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STRONG CONVERGENCE THEOREM FOR A GENERAL SYSTEM OF VARIATIONAL INEQUALITIES, EQUILIBRIUM PROBLEMS, AND FIXED POINT PROBLEMS

MARYAM YAZDI* AND SAEED HASHEMI SABABE**

*Young Researchers and Elite Club, Malard Branch,
Islamic Azad University, Malard, Iran
E-mail: Msh_yazdi@yahoo.com; M_yazdi@iaumalard.ac.ir

**Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada
and Young Researchers and Elite Club, Malard Branch, Islamic Azad University, Malard, Iran.
E-mail: Hashemi_1365@yahoo.com, S.Hashemi@ualberta.ca

Abstract. A new iterative scheme is proposed for finding a common element of the solution set of a general system of variational inequalities, the solution set of an equilibrium problem and the common fixed point set of a countable family of nonexpansive mappings in a real Hilbert space. Under some suitable conditions imposed on the parameters, a strong convergence theorem is proved. Moreover, a numerical result is given to show the effectiveness of the scheme.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $Fix(T)$ to denote the set of fixed points T, i.e., $Fix(T) = \{x \in C : Tx = x\}$. Also, a contraction on C is a self-mapping f of C such that $\|f(x) - f(y)\| \leq \kappa \|x - y\|$ for all $x, y \in C$ and some constant $\kappa \in [0, 1)$. In this case f is said to be a κ-contraction.

Consider an equilibrium problem (EP) which is to find a point $u \in C$ satisfying the property:

$$\phi(x, y) \geq 0 \quad \text{for all } y \in C, \tag{1.1}$$

where $\phi : C \times C \to \mathbb{R}$ is a bifunction of C. We use $EP(\phi)$ to denote the set of solutions of EP (1.1), that is, $EP(\phi) = \{x \in C : (1.1) \text{ holds}\}$. The EP (1.1) includes, as special cases, numerous problems in physics, optimization and economics. Some authors (e.g., [19, 21, 20, 23, 25, 24, 26, 27]) have proposed some useful methods for solving the EP (1.1). Set $\phi(x,y) = \langle Ax, y-x\rangle$ for all $x,y \in C$, where $A: C \to H$ is a nonlinear mapping. Then, $x^* \in EP(\phi)$ if and only if

$$\langle Ax^*, y-x^*\rangle \geq 0 \quad \text{for all } y \in C, \tag{1.2}$$

that is, x^* is a solution of the variational inequality. The (1.2) is well known as the classical variational inequality. The set of solutions of (1.2) is denoted by $VI(A,C)$.

In 2008, Ceng et al. [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ satisfying

$$\begin{cases} \langle \nu Ay^* + x^* - y^*, x - x^*\rangle \geq 0 & \text{for all } x \in C \\ \langle \mu Bx^* + y * -x^*, x - y^*\rangle \geq 0 & \text{for all } x \in C, \end{cases} \tag{1.3}$$

which is called a general system of variational inequalities, where $A, B : C \to H$ are two nonlinear mappings, $\lambda > 0$ and $\mu > 0$ are two fixed constants. Precisely, they introduced the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n), \end{cases}$$

and obtained strong convergence theorem.

Recently, Cai et al. [3] introduced the following modified viscosity implicit rules

$$\begin{cases} x_1 \in C \\ u_n = s_n x_n + (1-s_n)y_n \\ z_n = P_C(I - \mu B)u_n, \\ y_n = P_C(I - \lambda A)z_n, \\ x_{n+1} = P_C(\alpha_n f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \rho F)Ty_n), \quad n \geq 1, \end{cases}$$

where F is a Lipschitzian and strongly monotone map. Under some suitable assumptions imposed on the parameters, they obtained some strong convergence theorems.

In this paper, motivated by the above results, we propose a new composite iterative scheme for finding a common element of the set of solutions of a general system of variational inequalities, an equilibrium problem and the set of common fixed points of a countable family of nonexpansive mappings in Hilbert spaces. A numerical example is given for supporting our main result.

2. Preliminaries

Let H be a real Hilbert space. We use $\rightharpoonup$ and $\to$ to denote the weak and strong convergence in H, respectively. The following identity holds:

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x-y\|^2,$$

for all $x, y \in H$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and satisfies

$$\langle x-y, P_C(x)-P_C(y)\rangle\| \geq \|P_Cx-P_Cy\|^2 \quad \text{for all } x,y\in H. \tag{2.1}$$

Further, for $x \in H$ and $z \in C$, we have

$$z = P_C(x) \iff \langle x-z, z-y\rangle \geq 0 \quad \text{for all } y \in C.$$

Definition 2.1. A mapping $T : H \to H$ is called firmly nonexpansive if for any $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Lemma 2.2. [2] *Let C be a nonempty closed convex subset of H and $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

- (A_1) *$\phi(x, x) = 0$ for all $x \in C$;*
- (A_2) *ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;*
- (A_3) *for each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y)$;*
- (A_4) *for each $x \in C, y \mapsto \phi(x, y)$ is convex and weakly lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$\phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq\ 0, \quad \textit{for all } y \in C.$$

Lemma 2.3. [15] *Assume $\phi : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)-(A_4). For $r > 0$, define a mapping $Q_r : H \to C$ by*

$$Q_r x := \{z \in C : \phi(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \quad \textit{for all } y \in C\} \tag{2.2}$$

for all $x \in H$. Then, the following hold:

- *(i) Q_r is single-valued;*
- *(ii) Q_r is firmly nonexpansive;*
- *(iii) $Fix(Q_r) = EP(\phi)$;*
- *(iv) $EP(\phi)$ is closed and convex.*

Definition 2.4. A nonlinear operator A with domain $D(A) \subseteq H$ and range $R(A) \subseteq H$ is said to be $\alpha-$ inverse strongly monotone (for short, $\alpha-$ism) if there exists $\nu > 0$ such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \quad \text{for all } x,y \in D(A).$$

Lemma 2.5. [16] *Let C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 2.6. [1] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n v_n + \mu_n,$$

where $\{\gamma_n\}$ is a sequence in $[0, 1]$, $\{\mu_n\}$ a sequence of nonnegative real numbers, and $\{v_n\}$ a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} \gamma_n = \infty, \limsup_{n\to\infty} v_n \leq 0$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. [12] *For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of problem (1.3) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C(P_C(x - \mu Bx) - \nu AP_C(x - \mu Bx)) \quad \text{for all } x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

3. Main result

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 1$, define a mapping W_n of H into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\
&\vdots \\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\
&\vdots \\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I.
\end{aligned} \tag{3.1}$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$; see [18].

Lemma 3.1. [22] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 3.1, one can define mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \tag{3.2}$$

for every $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Throughout this paper, we assume $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$.

Lemma 3.2. [22] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, $Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$.*

Theorem 3.3. *Let C be a closed convex subset of H, $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A_1)-(A_4)$ of Lemma 2.2, $A, B : C \to H$ be α-ism and β-ism, respectively, and f a κ-contraction on C for some $\kappa \in [0, 1)$. Set $\Omega := \cap_{n=1}^{\infty} Fix(T_n) \cap Fix(G) \cap EP(\phi)$ and assume $\Omega \neq \emptyset$. Suppose $\{\alpha_n\}$ and $\{r_n\}$ are real sequences satisfying the following conditions:*

(B_1) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B_2) $\{r_n\} \subset (a, \infty)$ *for some* $a > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Let $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all} \;\; y \in C, \\ z_n = P_C(I - \mu B)u_n, \\ y_n = P_C(I - \nu A)z_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)W_n y_n, \quad\; n \geq 0, \end{cases} \tag{3.3}$$

where the initial guess $x_0 \in C$ *is arbitrary,* $\nu \in (0, 2\alpha)$*, and* $\mu \in (0, 2\beta)$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $q \in \Omega$*, where* $q = P_\Omega f(q)$*, which solves the following variational inequality (VI):*

$$\langle (I - f)q, q - x\rangle \leq 0 \quad \text{for all } x \in \Omega. \tag{3.4}$$

Proof. Since $P_\Omega f$ is a contractive self-mapping on C, there exists a unique element $q \in \Omega$ such that $q = P_\Omega f(q)$; equivalently, q is the unique solution of VI (3.4). Also, observe that $u_n = Q_{r_n} x_n$, where Q_r is defined by (2.2). For $x, y \in C$, we have

$$\begin{aligned} \|(I-\nu A)x - (I-\nu A)y\|^2 =& \|x - y - \nu(Ax - Ay)\|^2 \\ =& \|x-y\|^2 - 2\nu\langle x - y, Ax - Ay\rangle \\ & \quad + \nu^2\|Ax - Ay\|^2 \\ \leq& \|x-y\|^2 - 2\alpha\nu\|Ax - Ay\|^2 + \nu^2\|Ax - Ay\|^2 \\ =& \|x-y\|^2 + \nu(\nu - 2\alpha)\|Ax - Ay\|^2 \\ \leq& \|x - y\|^2, \end{aligned} \tag{3.5}$$

by $\nu \in (0, 2\alpha)$. This implies $I - \nu A$ is nonexpansive. In the following six steps, we can show $I - \mu B$ is also nonexpansive in a similar way.

Step 1: First, we claim $\{x_n\}$ and $\{u_n\}$ are bounded. Suppose that $x^* \in \Omega$ and $y^* = P_C(x^* - \mu B x^*)$. Noticing $u_n = Q_{r_n} x_n$ and $Q_{r_n} x^* = x^*$, we get

$$\|u_n - x^*\| \leq \|x_n - x^*\|.$$

Then

$$\|y_n - x^*\| = \|Gu_n - x^*\| = \|Gu_n - Gx^*\| \leq \|x_n - x^*\|. \tag{3.6}$$

From (3.3), we get

$$\begin{aligned} \|x_{n+1} - x^*\| \leq& \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(W_n y_n - x^*)\| \\ \leq& \alpha_n(\|f(x_n) - f(x^*)\| + \|f(x^*) - x^*\|) + (1-\alpha_n)\|y_n - x^*\| \\ \leq& \alpha_n\kappa\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ \leq& (1 - (1-\kappa)\alpha_n)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ \leq& \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\kappa}\right\}. \end{aligned}$$

By induction,

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\kappa}\right\} \text{ for all } n \ge 1.$$

Hence $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{f(x_n)\}$ and $\{W_n y_n\}$. Set

$$M_1 = \sup\left\{\|f(x_n)\|, \|W_n y_n\|, \frac{1}{a}\|u_n - x_n\| : n \in \mathbb{N}\right\}.$$

Step 2: We claim $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. By the definition of $\{x_n\}$, we have

$$\begin{aligned}
&\|x_{n+1} - x_{n+2}\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)W_n y_n - \alpha_{n+1} f(x_{n+1}) - (1-\alpha_{n+1})W_{n+1}y_{n+1}\| \\
&= \|\alpha_n(f(x_n) - f(x_{n+1})) + (\alpha_n - \alpha_{n+1})f(x_{n+1}) \\
&\quad + (1-\alpha_n)(W_n y_n - W_{n+1}y_{n+1}) + (\alpha_{n+1} - \alpha_n)W_{n+1}y_{n+1}\| \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| + (1-\alpha_n)(\|W_n y_n - W_{n+1}y_n\| \\
&\quad + \|W_n y_n - W_n y_{n+1}\|) \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| + (1-\alpha_n)(\|W_n y_n - W_{n+1}y_n\| \\
&\quad + \|y_n - y_{n+1}\|), \\
&\le \alpha_n\kappa\|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| + (1-\alpha_n)(\|W_n y_n - W_{n+1}y_n\| \\
&\quad + \|u_n - u_{n+1}\|),
\end{aligned} \tag{3.7}$$

for all $n \in \mathbb{N}$. From (3.1), since T_i and $U_{n,i}$ are nonexpansive, we obtain

$$\begin{aligned}
\|W_{n+1}y_n - W_n y_n\| =& \|\lambda_1 T_1 U_{n+1,2}y_n - \lambda_1 T_1 U_{n,2}y_n\| \\
\le& \lambda_1\|U_{n+1,2}y_n - U_{n,2}y_n\| \\
=& \lambda_1\|\lambda_2 T_2 U_{n+1,3}y_n - \lambda_2 T_2 U_{n,3}y_n\| \\
\le& \lambda_1\lambda_2\|U_{n+1,3}y_n - U_{n,3}y_n\| \\
\le& \dots \\
\le& \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \\
\le& M_2\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{3.8}$$

where $M_2 \ge 0$ is a constant such that $\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\| \le M_2$ for all $n \ge 0$. Let $u_n = Q_{r_n}x_n$ and $u_{n+1} = Q_{r_{n+1}}x_{n+1}$. So

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \quad \text{for all } y \in C \tag{3.9}$$

and

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0 \quad \text{for all } y \in C. \tag{3.10}$$

Set $y = u_{n+1}$ in (3.9) and $y = u_n$ in (3.10). Then by adding these two inequalities and using (A_2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_n - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

This implies

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \frac{1}{a}|r_n - r_{n+1}|\|u_{n+1} - x_{n+1}\| \right\}. \end{aligned}$$

Therefore

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| M_1. \tag{3.11}$$

Substituting (3.8) and (3.11) into (3.7), we get

$$\begin{aligned} \|x_{n+1} - x_{n+2}\| &\leq \alpha_n \kappa \|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| \\ &\quad + (1 - \alpha_n)\left(M_2 \prod_{i=1}^{n} \lambda_i + \|x_{n+1} - x_n\| + |r_{n+1} - r_n| M_1 \right) \\ &\leq (1 - (1-\kappa)\alpha_n)\|x_n - x_{n+1}\| + 2M_1|\alpha_n - \alpha_{n+1}| + M_2 b^n \\ &\quad + |r_{n+1} - r_n| M_1, \end{aligned}$$

for all $n \in \mathbb{N}$. So, from Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

Step 3: We claim $\lim_{n\to\infty} \|y_n - u_n\| = 0$. By (3.5), we have

$$\begin{aligned} \|z_n - y^*\|^2 &= \|P_C(I - \mu B)u_n - P_C(I - \mu B)x^*\|^2 \\ &\leq \|(I - \mu B)u_n - (I - \mu B)x^*\|^2 \\ &\leq \|u_n - x^*\|^2 - \mu(2\beta - \mu)\|Bu_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \mu(2\beta - \mu)\|Bu_n - Bx^*\|^2. \end{aligned} \tag{3.13}$$

In a similar way, we get

$$\|y_n - x^*\|^2 \leq \|z_n - y^*\|^2 - \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2. \tag{3.14}$$

Substituting (3.13) into (3.14), we obtain

$$\begin{aligned} \|y_n - x^*\|^2 \leq& \|x_n - x^*\|^2 - \mu(2\beta - \mu)\|Bu_n - Bx^*\|^2 \\ &- \nu(2\alpha - \nu)\|Az_n - Ay^*\|^2. \end{aligned} \tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(W_ny_n-x^*)\|^2\\
&\leq \alpha_n\|f(x_n)-f(x^*)+f(x^*)-x^*\|^2\\
&\quad+(1-\alpha_n)\|W_ny_n-x^*\|^2\\
&\leq \alpha_n\kappa^2\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|^2\\
&\quad+(1-\alpha_n)\|y_n-x^*\|^2+2\alpha_n\langle f(x_n)-f(x^*),f(x^*)-x^*\rangle\\
&\leq \alpha_n\kappa^2\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|^2\\
&\quad+2\alpha_n\kappa\|x_n-x^*\|\|f(x^*)-x^*\|\\
&\quad+(1-\alpha_n)(\|x_n-x^*\|^2-\mu(2\beta-\mu)\|Bu_n-Bx^*\|^2\\
&\quad-\nu(2\alpha-\nu)\|Az_n-Ay^*\|^2)\\
&\leq (1-(1-\kappa^2)\alpha_n)\|x_n-x^*\|^2+\alpha_n\|f(x^*)-x^*\|^2\\
&\quad+2\alpha_n\kappa\|x_n-x^*\|\|f(x^*)-x^*\|+(1-\alpha_n)\\
&\quad-\mu(2\beta-\mu)\|Bu_n-Bx^*\|^2-\nu(2\alpha-\nu)\|Az_n-Ay^*\|^2).
\end{aligned} \tag{3.16}$$

which implies

$$\begin{aligned}
&(1-\alpha_n)(\mu(2\beta-\mu)\|Bu_n-Bx^*\|+\nu(2\alpha-\nu)\|Az_n-Ay^*\|^2)\\
&\leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nM_3\\
&\leq (\|x_n-x^*\|-\|x_{n+1}-x^*\|)(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_nM_3\\
&\leq \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+\alpha_nM_3,
\end{aligned}$$

where $M_3=\sup\{\|f(x^*)-x^*\|^2+2\kappa\|x_n-x^*\|\|f(x^*)-x^*\|:\ n\in\mathbb{N}\}$. From (B_1) and (3.12), we have

$$\lim_{n\to\infty}\|Bu_n-Bx^*\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|Az_n-Ay^*\|=0. \tag{3.17}$$

On the other hand by (2.1), we get

$$\begin{aligned}
\|y_n-x^*\|^2 &= \|P_C(I-\nu A)z_n-P_C(I-\nu A)y^*,y_n-x^*\|^2\\
&\leq \langle (I-\nu A)z_n-(I-\nu A)y^*,y_n-x^*\rangle\\
&= \frac{1}{2}[\|(I-\nu A)z_n-(I-\nu A)y^*\|^2+\|y_n-x^*\|^2\\
&\quad-\|z_n-y_n+x^*-y^*-\nu(Az_n-Ay^*)\|^2].
\end{aligned}$$

This implies

$$\begin{aligned}
\|y_n-x^*\|^2 &\leq \|z_n-y^*\|^2-\|z_n-y_n+x^*-y^*-\nu(Az_n-Ay^*)\|^2\\
&= \|z_n-y^*\|^2-[\|z_n-y_n+x^*-y^*\|^2+\nu^2\|Az_n-Ay^*\|^2\\
&\quad-2\nu\langle z_n-y_n+x^*-y^*,Az_n-Ay^*\rangle]\\
&\leq \|z_n-y^*\|^2-\|z_n-y_n+x^*-y^*\|^2\\
&\quad+2\nu\|z_n-y_n+x^*-y^*\|\|Az_n-Ay^*\|.
\end{aligned} \tag{3.18}$$

Again by (2.1), we obtain

$$\begin{aligned}\|z_n - y^*\|^2 &= \|P_C(I-\mu B)u_n - P_C(I-\mu B)x^*\|^2\\ &\le \langle (I-\mu B)u_n - (I-\mu B)x^*, z_n - y^*\rangle\\ &= \frac{1}{2}[\|(I-\mu B)u_n - (I-\mu B)x^*\|^2 + \|z_n - y^*\|^2\\ &\quad - \|u_n - z_n + y^* - x^* - \mu(Bu_n - Bx^*)\|^2]\end{aligned}$$

which implies

$$\begin{aligned}&\|z_n - y^*\|^2\\ &\le \|u_n - x^*\|^2 - \|u_n - z_n + y^* - x^* - \mu(Bu_n - Bx^*)\|^2\\ &= \|u_n - x^*\| - [\|u_n - z_n + y^* - x^*\|^2\rangle\\ &\quad - 2\mu\langle u_n - z_n + y^* - x^*, Bu_n - Bx^* + \mu^2\|Bu_n - Bx^*\|^2]\\ &\le \|x_n - x^*\|^2 - \|u_n - z_n + y^* - x^*\|^2\\ &\quad + 2\mu\|u_n - z_n + y^* - x^*\|\ \|Bu_n - Bx^*\|.\end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\begin{aligned}&\|y_n - x^*\|^2\\ &\le \|x_n - x^*\|^2 - \|u_n - z_n + y^* - x^*\|^2\\ &\quad - \|z_n - y_n + y^* - x^*\|^2\\ &\quad + 2\mu\|u_n - z_n + y^* - x^*\|\ \|Bu_n - Bx^*\|\\ &\quad + 2\nu\|z_n - y_n + x^* - y^*\|\ \|Az_n - Ay^*\|.\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.16), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 = \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(W_n y_n - x^*)\|^2\\ &\le \alpha_n\|f(x_n) - f(x^*) + f(x^*) - x^*\|^2 + (1-\alpha_n)\|W_n y_n - x^*\|^2\\ &\le \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2\\ &\quad + (1-\alpha_n)\|y_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), f(x^*) - x^*\rangle\\ &\le \alpha_n\kappa^2\|x_n - x^*\|^2 + \alpha_n\|f(x^*) - x^*\|^2 + 2\alpha_n\kappa\|x_n - x^*\|\|f(x^*) - x^*\|\\ &\quad + (1-\alpha_n)(\|x_n - x^*\|^2 - \|u_n - z_n + y^* - x^*\|^2 - \|z_n - y_n + y^* - x^*\|^2\\ &+ 2\mu\|u_n - z_n + y^* - x^*\|\|Bu_n - Bx^*\| + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\|)\\ &\le \|x_n - x^*\|^2 + (1-\alpha_n)(-\|u_n - z_n + y^* - x^*\|^2 - \|z_n - y_n + y^* - x^*\|^2\\ &+ 2\mu\|u_n - z_n + y^* - x^*\|\|Bu_n - Bx^*\| + 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\|)\end{aligned}$$

This implies

$$\begin{aligned}
&(1-\alpha_n)\|u_n - z_n + y^* - x^*\|^2 + (1-\alpha_n)\|z_n - y_n + x^* - y^*\|^2 \\
\le\ & \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\mu\|u_n - z_n + y^* - x^*\|\|Bu_n - Bx^*\| \\
&+ 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\| + 2\alpha_n M_3 \\
\le\ & \|x_{n+1} - x_n\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&+ 2\mu\|u_n - z_n + y^* - x^*\|\|Bu_n - Bx^*\| \\
&+ 2\nu\|z_n - y_n + x^* - y^*\|\|Az_n - Ay^*\| + 2\alpha_n M_3
\end{aligned}$$

From (B_1), (3.12) and (3.17), we get

$$\lim_{n\to\infty} \|u_n - z_n + y^* - x^*\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|z_n - y_n + x^* - y^*\| = 0. \tag{3.21}$$

By (3.21) and

$$\|u_n - y_n\| \le \|u_n - z_n + y^* - x^*\| + \|z_n - y_n + x^* - y^*\|$$

we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.22}$$

Step 4: We claim $\lim_{n\to\infty} \|x_n - u_n\| = 0$. From Lemma 2.3, we have

$$\begin{aligned}
\|u_n - x^*\|^2 =& \|Q_{r_n} x_n - Q_{r_n} x^*\|^2 \le \langle x_n - x^*, u_n - x^* \rangle \\
=& \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - x_n\|^2).
\end{aligned}$$

This implies

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - x_n\|^2. \tag{3.23}$$

So, we derive from (3.23) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(W_n y_n - x^*)\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 - \|u_n - x_n\|^2) \\
&\le \alpha_n\|f(x_n) - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|u_n - x_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|f(x_n) - x^*\|^2 \\
&\le \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + \alpha_n\|f(x_n) - x^*\|^2.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \|x_n - u_n\| = 0$. So, from (3.22), $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Also, from (3.3), we have

$$\lim_{n\to\infty} \|x_{n+1} - W_n y_n\| = \lim_{n\to\infty} \alpha_n\|f(x_n) - W_n y_n\| = 0.$$

Hence $\lim_{n\to\infty} \|x_n - W_n y_n\| = 0$. Since

$$\begin{aligned}
\|u_n - W_n u_n\| \le& \|W_n u_n - W_n y_n\| + \|W_n y_n - x_n\| \\
\le& \|u_n - y_n\| + \|W_n y_n - x_n\|,
\end{aligned}$$

from (3.22), we obtain $\lim_{n\to\infty}\|u_n - W_n u_n\| = 0$. From

$$\|u_n - Wu_n\| \le \|W_n u_n - Wu_n\| + \|u_n - W_n u_n\|,$$

and (3.2), we have

$$\lim_{n\to\infty}\|u_n - Wu_n\| = 0. \tag{3.24}$$

Step 5: We claim $\limsup_{n\to\infty}\langle (I-f)q, q - W_n y_n\rangle \le 0$. To show this, choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle (I-f)q, q-u_n\rangle = \lim_{i\to\infty}\langle (I-f)q, q-u_{n_i}\rangle.$$

Since $\{u_{n_i}\}$ is bounded, without loss of generality, we assume $u_{n_i} \rightharpoonup z$. We show $z \in \Omega$. From (3.24) and Lemma 2.5, we get $z \in Fix(W)$. Now, we show $z \in EP(\phi)$. Since $u_n = Q_{r_n} x_n$, we obtain

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \ \text{ for all } y \in C.$$

From (A_2), we get $\frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \ge \phi(y, u_n)$ for all $y \in C$. Replacing n by n_i, we have

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge \phi(y, u_{n_i}) \ \text{ for all } y \in C.$$

Since $u_{n_i} \rightharpoonup z$ and $\lim_{i\to\infty}\|x_{n_i} - u_{n_i}\| = 0$, it follows from $(A4)$ and (B_2) that $\phi(y,z) \le 0$ for all $y \in C$. Set $y_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Then $y_t \in C$ and hence $\phi(y_t, z) \le 0$. From (A_1) and (A_2), we obtain

$$0 = \phi(y_t, y_t) \le t\phi(y_t, y) + (1-t)\phi(y_t, z) \le t\phi(y_t, y).$$

Therefore $\phi(y_t, y) \ge 0$. Letting $t \to 0$, we get $\phi(z, y) \ge 0$ for all $y \in C$. This implies $z \in EP(\phi)$. Moreover, we know

$$\lim_{i\to\infty}\|u_{n_i} - Gu_{n_i}\| = \lim_{i\to\infty}\|u_{n_i} - y_{n_i}\| = 0.$$

From Lemma 2.5, we have $z \in Fix(G)$. So $z \in \Omega$. Since $q = P_\Omega f(q)$, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)q, q - W_n y_n\rangle &= \lim_{i\to\infty}\langle (I-f)q, q - W_{n_i} y_{n_i}\rangle\\ &= \lim_{i\to\infty}\langle (I-f)q, q - y_{n_i}\rangle\\ &= \lim_{i\to\infty}\langle (I-f)q, q - u_{n_i}\rangle\\ &= \lim_{i\to\infty}\langle (I-f)q, q - z\rangle \le 0.\end{aligned}$$

Step 6: We claim $\{x_n\}$ converges strongly to q. By using (3.3) and (3.6), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\alpha_n(f(x_n)-q)+(1-\alpha_n)(W_ny_n-q)\|^2\\
&= \alpha_n^2\|(f(x_n)-f(q))+(f(q)-q)\|^2+(1-\alpha_n)^2\|(W_ny_n-q)\|^2\\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-q, W_ny_n-q\rangle\\
&\le \alpha_n^2(\kappa^2\|x_n-q\|^2+\|f(q)-q\|^2)+(1-\alpha_n)^2\|x_n-q\|^2\\
&\quad +2\alpha_n^2\langle f(x_n)-f(q), f(q)-q\rangle\\
&\quad +2\alpha_n(1-\alpha_n)\langle f(x_n)-f(q), W_ny_n-q\rangle\\
&\quad +2\alpha_n(1-\alpha_n)\langle f(q)-q, W_ny_n-q\rangle\\
&\le (\alpha_n^2\kappa^2+(1-\alpha_n)^2)\|x_n-q\|^2+2\alpha_n^2\kappa\|x_n-q\|\|f(q)-q\|\\
&\quad +\alpha_n^2\|f(q)-q\|^2+2\alpha_n(1-\alpha_n)\kappa\|x_n-q\|^2\\
&\quad +2\alpha_n(1-\alpha_n)\langle f(q)-q, W_ny_n-q\rangle\\
&\le (1-(1-\kappa)\alpha_n)^2\|x_n-q\|^2+2\alpha_n^2\kappa\|x_n-q\|\|f(q)-q\|\\
&\quad +\alpha_n^2\|f(q)-q\|^2+2\alpha_n(1-\alpha_n)\langle f(q)-q, W_ny_n-q\rangle\\
&= (1-(1-\kappa)\alpha_n)^2\|x_n-q\|^2+\alpha_n(1-\kappa)[\frac{1}{1-\kappa}(\alpha_n\|f(q)-q\|^2\\
&\quad +2\alpha_n\kappa\|x_n-q\|\|f(q)-q\|+2(1-\alpha_n)\langle f(q)-q, W_ny_n-q\rangle)].
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1}-q\|^2 \le&(1-\gamma_n)\|x_n-q\|^2+\gamma_n[\frac{1}{1-\kappa}(\alpha_n\|f(q)-q\|^2\\
&+2\alpha_n\kappa\|x_n-q\|\|f(q)-q\|+2(1-\alpha_n)\langle f(q)-q, W_ny_n-q\rangle)],
\end{aligned} \tag{3.25}$$

where $\gamma_n=\alpha_n(1-\kappa)$, we may apply Lemma 2.6 to (3.25) to obtain that $\|x_n-q\|\to 0$, that is, $x_n\to q$ in norm. □

Table 1. The values of the sequence $\{x_n\}$

Numerical results for $x_1=-49$ and $x_1=38$			
n	x_n	n	x_n
1	-49	1	38
2	-24.5	2	19
3	-7.175	3	5.5643
⋮	⋮	⋮	⋮
20	$-1.6249e^{-14}$	20	$1.2601e^{-14}$
21	$-1.8252e^{-15}$	21	$1.4154e^{-15}$
22	$-2.033e^{-16}$	22	$1.5766e^{-16}$
⋮	⋮	⋮	⋮
38	$-5.3901e^{-32}$	38	$4.1801e^{-32}$
39	$-5.5502e^{-33}$	39	$4.3042e^{-33}$
40	$-5.7003e^{-34}$	40	$4.4206e^{-34}$

4. Numerical test

In this section, we give a numerical example to illustrate the convergence of the algorithm (3.3) in Theorem 3.3. Let $C = [-50, 50] \subset H = \mathbb{R}$ and define

$$\phi(x, y) = -7x^2 + xy + 6y^2.$$

First, we verify that ϕ satisfies the conditions $(A_1) - (A_4)$ as follows:
(A_1) $\phi(x, x) = -7x^2 + x^2 + 6x^2 = 0$ for all $x \in [-50, 50]$;
(A_2) $\phi(x, y) + \phi(y, x) = -(y - x)^2 \leq 0$ for all $x, y \in [-50, 50]$;
(A_3) For all $x, y, z \in [-50, 50]$,

$$\begin{aligned}\limsup_{t\to 0^+} \phi(tz + (1-t)x, y) =& \limsup_{t\to 0^+} (-7(tz + (1-t)x)^2 \\ &+ (tz + (1-t)x)y + 6y^2) \\ =& \phi(x, y).\end{aligned}$$

(A_4) For all $x \in [-50, 50]$, $\Phi(y) = \phi(x, y) = -7x^2 + xy + 6y^2$ is a lower semicontinuous and convex function.

From Lemma 2.3, Q_r is single-valued for all $r > 0$. Now, we deduce a formula for $Q_r(x)$. For any $y \in [-50, 50]$ and $r > 0$, we have

$$\begin{aligned}&\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \\ \Leftrightarrow\ & 6ry^2 + ((r + 1)z - x)y + xz - (7r + 1)z^2 \geq 0.\end{aligned}$$

Set

$$G(y) = 6ry^2 + ((r + 1)z - x)y + xz - (7r + 1)z^2.$$

Then $G(y)$ is a quadratic function of y with coefficients $a = 6r$, $b = (r + 1)z - x$ and $c = xz - (7r + 1)z^2$. So its discriminate $\Delta = b^2 - 4ac$ is

$$\begin{aligned}\Delta =& [(r + 1)z - x]^2 - 24r(xz - (7r + 1)z^2) \\ =& [(13r + 1)z - x]^2.\end{aligned}$$

Since $G(y) \geq 0$ for all $y \in C$, this is true if and only if $\Delta \leq 0$. That is,

$$[(13r + 1)z - x]^2 \leq 0.$$

Therefore,

$$z = \frac{x}{13r + 1},$$

which yields

$$Q_r(x) = \frac{x}{13r + 1}.$$

So, from Lemma 2.3, we get $EP(\phi) = \{0\}$. Let

$$\alpha_n = \frac{1}{n},\ r_n = \frac{n}{5n - 1},\ \lambda_n = \gamma \in (0, 1),\ \text{and } T_n x = x$$

for all $n \in \mathbb{N}$. Suppose $f(x) = \frac{1}{2}x$, $Ax = \frac{x}{10}$ is $5-$ism , $Bx = \frac{x}{5}$ is $2-$ism, $\nu = 5$, and $\mu = 2$. Hence

$$\Omega = \cap_{n=1}^{\infty} Fix(T_n) \cap EP(\phi) \cap Fix(G) = \{0\}.$$

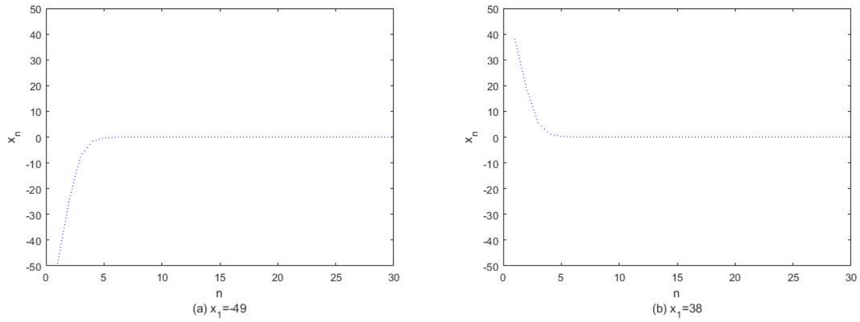

Figure 1. The convergence of $\{x_n\}$ with different initial values x_1.

Also, from (3.1), we have

$$
\begin{aligned}
W_1 &= U_{1,1} = \lambda_1 T_1 U_{1,2} + (1-\lambda_1)I,\\
W_2 &= U_{2,1} = \lambda_1 T_1 U_{2,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{2,3} + (1-\lambda_2)I) + (1-\lambda_1)I\\
&= \lambda_1\lambda_2 T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I,\\
W_3 &= U_{3,1} = \lambda_1 T_1 U_{3,2} + (1-\lambda_1)I = \lambda_1 T_1(\lambda_2 T_2 U_{3,3} + (1-\lambda_2)I) + (1-\lambda_1)I\\
&= \lambda_1\lambda_2 T_1 T_2 U_{3,3} + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I\\
&= \lambda_1\lambda_2 T_1 T_2(\lambda_3 T_3 U_{3,4} + (1-\lambda_3)I) + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I\\
&= \lambda_1\lambda_2\lambda_3 T_1 T_2 T_3 + \lambda_1\lambda_2(1-\lambda_3)T_1 T_2 + \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I.
\end{aligned}
$$

By computing in this way by (3.1), we obtain

$$
\begin{aligned}
W_n = U_{n,1} =& \lambda_1\lambda_2 ... \lambda_n T_1 T_2 ... T_n + \lambda_1\lambda_2 ... \lambda_{n-1}(1-\lambda_n)T_1 T_2 ... T_{n-1}\\
&+ \lambda_1\lambda_2 ... \lambda_{n-2}(1-\lambda_{n-1})T_1 T_2 ... T_{n-2} + ... +\\
&+ \lambda_1(1-\lambda_2)T_1 + (1-\lambda_1)I.
\end{aligned}
$$

Since $T_n = I, \lambda_n = \gamma$ for all $n \in \mathbb{N}$, we get

$$W_n = (\gamma^n + \gamma^{n-1}(1-\gamma) + ... + \gamma(1-\gamma) + (1-\gamma))I = I.$$

Then, from Theorem 3.3, the sequence $\{x_n\}$, generated iteratively by

$$
\begin{cases}
u_n = Q_{r_n} x_n = \frac{5n-1}{18n-1} x_n,\\
z_n = P_C(I-\mu B)u_n = P_C(\frac{2}{3}u_n) = \frac{2}{3}u_n\\
y_n = P_C(I-\nu A)z_n = P_C(\frac{1}{2}z_n) = \frac{1}{3}u_n\\
x_{n+1} = \frac{1}{2n}x_n + (1-\frac{1}{n})y_n = \frac{10n^2+42n-1}{108n^2-6n}x_n,
\end{cases}
\tag{4.1}
$$

converge strongly to $0 \in \Omega$, where $0 = P_\Omega(f)(0)$.
The Table 1 indicates the values of sequence $\{x_n\}$ for algorithm (4.1) where $x_1 = -49$, $x_1 = 38$, and $n = 40$.
The Figure 1 presents the behavior of $\{x_n\}$ that corresponds to the Table 1 and shows the sequence $\{x_n\}$ converges to $0 \in \Omega$.

Finally, we note that there have now been many results on the various composite iterative schemes related closely to the extragradient method in the present literature. We refer the readers to compare other composite iterative schemes to their iterative one (see [4, 8, 9, 13, 7, 14, 10, 11, 5, 6]).

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