'$ are pencils over $\mathbb{C}$, find a necessary and sufficient condition in terms of their classical Kronecker invariants for $A'+XB'$ to be a subpencil of $A+XB$. Also construct the completion pencils $A_{12}+XB_{12}$, $A_{21}+XB_{21}$, $A_{22}+XB_{22}$.

Next we will translate all the terms above (taken from pencil theory) into the language of Kronecker modules (representations). Indeed one can easily see that a matrix pencil $A+XB \in M_{m,n}(k[X])$ corresponds to the Kronecker module $M_{A,B} = (k^m, k^n; \alpha_A, \alpha_B)$ where choosing the canonical basis in k^n and k^m the matrix of $\alpha_A : k^n \to k^m$ (respectively of $\alpha_B : k^n \to k^m$) is A (respectively B). The strict equivalence $A+XB \sim A'+XB'$ means the isomorphism of modules $M_{A,B} \cong M_{A',B'}$. It follows easily that a pencil $A'+XB'$ is a subpencil of $A+XB$ if and only if the module $M_{A',B'}$ is a subfactor of $M_{A,B}$ (i.e. there is a module N such that $M_{A',B'} \twoheadleftarrow N \hookrightarrow M_{A,B}$ or equivalently there is a module L such that $M_{A,B} \twoheadrightarrow L \hookleftarrow M_{A',B'}$, see [4] Proposition 1. for details). A preprojective module $P_{d_1} \oplus ... \oplus P_{d_m}$, where $(d_1, ..., d_m)$ is a finite increasing sequence of non-negative integers corresponds to the pencil with the following classical Kronecker invariants:

- minimal indices for rows: $d_1, ..., d_m$;
- no minimal indices for columns, no finite elementary divisors, no infinite elementary divisors.

So pencils with only minimal indices for rows corresponds to preprojective modules. Dually, pencils with only minimal indices for columns correspond to preinjective modules.

Let M, M' be Kronecker modules. Having in mind the **Challenge** above and its modular translation it is a natural question to find a numerical criterion in terms of Kronecker invariants for $M \twoheadrightarrow M'$ (and dually $M' \hookrightarrow M$). In the first part of the paper we will give this numerical criterion for $P \twoheadrightarrow P'$ (where P', P are preprojectives) working over an arbitrary field k and using a purely modular approach. Dually one gets the numerical criterion for $I' \hookrightarrow I$ (where I', I are preinjectives). Note that a different criterion was given by Han Yang in [4] working over an algebraically closed field. He uses calculation of ranks of matrices over polynomial rings and a so called generalization and specialization approach.

As an application we describe the possible middle terms in certain preprojective short exact sequences. At the end we prove that the possible middle terms in preprojective short exact sequences do not depend on the base field k. Again all these results can be formulated dually for preinjective modules.

2. **Epimorphisms between preprojectives**

Our aim is to give a numerical criterion in terms of Kronecker invariants for $P \twoheadrightarrow P'$ (where P', P are preprojectives). Dually we obtain a numerical criterion for $I' \hookrightarrow I$ (where I', I are preinjectives).

We begin with two (dual) lemmas which permits us to split the "smaller" module P'.

Lemma 2.1. *Let N_1, N_2, M_1, M_2 be finite dimensional right modules over the Kronecker algebra kK (where k is a field) such that* $\mathrm{Ext}^1(N_1, N_2) = 0$ *and* $\mathrm{Hom}(M_2, N_1) = 0$*. Then there exists an exact sequence of the form*

$$0 \to Y \to M_1 \oplus M_2 \to N_1 \oplus N_2 \to 0$$

if and only if there is a module X with exact sequences

$$0 \to X \to M_1 \to N_1 \to 0$$

$$0 \to Y \to X \oplus M_2 \to N_2 \to 0.$$

Proof. "$\Rightarrow$" Suppose we have an exact sequence

$$0 \longrightarrow Y \longrightarrow M_1 \oplus M_2 \xrightarrow{\begin{pmatrix} u & v \\ w & t \end{pmatrix}} N_1 \oplus N_2 \longrightarrow 0\ .$$

Since $\mathrm{Hom}(M_2, N_1) = 0$ then $v = 0$ and u is an epimorphism with kernel denoted by X so we have an exact sequence

$$0 \longrightarrow X \longrightarrow M_1 \xrightarrow{u} N_1 \longrightarrow 0\ .$$

Consider now the direct sum of this exact sequence with the trivial one

$$0 \to M_2 \to M_2 \to 0 \to 0$$

so we get the exact sequence

$$0 \longrightarrow X \oplus M_2 \longrightarrow M_1 \oplus M_2 \xrightarrow{(u\ 0)} N_1 \longrightarrow 0\ .$$

Finally we can construct the following commutative diagram (with exact rows and columns) the first column being our second desired exact sequence.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & = & Y & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \oplus M_2 & \longrightarrow & M_1 \oplus M_2 & \xrightarrow{(u\ 0)} & N_1 & \longrightarrow & 0 \\
 & & \vdots\ \mathrm{ker} & & \downarrow \begin{pmatrix} u & 0 \\ w & t \end{pmatrix} & & \| & & \\
0 & \longrightarrow & N_2 & \longrightarrow & N_1 \oplus N_2 & \xrightarrow{(1\ 0)} & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

"$\Leftarrow$" Suppose we have two exact sequences of the form

$$0 \to X \to M_1 \to N_1 \to 0$$

and

$$0 \to Y \to X \oplus M_2 \to N_2 \to 0.$$

Consider now the direct sum of the first one with the trivial one

$$0 \to M_2 \to M_2 \to 0 \to 0$$

so we get the exact sequence

$$0 \longrightarrow X \oplus M_2 \longrightarrow M_1 \oplus M_2 \longrightarrow N_1 \longrightarrow 0\ .$$

Then we can construct the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & = & Y & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X \oplus M_2 & \longrightarrow & M_1 \oplus M_2 & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N_2 & \longrightarrow & Po & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $Po \in \operatorname{Ext}^1(N_1, N_2) = 0$ so $Po \cong N_1 \oplus N_2$ and we are done (the desired exact sequence being the middle column). □

Dually we have

Lemma 2.2. *Let N_1, N_2, M_1, M_2 be finite dimensional right modules over the Kronecker algebra kK (where k is a field) such that $\operatorname{Ext}^1(N_1, N_2) = 0$ and $\operatorname{Hom}(N_2, M_1) = 0$. Then there exists an exact sequence of the form*

$$0 \to N_1 \oplus N_2 \to M_1 \oplus M_2 \to Y \to 0$$

if and only if there is a module X with exact sequences

$$0 \to N_2 \to M_2 \to X \to 0$$

$$0 \to N_1 \to M_1 \oplus X \to Y \to 0.$$

The following lemma gives the criterion for the existence of an epimorphism $f : P_{n_1} \oplus \cdots \oplus P_{n_p} \to P_n$ with $n \geq n_p \geq \cdots \geq n_1 \geq 0$ in $\mod kK$ where k is an arbitrary field.

Lemma 2.3. *Let $n \geq n_p \geq \cdots \geq n_1 \geq 0$ be integers. Then there exists an epimorphism $f : P_{n_1} \oplus \cdots \oplus P_{n_p} \to P_n$ iff $\sum_{i=1}^{p} n_i \geq n$. Moreover in this case $\operatorname{Ker} f \cong P_{m_1} \oplus ... \oplus P_{m_{p-1}}$ where $n \geq m_{p-1} \geq \cdots \geq m_1 \geq 0$.*

Proof. The only if part follows directly from looking at the dimension vectors. For the if part write $P_n = (k^{n+1}, k^n; \alpha, \beta)$ with bases $x_1, ..., x_n$ in k^n and $y_0, ..., y_n$ in k^{n+1} such that $\alpha(x_i) = y_{i-1}$ and $\beta(x_i) = y_i$. Let $m_j = \sum_{i=1}^{j-1} n_j$ (so $m_1 = 0$). We can assume that $m_p < n$. For $1 \leq j < p$ consider the subrepresentation $U_j = (\langle y_{m_j}, ..., y_{m_j+n_j}\rangle, \langle x_{m_j+1}, ..., x_{m_j+n_j}\rangle, \alpha, \beta)$ of P_n and let $U_p = (\langle y_{n-n_p}, ..., y_n\rangle, \langle x_{n-n_p+1}, ..., x_n\rangle, \alpha, \beta)$. Then U_i is isomorphic to P_{n_i} for $i = \overline{1,p}$ and obviously $\sum_{i=1}^{p} U_i = P_n$ (since $m_p \geq n - n_p$). The natural map $U_1 \oplus \cdots \oplus U_p \to \sum_{i=1}^{p} U_i$ is then our epimorphism.

The conditions on $\operatorname{Ker} f$ are easily verified. Since $\operatorname{Ker} f$ embeds into $P_{n_1} \oplus \cdots \oplus P_{n_p}$ it follows by Lemma 1.1 a) that $\operatorname{Ker} f$ is preprojective and moreover its defect $\partial \operatorname{Ker} f = -(p-1)$ so $\operatorname{Ker} f \cong P_{m_1} \oplus ... \oplus P_{m_{p-1}}$ with $m_{p-1} \geq \cdots \geq m_1 \geq 0$. If $m_{p-1} > n$ then $P_{m_{p-1}}$ embeds into $P_{n_1} \oplus \cdots \oplus P_{n_p}$ where $m_{p-1} > n \geq n_p \geq \cdots \geq n_1 \geq 0$, so we contradict Lemma 1.1 b). □

We are ready now to give the numerical criterion for the existence of an epimorphism $f : P \to P'$ where P, P' are preprojectives.

Theorem 2.4. *Suppose $d_n \geq ... \geq d_1 > 0$ and $c_m \geq ... \geq c_1 > 0$ are integers. We have an epimorphism*

$$f : cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m} \to dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n}$$

iff $d \leq c$ and $d_1 + ... + d_i \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1,n}$ (the empty sum being 0).

Proof. Using Lemma 1.1. a) one can see that the existence of an epimorphism f is equivalent with the existence of a short exact sequence

$$0 \to X \to cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m} \to dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n} \to 0,$$

with X preprojective. Using Lemma 1.1. b) and Lemma 2.1. inductively this is equivalent with the existence of the exact sequences

$$(0) \;\; 0 \to X_0 \to cP_0 \to dP_0 \to 0$$
$$(1) \;\; 0 \to X_1 \to X_0 \oplus \bigoplus_{c_j \leq d_1} P_{c_j} \to P_{d_1} \to 0$$
$$(2) \;\; 0 \to X_2 \to X_1 \oplus \bigoplus_{d_1 < c_j \leq d_2} P_{c_j} \to P_{d_2} \to 0$$
$$\cdots$$
$$(n-1) \;\; 0 \to X_{n-1} \to X_{n-2} \oplus \bigoplus_{d_{n-2} < c_j \leq d_{n-1}} P_{c_j} \to P_{d_{n-1}} \to 0$$
$$(n) \;\; 0 \to X \to X_{n-1} \oplus \bigoplus_{d_{n-1} < c_j} P_{c_j} \to P_{d_n} \to 0,$$

with $X_0 \cong (c-d)P_0$ and X_i a preprojective module with indecomposable components P_m satisfying $m \leq d_i \leq d_{i+1}$.

"$\Rightarrow$" Suppose we have the exact sequences (0),...,(n). We notice that the second component of the dimension vector of X_i is $(\sum_{c_j \leq d_i} c_j) - d_1 - ... - d_i$. Looking again at the second component of the dimension vectors it follows that $d \leq c$ (from (0)), $d_1 \leq \sum_{c_j \leq d_1} c_j$ (from (1)), $d_2 \leq (\sum_{c_j \leq d_2} c_j) - d_1$ (from (2)),..., $d_{n-1} \leq (\sum_{c_j \leq d_{n-1}} c_j) - d_1 - ... - d_{n-2}$ (from (n-1)). In the exact sequence (1) due to Lemma 1.1. b) we have that $X_{n-1} \oplus \bigoplus_{d_{n-1} < c_j \leq d_n} P_{c_j}$ projects on P_{d_n} so we have $d_n \leq (\sum_{c_j \leq d_n} c_j) - d_1 - ... - d_{n-1}$.

"$\Leftarrow$" Suppose that $d \leq c$ and $d_1 + ... + d_i \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1,n}$. We need to construct short exact sequences of the form (0),...,(n). Since $d \leq c$ we have an exact sequence

$$(0) \;\; 0 \to X_0 \to cP_0 \to dP_0 \to 0,$$

with $X_0 \cong (c-d)P_0$.

Since $d_1 \leq \sum_{c_j \leq d_1} c_j$ using Lemma 2.3. we have an exact sequence

$$(1)\quad 0 \to X_1 \to X_0 \oplus \bigoplus_{c_j \leq d_1} P_{c_j} \to P_{d_1} \to 0$$

with X_1 preprojective with the second component of its dimension $(\sum_{c_j \leq d_1} c_j) - d_1$ and with indecomposable components P_m satisfying $m \leq d_1 \leq d_2$.

Since $d_2 \leq (\sum_{c_j \leq d_2} c_j) - d_1$ using Lemma 2.3. we have an exact sequence

$$(2)\quad 0 \to X_2 \to X_1 \oplus \bigoplus_{d_1 < c_j \leq d_2} P_{c_j} \to P_{d_2} \to 0,$$

with X_2 preprojective with the first component of its dimension $(\sum_{c_j \leq d_2} c_j) - d_1 - d_2$ and with indecomposable components P_m satisfying $m \leq d_2 \leq d_3$.

Continuing in this way we can construct the exact sequences (3),...,(n-1) where X_{n-1} is a preprojective with the first component of its dimension $(\sum_{c_j \leq d_{n-1}} c_j) - d_1 - \cdots - d_{n-1}$ and with indecomposable components P_m satisfying $m \leq d_{n-1} \leq d_n$. Since $d_n \leq (\sum_{c_j \leq d_n} c_j) - d_1 - \cdots - d_{n-1}$ using Lemma 2.3. we have an exact sequence

$$0 \to X_n \to X_{n-1} \oplus \bigoplus_{d_{n-1} < c_j \leq d_n} P_{c_j} \to P_{d_n} \to 0.$$

Consider now the direct sum of this sequence with the trivial one

$$0 \to \bigoplus_{d_n < c_j} P_{c_j} \to \bigoplus_{d_n < c_j} P_{c_j} \to 0 \to 0$$

so we get the exact sequence

$$(n)\quad 0 \to X_n \oplus \bigoplus_{d_n < c_j} P_{c_j} \to X_{n-1} \oplus \bigoplus_{d_{n-1} < c_j} P_{c_j} \to P_{d_n} \to 0.$$

□

Remark 2.5. Using the notation $P' = (a_0 P_0) \oplus ... \oplus (a_n P_n) \oplus ...$, $P = (b_0 P_0) \oplus ... \oplus (b_n P_n) \oplus ...$ we have an epimorphism $f : P \to P'$ iff

$$a_0 \leq b_0$$

$$a_1 \leq b_1$$

$$a_1 + 2a_2 \leq b_1 + 2b_2$$

$$...$$

$$a_1 + 2a_2 + ... + na_n \leq b_1 + 2b_2 + ... + nb_n$$

$$...$$

So one can see that in the preprojective case "a kind of" weighted dominance describes the numerical criterion for the embedding.

Theorem 2.4. can be easily dualized for preinjectives.

Theorem 2.6. *Suppose $d_1 \geq ... \geq d_n > 0$ and $c_1 \geq ... \geq c_m > 0$ are integers. We have a monomorphism*

$$f : I_{d_1} \oplus ... \oplus I_{d_n} \oplus dI_0 \to I_{c_1} \oplus ... \oplus I_{c_m} \oplus cI_0$$

iff $d \leq c$ and $d_i + ... + d_n \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1, n}$ (the empty sum being 0).

3. Middle terms in preprojective short exact sequences

Applying Theorem 2.4. we describe first the possible middle terms in certain preprojective short exact sequences.

The proposition below is well-known.

Proposition 3.1. *If P', P'' are preprojectives and Y is a middle term in $\mathrm{Ext}^1(P', P'')$ then Y is also preprojective.*

Corollary 3.2. *Suppose $d_n \geq ... \geq d_1 > 0$, $c_m \geq ... \geq c_1 > 0$ and $a > 0$ are integers. Then we have:*

a) *$cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m}$ is a middle term in $\mathrm{Ext}^1(dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n}, P_a)$ iff $m + c = n + d + 1$, $\sum_{i=1}^m c_i = a + \sum_{j=1}^n d_j$, $d \leq c$ and $d_1 + ... + d_i \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1, n}$.*

b) *$cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m}$ is a middle term in $\mathrm{Ext}^1(dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n}, aP_0)$ iff $m + c = n + d + a$, $\sum_{i=1}^m c_i = \sum_{j=1}^n d_j$, $d \leq c$ and $d_1 + ... + d_i \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1, n}$.*

Proof. a) "$\Rightarrow$" We have an exact sequence

$$0 \to P_a \to cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m} \to dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n} \to 0.$$

Applying Theorem 2.4. and looking at the dimensions and defects the assertion follows.

"$\Leftarrow$" Using Theorem 2.4. the condition $d \leq c$ and $d_1 + ... + d_i \leq \sum_{c_j \leq d_i} c_j$ for $i = \overline{1, n}$ implies the existence of an exact sequence

$$0 \to X \to cP_0 \oplus P_{c_1} \oplus ... \oplus P_{c_m} \to dP_0 \oplus P_{d_1} \oplus ... \oplus P_{d_n} \to 0,$$

with X preprojective. From $m+c = n+d+1$ it follows that $\partial X = -1$ (so X is indecomposable preprojective) and from $\sum_{i=1}^m c_i = a + \sum_{j=1}^n d_j$ that $\underline{\dim} X = (a + 1, a)$. So $X \cong P_a$.

b) See a). □

We are ready now to prove that the possible middle terms in preprojective short exact sequences do not depend on the base field k. More precisely denote now by P_n^k the preprojective indecomposable in $\mod kK$ of dimension $(n + 1, n)$. Then we have that

Theorem 3.3. *Suppose $d_n \geq ... \geq d_1 \geq 0$, $c_m \geq ... \geq c_1 \geq 0$ and $e_p \geq ... \geq e_1 \geq 0$ are integers, k, k' are fields and we have the short exact sequence in $\mod kK$*

$$0 \to P_{e_1}^k \oplus ... \oplus P_{e_p}^k \to P_{c_1}^k \oplus ... \oplus P_{c_m}^k \to P_{d_1}^k \oplus ... \oplus P_{d_n}^k \to 0.$$

Then we have a similar short exact sequence in $\mod k'K$

$$0 \to P_{e_1}^{k'} \oplus ... \oplus P_{e_p}^{k'} \to P_{c_1}^{k'} \oplus ... \oplus P_{c_m}^{k'} \to P_{d_1}^{k'} \oplus ... \oplus P_{d_n}^{k'} \to 0.$$

Proof. We proceed by induction on p. By Corollary 3.2. for $p = 1$ the existence of an exact sequence

$$0 \to P_{e_1}^k \to P_{c_1}^k \oplus ... \oplus P_{c_m}^k \to P_{d_1}^k \oplus ... \oplus P_{d_n}^k \to 0$$

is equivalent with a field independent numerical criterion depending on $d_1, ..., d_n, c_1, ..., c_m, e_1$. So in this case we are done. Suppose now that $p > 1$ and the assertion is true for $p - 1$. Using Lemma 2.2. with $Y = P_{d_1}^k \oplus ... \oplus P_{d_n}^k$, $M_2 = P_{c_1}^k \oplus ... \oplus P_{c_m}^k$, $M_1 = 0$, $N_1 = P_{e_1}^k$, $N_2 = P_{e_2}^k \oplus ... \oplus P_{e_p}^k$ from the exact sequence

$$0 \to P_{e_1}^k \oplus ... \oplus P_{e_p}^k \to P_{c_1}^k \oplus ... \oplus P_{c_m}^k \to P_{d_1}^k \oplus ... \oplus P_{d_n}^k \to 0.$$

we obtain the exact sequences

$$0 \to P_{e_2}^k \oplus ... \oplus P_{e_p}^k \to P_{c_1}^k \oplus ... \oplus P_{c_m}^k \to X^k \to 0$$

$$0 \to P^k_{e_1} \to X^k \to P^k_{d_1} \oplus ... \oplus P^k_{d_n} \to 0.$$

Due to Proposition 3.1. we have that X^k is preprojective so $X^k = P^k_{a_1} \oplus ... \oplus P^k_{a_l}$ but then by our induction hypothesis we have the similar exact sequences in $\mod k'K$

$$0 \to P^{k'}_{e_2} \oplus ... \oplus P^{k'}_{e_p} \to P^{k'}_{c_1} \oplus ... \oplus P^{k'}_{c_m} \to P^{k'}_{a_1} \oplus ... \oplus P^{k'}_{a_l} \to 0$$

$$0 \to P^{k'}_{e_1} \to P^{k'}_{a_1} \oplus ... \oplus P^{k'}_{a_l} \to P^{k'}_{d_1} \oplus ... \oplus P^{k'}_{d_n} \to 0.$$

Using again Lemma 2.2. we obtain an exact sequence

$$0 \to P^{k'}_{e_1} \oplus ... \oplus P^{k'}_{e_p} \to P^{k'}_{c_1} \oplus ... \oplus P^{k'}_{c_m} \to P^{k'}_{d_1} \oplus ... \oplus P^{k'}_{d_n} \to 0.$$

□

We remark that all the results of this section can be easily dualized for preinjective modules.

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