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SOLVABILITY AND OPTIMAL CONTROL OF SEMILINEAR FRACTIONAL EVOLUTION EQUATIONS WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. This paper is devoted to a class of semilinear Riemann-Liouville fractional evolution equations in Banach spaces. Using the Banach fixed point theorem and semigroup theory, we first establish an existence and uniqueness theorem of the mild solution, which improves the existing results in literature. Then, we consider an optimal control problem governed by semilinear fractional diffusion equations. The existence of optimal pairs and the compactness of the states are obtained. Moreover, the necessary optimality conditions of first order are derived.
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1. Introduction

In recent years, fractional differential equations have been of great interest to researchers due to their wide applications to problems in physics, electroanalytical chemistry, biology, control theory, signal processing, aerodynamics; see monographs [23, 21, 11] and the references therein. One significant branch of the study is the theory of fractional evolution equations, which, motivated by practical problems arising in viscoelasticity, electrodynamics and heat conduction in materials with memory, has been attracting increasing attention. The purpose of this paper is to deal with the solvability and optimal control of semilinear fractional evolution equations in vector-valued function spaces.

There is considerable literature on the existence of mild solutions for semilinear fractional evolution equations with the Caputo and Riemann-Liouville fractional derivatives; see, e.g. [32, 2, 34, 27, 13]. Let $0 < \alpha < 1$ and $J = [0, T]$ with $T > 0$

being terminal time. In this work, we first study the existence and uniqueness of the mild solution to the initial-value problem of a semilinear fractional evolution equation

$$\begin{cases} {}^{L}D_t^{\alpha} y - Ay = f(t,y) \quad \text{a.e. } t \in J \\ I^{1-\alpha} y(0^+) = y^0, \end{cases} \tag{1.1}$$

where ${}^{L}D_t^{\alpha}$ denotes the Riemann-Liouville fractional derivative with respect to time, $A : D(A) \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)\,(t \geq 0)$ on a Banach space X, and $f : J \times X \to X$ is a nonlinear term to be specified later. Moreover, $y^0 \in X$ and $I^{1-\alpha}y(0^+) = \lim_{t\to 0^+} I^{1-\alpha}y(t)$ where $I^{1-\alpha}$ stands for the fractional integration of order $1-\alpha$. Note that, unlike fractional evolution equations with Caputo derivatives, the mild solution of (1.1) involves a singular term with respect to the initial value. For the existence results of (1.1), the conditions of the nonlinear term $f(t,y)$ in existing literature are restrictive, see, e.g. [34, 13, 18, 9]. In this work, we show that (1.1) has a unique mild solution where the growth condition of $f(t,y)$ has been improved. Meanwhile, a uniform boundedness of mild solutions to the data is given.

Then, following the existence result of (1.1), we deal with an optimal control problem of semilinear fractional diffusion equations with constraints. As is known, optimal control of differential equations of integer order have been studied extensively by numerous authors. We refer readers to Lions [12] and Tröltzsch [26] for optimal control problems governed by partial differential equations, and to Barbu [4], Peng-Kunisch [20], Peng [19] and Xiao-Sofonea [29] for optimal control problems governed by variational inequalities. Optimal control problems governed by linear fractional differential equations can be found in [17, 7, 6, 25]. Optimal control of semilinear fractional differential equations have been considered in [27, 18] which focused on the existence of optimal solutions. In addition, optimal feedback control, the exact and approximate controllability of fractional evolution equations can be found in [13, 28, 14, 22, 30, 1], etc. To the best of our knowledge, however, no literature has yet studied the optimality system governed by the semilinear fractional evolution equation (1.1).

The novelty of this paper is two-fold. On the one hand, the work generalizes the existence theorems established in [13, 18] since one restrictive growth condition of $f(t,y)$ used in [13, 18] has been dropped (Theorem 3.3, Remark 3.7, Remark 3.8). Also, the hypothesis of $f(t,y)$ here is more direct and simpler than that considered in [34, 9]. This makes our result entail wider applications in practical problems. On the other hand, an optimal control problem governed by semilinear diffusion equations has been considered. In addition to the existence of optimal solutions and the compactness of the states, the first-order necessary optimality conditions are derived (Theorem 4.3, 4.4).

The paper is organized as follows. In section 2, we recall some vector-valued function spaces and the preliminaries concerning fractional calculus. Section 3 is devoted to the solvability and uniform boundedness of the mild solutions to problem (1.1). On the basis of Section 3, an optimal control problem governed by semilinear fractional diffusion equations is considered in Section 4. We first show the existence of optimal pairs and the compactness of the states, and then derive the necessary optimality condition of first order by the Lagrange multiplier method.

2. Preliminaries

We begin with some definitions and preliminaries concerning function spaces and fractional calculus which are used throughout the paper. Let X be a real Banach space with norm $\|\cdot\|$. We denote by $L^p(J;X)$ the Banach space of all measurable vector-valued functions $v: J \to X$ such that $\|v(\cdot)\|$ belongs to $L^p(J)$ with the norm

$$\|v\|_{L^p(J;X)} = \left(\int_0^T \|v(t)\|^p dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

The space $C(J;X)$ comprises all continuous functions $v: J \to X$ with the norm

$$\|v\|_{C(J:X)} = \sup_{t \in J} \|v(t)\|.$$

Furthermore, $AC(J;X)$ consists of all functions $v: J \to X$ that are absolutely continuous, and $AC^n(J;X) := \{v \in C(J;X) : \ v^{(n-1)} \in AC(J;X)\}$ for $n \in Z^+$. According to monographs [23, 21, 11, 33], we present the definitions of fractional derivatives and integrals for vector-valued functions.

Definition 2.1. If $Re(z) > 0$, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{2.1}$$

If $-m < Re(z) < -m+1$ where m is a positive integer, then

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}. \tag{2.2}$$

Definition 2.2. Let $f \in L^1(J;X)$ and $\alpha > 0$. The Riemann-Liouville fractional integral $I^\alpha f$ of order α is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0.$$

Definition 2.3. Let $n-1 < \alpha < n$, $n \in Z^+$ and $f \in L^1(J;X)$ with $I^{1-\alpha} f \in AC^n(J;X)$. The Riemann-Liouville fractional derivative ${}^L D_t^\alpha f$ of order α is defined by

$${}^L D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \ t > 0.$$

Definition 2.4. Let $n-1 < \alpha < n$, $n \in Z^+$ and $f \in AC^n(J;X)$. The Caputo fractional derivative ${}^C D_t^\alpha f$ of order α is defined by

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \ t > 0.$$

From now on, we set

$$\mathfrak{D}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds, \ 0 < t < T, \tag{2.3}$$

where $0 < \alpha < 1$, $f \in AC(J;X)$. Note that the notation $-\mathfrak{D}^\alpha f$ is the so-called right fractional Caputo derivative of f.

What follows is a Banach space used frequently in fractional evolution equations with Riemann-Liouville fractional derivative; see, [34, 13] for examples.

Definition 2.5. The Banach space $C_{1-\alpha}(J;X)$ is defined by

$$C_{1-\alpha}(J;X) = \{y : J' \to X;\ t^{1-\alpha}y \in C(J;X)\},$$

with the norm

$$\|y\|_{C_{1-\alpha}(J;X)} = \sup_{t\in J}\{t^{1-\alpha}\|y(t)\|\},$$

where $J' = (0,T]$. Note that $t^{1-\alpha}y \in C(J;X)$ is understood that the limit $\eta = \lim_{t\to 0} t^{1-\alpha}y$ exists, and the function $t^{1-\alpha}y$ is continuous on J by taking the value at $t = 0$ with the limit η.

Lemma 2.6. ([11]) Let $0 < \alpha < 1$, $g \in L^p(J)$, $1 \le p \le \infty$ and $\psi : J' \to R^+$ be the function defined by:

$$\psi(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Then for almost every $t \in J$, the function $s \mapsto \psi(t-s)g(s)$ is integrable on J. Moreover, the convolution $\psi * g$, given by

$$\psi * g(t) = \int_0^t \psi(t-s)g(s)ds,$$

belongs to $L^p(J)$ and

$$\|\psi * g\|_{L^p(J)} \le \|\psi\|_{L^1(J)}\|g\|_{L^p(J)}.$$

Note that if, in addition, $p > \frac{1}{1-\alpha}$, then $\psi * g$ is continuous on $[0,T]$.

lemma 2.7. ([24, Theorem 1]) Let $\Sigma \subset L^p(J;X)$. Then Σ is relatively compact in $L^p(J;X)$ for $1 \le p < \infty$, if and only if

(i) $\{\int_{t_1}^{t_2} \eta(t)dt : \eta \in \Sigma\}$ is relatively compact in X, $\forall\ 0 < t_1 < t_2 < T$.

(ii) $\int_0^{T-h} \|\eta(t+h) - \eta(t)\|_X^p dt \to 0$ as $h \to 0^+$, uniformly for $\eta \in \Sigma$.

Here conditions (i) and (ii) are called the space and time criterions, respectively. Note that if $\Sigma = \{\eta\}$ with $\eta \in L^p(J;X)$. Then Σ is compact, and thus (i) and (ii) are satisfied.

3. Existence of mild solution

This section is devoted to the solvability of problem (1.1). We begin with the hypotheses on the data of the problem.

(H1) The operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ with $\|T(t)\| \le M$ for some constant $M > 0$ and all $t > 0$.

(H2) The operator $f(t,y)$ is measurable with respect to t for each fixed $y \in X$ with $\phi(\cdot) := \|f(\cdot,0)\| \in L^2(J)$, and there exists a constant $L > 0$ such that

$$\|f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\|,\ \text{a.e. } t \in J,\ \forall\ y_1, y_2 \in X.$$

According to [13, lemma2.4], we give the definition of mild solutions to (1.1).

Definition 3.1. A function $y \in L^2(J;X)$ is called a mild solution to problem (1.1) if it satisfies the following fractional integration equation

$$y(t) = t^{\alpha-1}T_\alpha(t)y^0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s))ds, \tag{3.1}$$

where

$$T_\alpha(t) = \alpha \int_0^\infty \theta\Phi_\alpha(\theta)T(t^\alpha\theta)d\theta. \tag{3.2}$$

Recall that $\Phi_\alpha(\theta)$ the is the so-called Wright function given by

$$\Phi_\alpha(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^n}{\Gamma(-\alpha n + 1 - \alpha)},\ 0 < \alpha < 1.$$

It's well known that ([16, A.39])

$$\int_0^\infty \theta\Phi_\alpha(\theta)d\theta = \frac{1}{\Gamma(\alpha+1)}. \tag{3.3}$$

Lemma 3.2. Let $\frac{1}{2} < \alpha < 1$ and the hypothesis (H1) hold. Define a vector-valued function $\xi : J \to X$ by $\xi(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\vartheta(s)ds$ for $t \in J$, where $\vartheta \in L^2(J;X)$ is given. Then $\xi \in C(J;X)$.

Proof. In fact, using (3.3) and assumption (H1), we have

$$\|T_\alpha(t)\| \le \frac{M}{\Gamma(\alpha)}. \tag{3.4}$$

Take $t \in [0,T)$ and $h > 0$ with $0 < t+h \le T$. Since

$$\begin{aligned}
\xi(t+h) &= \int_0^{t+h} (t+h-s)^{\alpha-1}T_\alpha(t+h-s)\vartheta(s)ds \\
&= \int_{-h}^{t} (t-\tau)^{\alpha-1}T_\alpha(t-\tau)\vartheta(\tau+h)d\tau \\
&= \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\vartheta(s+h)ds \\
&\quad + \int_{-h}^{0} (t-s)^{\alpha-1}T_\alpha(t-s)\vartheta(s+h)ds,
\end{aligned}$$

we calculate

$$\begin{aligned}
&\|\xi(t+h) - \xi(t)\| \\
\le&\frac{M}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}\|\vartheta(s+h)-\vartheta(s)\|ds + \int_{-h}^{0}(t-s)^{\alpha-1}\|\vartheta(s+h)\|ds\right).
\end{aligned} \tag{3.5}$$

On the one hand, using the Hölder inequality, we have

$$\int_0^t (t-s)^{\alpha-1}\|\vartheta(s+h)-\vartheta(s)\|ds \le \left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}\left(\int_0^t \|\vartheta(s+h)-\vartheta(s)\|^2ds\right)^{\frac{1}{2}}.$$

From Lemma 2.7(ii), it follows that the right-hand side of the last inequality tends zero as $h \to 0^+$. On the other hand,

$$\lim_{h\to 0^+}\int_{-h}^{0}(t-s)^{\alpha-1}\|\vartheta(s+h)\|ds = \lim_{h\to 0^+}\int_0^h (t+h-\tau)^{\alpha-1}\|\vartheta(\tau)\|d\tau = 0.$$

Consequently,

$$\lim_{h\to 0^+}\|\xi(t+h)-\xi(t)\| = 0, \quad t\in[0,T). \tag{3.6}$$

Next, take $h>0$ with $0\le t-h<T$. we see that

$$\begin{aligned}&\|\xi(t-h)-\xi(t)\|\\ \le&\frac{M}{\Gamma(\alpha)}\int_0^{t-h}\big((t-h-s)^{\alpha-1}-(t-s)^{\alpha-1}T_\alpha(h)\big)\|\vartheta(s)\|ds\\ &+\frac{M}{\Gamma(\alpha)}\int_{t-h}^{t}(t-s)^{\alpha-1}\|\vartheta(s)\|ds.\end{aligned} \tag{3.7}$$

On the one hand,

$$\begin{aligned}&\lim_{h\to 0^+}\int_0^{t-h}\big((t-h-s)^{\alpha-1}-(t-s)^{\alpha-1}T_\alpha(h)\big)\|\vartheta(s)\|ds\\ \le&\lim_{h\to 0^+}\|\vartheta\|_{L^2(Q)}\int_0^{t-h}\big((t-h-s)^{\alpha-1}-(t-s)^{\alpha-1}\big)^2 ds.\end{aligned} \tag{3.8}$$

We set

$$\varpi_h(s) = \big((t-h-s)^{\alpha-1}-(t-s)^{\alpha-1}\big)^2,$$

Obviously, $|\varpi_h(s)|\le\varpi_1(s)\in L(J)$, as $h<1$, and $\lim_{h\to 0}\varpi_h(s)=0$ a.e. $s\in J$. According to the Lebesgue dominated convergence theorem, we have

$$\lim_{h\to 0^+}\int_0^T\varpi_h(s)ds = 0. \tag{3.9}$$

On the other hand, since $(t-s)^{\alpha-1}, \|\vartheta(s)\|\in L^2(J)$, we get

$$\lim_{h\to 0^+}\int_{t-h}^{t}(t-s)^{\alpha-1}\|\vartheta(s)\|ds = 0. \tag{3.10}$$

Using (3.7)-(3.10), we have

$$\lim_{h\to 0^+}\|\xi(t-h)-\xi(t)\| = 0, \quad t\in(0,T]. \tag{3.11}$$

Thus, from (3.6) and (3.11) we get $\xi\in C(J;X)$. The proof is complete. □

We are now in position to present the main result of this section.

Theorem 3.3. Let (H1), (H2) hold and $\frac{1}{2}<\alpha<1$. Then problem (1.1) has a unique mild solution $y\in C(J';X)\cap L^2(J;X)$ given by (3.1). Moreover,

$$\|y\|^2_{L^2(J;X)}\le k_1(1+k_2e^{k_2})\left(\alpha^2\|y^0\|^2+(2\alpha-1)T\|\phi\|^2_{L^2(J)}\right), \tag{3.12}$$

where

$$k_1=\frac{3M^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha+1))^2}, \quad k_2=\frac{3(MLT^\alpha)^2}{(2\alpha-1)(\Gamma(\alpha))^2}.$$

Proof. Consider an operator $\mathcal{F}$ defined by

$$\mathcal{F}y(t) = t^{\alpha-1}T_\alpha(t)y^0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s))ds,\ t > 0. \tag{3.13}$$

The proof is divided into four steps.

Step 1. We show that $\mathcal{F}$ maps $L^2(J;X)$ into itself. In fact, according to (3.4) and the assumption (H2), we have

$$\begin{aligned} \|\mathcal{F}y(t)\| &\le \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\|y^0\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|)ds \\ &\le \frac{Mt^{\alpha-1}}{\Gamma(\alpha)}\|y^0\| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\phi(s) + L\|y(s)\|)ds, \end{aligned} \tag{3.14}$$

which implies that

$$\begin{aligned} \|\mathcal{F}y(t)\|^2 \le& \frac{3M^2t^{2\alpha-2}}{(\Gamma(\alpha))^2}\|y^0\|^2 + \frac{3M^2}{(\Gamma(\alpha))^2}\left(\int_0^t (t-s)^{\alpha-1}\phi(s)ds\right)^2 \\ &+ \frac{3M^2}{(\Gamma(\alpha))^2}\left(\int_0^t (t-s)^{\alpha-1}L\|y(s)\|ds\right)^2. \end{aligned} \tag{3.15}$$

Since $\phi(\cdot), \|y(\cdot)\|$ belong to $L^2(J)$, by using Lemma 2.6, we have

$$\begin{aligned} &\int_0^T \|\mathcal{F}y(t)\|^2dt \\ \le& \frac{3M^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}\|y^0\|^2 + \frac{3(MT^\alpha)^2}{(\Gamma(\alpha+1))^2}\|\phi\|^2_{L^2(J)} + \frac{3(MLT^\alpha)^2}{(\Gamma(\alpha+1))^2}\|y\|^2_{L^2(J;X)}. \end{aligned}$$

Therefore, $\mathcal{F}y$ belongs to $L^2(J;X)$ for each $y \in L^2(J;X)$, i.e., $\mathcal{F}$ maps $L^2(J;X)$ into itself.

From now on, for each $g \in L^2(J)$ and $n \in Z^+$, we set

$$\mathcal{I}^n g(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} ((t-t_1)(t_1-t_2)\cdots(t_{n-1}-t_n))^{\alpha-1} g(t_n)dt_n\cdots dt_2dt_1.$$

Step 2. We claim that for any $y_1,\ y_2 \in L^2(J;X)$, the following inequality holds:

$$\|\mathcal{F}^n y_1(t) - \mathcal{F}^n y_2(t)\| \le \left(\frac{ML}{\Gamma(\alpha)}\right)^n \mathcal{I}^n\|y_1(t) - y_2(t)\|,\quad n \in Z^+,\ t > 0. \tag{3.16}$$

This inequality will be proved by mathematical induction. Using (H2), we have

$$\begin{aligned} \|\mathcal{F}y_1(t) - \mathcal{F}y_2(t)\| &= \left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)(f(s,y_1(s)) - f(s,y_2(s)))ds\right\| \\ &\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y_1(s)) - f(s,y_2(s))\|ds \\ &\le \frac{ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_1(s) - y_2(s)\|ds. \end{aligned}$$

This implies that the inequality (3.16) holds for $n = 1$. Assume that (3.16) holds for $n = k$, $(k \geq 1)$ i.e.,

$$\|\mathcal{F}^k y_1(t) - \mathcal{F}^k y_2(t)\| \leq \left(\frac{ML}{\Gamma(\alpha)}\right)^k \mathcal{I}^k \|y_1(t) - y_2(t)\|.$$

Then we see that for $n = k+1$,

$$\begin{aligned}
& \|\mathcal{F}^{k+1} y_1(t) - \mathcal{F}^{k+1} y_2(t)\| \\
= \;& \left\| \int_0^t (t-t_1)^{\alpha-1} T_\alpha(t-t_1)(f(t_1, \mathcal{F}^k y_1(t_1)) - f(t_1, \mathcal{F}^k y_2(t_1)))dt_1 \right\| \\
\leq \;& \frac{ML}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} \|\mathcal{F}^k y_1(t_1) - \mathcal{F}^k y_2(t_1)\| dt_1 \\
\leq \;& \frac{ML}{\Gamma(\alpha)} \int_0^t (t-t_1)^{\alpha-1} \mathcal{I}^k \|y_1(t_1) - y_2(t_1)\| dt_1 \\
\leq \;& \left(\frac{ML}{\Gamma(\alpha)}\right)^{k+1} \mathcal{I}^{k+1} \|y_1(t) - y_2(t)\|.
\end{aligned}$$

Thus inequality (3.16) holds.

Step 3. The operator $\mathcal{F}^n$ is contractive on $L^2(J;X)$ for n being sufficiently large. Moreover, we have the bound (3.12). In fact, we see from (3.16) that

$$\begin{aligned}
& \|\mathcal{F}^n y_1 - \mathcal{F}^n y_2\|^2_{L^2(J;X)} = \int_0^T \|\mathcal{F}^n y_1(t) - \mathcal{F}^n y_2(t)\|^2 dt \\
\leq \;& \left(\frac{ML}{\Gamma(\alpha)}\right)^{2n} \int_0^T \left(\mathcal{I}^{n-1} \int_0^t (t-t_n)^{\alpha-1} \|y_1(t_n) - y_2(t_n)\| dt_n\right)^2 dt \\
\leq \;& \left(\frac{ML}{\Gamma(\alpha)}\right)^{2n} \|y_1 - y_2\|_{L^2(J;X)} \int_0^T \left(\mathcal{I}^{n-1} \left(\frac{t^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}\right)^2 dt.
\end{aligned}$$

Then, by the Hölder inequality, we have

$$\begin{aligned}
& \int_0^T \left(\mathcal{I}^{n-1} \Big(\frac{t^{2\alpha-1}}{2\alpha-1}\Big)^{\frac{1}{2}}\right)^2 dt \\
= \;& \int_0^T \left(\mathcal{I}^{n-2} \left(\int_0^t (t-t_{n-1})^{2\alpha-2} dt_{n-1}\right)^{\frac{1}{2}} \left(\int_0^t \frac{t_{n-1}^{2\alpha-1}}{2\alpha-1} dt_{n-1}\right)^{\frac{1}{2}}\right)^2 dt \\
\leq \;& \int_0^T \left(\mathcal{I}^{n-2} \left(\frac{t^{4\alpha-1}}{2\alpha(2\alpha-1)^2}\right)^{\frac{1}{2}}\right)^2 dt \\
\leq \;& \int_0^T \left(\mathcal{I}^{n-2} \left(\frac{t^{4\alpha-1}}{2\alpha(2\alpha-1)^2}\right)^{\frac{1}{2}}\right)^2 dt \\
= \;& \int_0^T \left(\mathcal{I}^{n-3} \int_0^t (t-t_{n-2})^{\alpha-1} \Big(\frac{t_{n-2}^{4\alpha-1}}{2\alpha(2\alpha-1)^2}\Big)^{\frac{1}{2}} dt_{n-2}\right)^2 dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T \left(\mathcal{I}^{n-3}\Big(\frac{t^{6\alpha-1}}{2\alpha\cdot 4\alpha(2\alpha-1)^3}\Big)^{\frac{1}{2}}\right)^2 dt \\
&\leq \int_0^T \frac{t^{2n\alpha-1}}{2\alpha\cdot 4\alpha\cdots 2(n-1)\alpha(2\alpha-1)^n}dt \\
&= \frac{T^{2n\alpha}}{n!(2\alpha(2\alpha-1))^n}.
\end{aligned}
$$

Thus, it follows that

$$
\|\mathcal{F}^n y_1 - \mathcal{F}^n y_2\|^2_{L^2(J;X)} \leq \left(\frac{ML}{\Gamma(\alpha)}\right)^{2n} \frac{T^{2n\alpha}}{n!(2\alpha(2\alpha-1))^n}\|y_1-y_2\|^2_{L^2(J;X)}.
$$

The last inequality gives

$$
\|\mathcal{F}^n y_1 - \mathcal{F}^n y_2\|_{L^2(J;X)} \leq \frac{(MLT^\alpha)^n}{(\Gamma(\alpha))^n\sqrt{(2\alpha(2\alpha-1))^n n!}}\|y_1-y_2\|_{L^2(J;X)}.
$$

Due to

$$
\frac{(MLT^\alpha)^n}{(\Gamma(\alpha))^n\sqrt{(2\alpha(2\alpha-1))^n n!}} \to 0,\ \ as\ n\to\infty,
$$

there exist a positive integer $\mathcal{N}$ such that

$$
\frac{(MLT^\alpha)^{\mathcal{N}}}{(\Gamma(\alpha))^{\mathcal{N}}\sqrt{(2\alpha(2\alpha-1))^{\mathcal{N}}\mathcal{N}!}} < 1.
$$

Hence, $\mathcal{F}^{\mathcal{N}}$ is a contraction operator on $L^2(J;X)$. By the Banach fixed point theorem of contraction mapping, we can deduce that $\mathcal{F}$ has a unique fixed point $y=\mathcal{F}y$ on $L^2(J;X)$, which is the desired mild solution of problem (1.1).

Next, we shall prove the estimate (3.12). Since the solution y is the fixed point of Fy, we see from (3.15) that

$$
\begin{aligned}
\|y(t)\|^2 \leq{}& \frac{3M^2t^{2\alpha-2}}{(\Gamma(\alpha))^2}\|y^0\|^2 + \frac{3M^2}{(\Gamma(\alpha))^2}\left(\int_0^t (t-s)^{\alpha-1}\phi(s)ds\right)^2 \\
&+ \frac{3M^2}{(\Gamma(\alpha))^2}\left(\int_0^t (t-s)^{\alpha-1}L\|y(s)\|ds\right)^2.
\end{aligned}
$$

Integrating the last inequality over $(0,\tau)$ with $\tau\in(0,T]$, and using Lemma 2.6 and the Hölder inequality, we have

$$
\begin{aligned}
\int_0^\tau \|y(t)\|^2dt \leq{}& \frac{3M^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}\|y^0\|^2 + \frac{3(MT^\alpha)^2}{(\Gamma(\alpha+1))^2}\|\phi\|^2_{L^2(J)} \\
&+ \frac{3(ML)^2T^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}\int_0^\tau\int_0^t \|y(s)\|^2dsdt.
\end{aligned}
\tag{3.17}
$$

Now, we set

$$
\zeta_1(\tau) = \int_0^\tau \|y(t)\|^2dt.
$$

Then, it follows that

$$
\zeta_1(\tau) \leq \rho_1 + \gamma_1\int_0^\tau \zeta_1(t)dt, \tag{3.18}
$$

where

$$\rho_1 = \frac{3M^2T^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}\|y^0\|^2 + \frac{3(MT^\alpha)^2}{(\Gamma(\alpha+1))^2}\|\phi\|^2_{L^2(J)},\ \gamma_1 = \frac{3(ML)^2T^{2\alpha-1}}{(\Gamma(\alpha))^2(2\alpha-1)}.$$

Finally, using the Gronwall inequality, we deduce that

$$\zeta_1(\tau) \le \rho_1(1+\gamma_1\tau e^{\gamma_1\tau}),\ \tau \in (0,T]. \tag{3.19}$$

Taking $\tau = T$ we obtain the estimate (3.12), where $k_2 = \gamma_1 T$.

Step 4. We show that $y \in C(J';X)$. In fact, since $y \in L^2(J;X)$, we have

$$f(\cdot, y(\cdot)) \in L^2(J;X)$$

by (H2). Thus

$$\int_0^{\cdot} (\cdot - s)^{\alpha-1}T_\alpha(\cdot - s)f(s,y(s))ds \in C(J;X)$$

by Lemma 3.2. Moreover, it is easy to see that $t^{\alpha-1}T_\alpha(t)y^0 \in C(J';X)$. Therefore, we conclude that the mild solution y given by (3.1) belongs to $C(J';X)$. The proof is complete. □

Theorem 3.4. Let (H1), (H2) hold and $y^0 = 0$. Then for all $\frac{1}{2} < \alpha < 1$, problem (1.1) has a unique mild solution $y \in C(J;X)$ given by

$$y(t) = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s))ds. \tag{3.20}$$

with the bound

$$\|y\|_{C(J;X)} \le \frac{M}{\Gamma(\alpha)}\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}} E_\alpha(MLT^\alpha)\|\phi\|_{L^2(J)}.$$

Proof. According to Theorem 3.3 and the proof in step 4, problem (1.1) has a unique mild solution $y \in C(J;X)$ which satisfies the inequality (3.14) with $\mathcal{F}y = y$, i.e.,

$$\|y(t)\| \le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\phi(s) + L\|y(s)\|)ds.$$

Using the Hölder inequality, we obtain

$$\|y(t)\| \le \frac{M}{\Gamma(\alpha)}\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}\|\phi\|_{L^2(J)} + \frac{ML}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds.$$

By the Gronwall inequality for fractional differential equations ([31, Corollary 2], [10, Lemma 7.1.1]), we have

$$\|y(t)\| \le \frac{M}{\Gamma(\alpha)}\left(\frac{T^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}} E_\alpha(MLT^\alpha)\|\phi\|_{L^2(J)},\ \forall\, t \in J.$$

The proof is complete. □

Remark 3.5. Let $\frac{1}{\alpha} < p < \frac{1}{1-\alpha}$ and hypothesis $(H1)$ be satisfied. Besides, assume $(H2)$ holds with the function $\|f(\cdot,0)\| \in L^p(J)$. Then the problem (1.1) admits a unique mild solution $y \in C(J';X) \cap L^p(J;X)$.

The condition $p < \frac{1}{1-\alpha}$ is used to ensure the singular term $t^{\alpha-1}T_\alpha(t)y^0$ in (3.1) belongs to $L^p(J;X)$. The proof is similar to Theorem 3.3, so we omit the details for simplicity.

Remark 3.6. Let $y \in C(J';X) \cap L^2(J;X)$ be the mild solution of problem (1.1) given by (3.1). Then it follows $y \in C_{1-\alpha}(J;X)$.

In fact, taking $z(t) = t^{1-\alpha}y(t)$ for $t > 0$ where y is the mild solution of (1.1), i.e.,

$$z(t) = T_\alpha(t)y^0 + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s))ds, \quad t > 0.$$

Since $\int_0^{\cdot}(\cdot - s)^{\alpha-1}T_\alpha(\cdot - s)f(s,y(s))ds \in C(J;X)$ from step 4 in the proof of Theorem 3.3, we see that the function $t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,y(s))ds$ is continuous on J. This implies $z \in C(J';X)$. Since $z(t)$ tends to y^0 as t goes to zero, setting $z(0) = y^0$ we conclude that $z \in C(J;X)$. From Definition 2.5, it follows $y \in C_{1-\alpha}(J;X)$. $\square$

Note that if $y \in C_{1-\alpha}(J;X)$ is a mild solution of problem (1.1) given by (3.1), as a rule, we could not necessarily deduce $y \in L^2(J;X)$.

Remark 3.7. For the existence of semilinear fractional evolution equations with Riemann-Liouville derivatives, the conditions posed on the nonlinear term $f(t,y)$ are, in general, quite strong in the literature because the mild solution (3.1) involves a singular term with respect to the initial value. For example, except for (H2), the following additional hypothesis is considered in [13, 18] to show the existence of a mild solution $y \in C_{1-\alpha}(J;X)$ to problem (1.1).

(H2-1) There exists $\phi \in L^p(J)$ with $p > \frac{1}{\alpha}$, and a constant $c > 0$ such that

$$\|f(t,y)\| \le \phi(t) + ct^{1-\alpha}\|y\| \quad \text{for a.e. } t \in J \text{ and all } y \in X. \tag{3.21}$$

If, in particular, $c = 0$, (H2-1) is considered in [15]. Besides, we also mention that problem (1.1) has been proved in [34, 9] to have a mild solution $y \in C_{1-\alpha}(J;X)$ under the following assumption.

(H2-2) There exists a function $\phi \in L^1(J;X)$ such that $I^\alpha\phi(t) \in C(J;X)$,

$$\lim_{t\to 0^+} t^{1-\alpha}I^\alpha\phi(t) = 0$$

and

$$\|f(t,y(t))\| \le \phi(t), \quad \text{for all } y \in B_r^\alpha \text{ and a.e. } t \in J,$$

where $B_r^\alpha := \{y \in C_{1-\alpha}(J,X) : \|y\|_{C_{1-\alpha}(J,X)} \le r\}$ and $r > 0$ is a constant related to the fractional integration of $\phi(t)$.

Note that (H2-1) is not a friendly condition since it requires that the second term on the right hand side of (3.21) tend to zero as t goes to zero. For instance, if $f(t,y) = e^{-t}z_0 + t\sin y + y$ where $z_0 \in X$ is given, then (H2) holds but (H2-1) does not hold. From Remark 3.6 and 3.7, we have the following conclusion.

Remark 3.8. Theorem 3.3 improves the existence theorems of the mild solution to (1.1) in [13, 18] because the condition (H2-1) is removed.

4. Optimal control problem

Based on Theorem 3.3, this section deals with an optimal control problem governed by semilinear fractional diffusion equations. For simplicity, from now on, we set

$X = L^2(\Omega)$ and $Q = \Omega \times (0,T)$ where Ω is a bounded subset of $\mathbb{R}^m$ with $m \geq 1$. It follows that $L^2(J;X) = L^2(Q)$. Let $\mathcal{U} = L^2(J;U)$ where U is a given reflexive Banach space. We consider an optimal control problem:
minimize the functional

$$\mathcal{J}(u) = \frac{1}{2}\|y - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|u\|^2_{\mathcal{U}}, \tag{4.1}$$

on all $(y,u) \in L^2(Q) \times \mathcal{U}_{ad}$, subject to

$$\begin{cases} {}^{L}D_t^\alpha y - \Delta y = f(y) + Bu \quad \text{in } Q \\ y = 0 \quad \text{ on } \ \partial\Omega \times (0,T) \\ I^{1-\alpha} y(0^+) = y^0 \quad \text{in } \Omega, \end{cases} \tag{4.2}$$

where y is the state, u is the control, z_d belongs to $L^2(Q)$, N is a positive constant, Δ is the Laplace operator, and $\mathcal{U}_{ad}$ is a nonempty, closed and convex subset of the control space $\mathcal{U}$. Moreover, the operator $B : U \to X$ is linear and continuous, but not necessarily compact. For the sake of brevity, from now on we assume that the nonlinear mapping f is independent of the time t.

Definition 4.1. A function $y : J \to X$ is called a mild solution of (4.2) if

$$y(t) = t^{\alpha-1} T_\alpha(t) y^0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(f(y(s)) + Bu(s))ds, \ t > 0.$$

Theorem 4.2. For a given $u \in \mathcal{U}_{ad}$ and $1/2 < \alpha < 1$, the problem (4.2) admits a unique mild solution $y(u) \in C(J';X) \cap L^2(Q)$ under (H2).

Proof. It is known that the Laplace operator with the homogeneous Dirichlet boundary condition is the infinitesimal generator of a C_0-semigroup in $L^2(\Omega)$ which is contractive and compact [5, 8]. Therefore, (4.2) can be transformed into the abstract evolution equation (1.1) where $f(t,y)$ is replaced by $f(y) + Bu$ and $A = \Delta$ with domain $H_0^1(\Omega) \cap H^2(\Omega)$. Therefore, this theorem is a direct corollary of Theorem 3.3 as Bu is fixed. □

Theorem 4.3. Under (H2), the optimal control problem (4.1)-(4.2) admits an optimal pair $(\bar{y}, \bar{u})$.

Proof. Since the functional $\mathcal{J}(u)$ is nonnegative, there exists a minimizing sequence $u_n \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(u_n) = \frac{1}{2}\|y_n - z_d\|^2_{L^2(Q)} + \frac{N}{2}\|u_n\|^2_{\mathcal{U}} \longrightarrow \inf_{u \in \mathcal{U}_{ad}} \mathcal{J}(u), \quad \text{as } n \to \infty, \tag{4.3}$$

where y_n, from Theorem 4.2, is given by

$$y_n(t) = t^{\alpha-1} T_\alpha(t) y^0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(f(y_n(s)) + Bu_n(s))ds. \tag{4.4}$$

Moreover, in view of (4.3), there exists a constant $c_1 > 0$ such that

$$\|y_n\|_{L^2(Q)} \leq c_1, \quad \|u_n\|_{\mathcal{U}} \leq c_1. \tag{4.5}$$

Therefore, by taking a subsequence there exists $\bar{u} \in \mathcal{U}$, $\bar{y} \in L^2(Q)$ such that

$$u_n \rightharpoonup \bar{u} \text{ in } \mathcal{U}, \quad y_n \rightharpoonup \bar{y} \text{ in } L^2(Q).$$

Since u_n belongs to $\mathcal{U}_{ad}$ which is closed and convex, we obtain $\bar{u} \in \mathcal{U}_{ad}$. Next, we aim to show that the pair $(\bar{y}, \bar{u})$ satisfies (4.2). To this end, we first shall prove that the sequence Gy_n, given by

$$Gy_n := \int_{t_1}^{t_2} y_n(t)dt,\ 0 < t_1 < t_2 < T,$$

is relatively compact in X. For simplicity, set

$$\omega_n(t) := f(y_n(t)) + Bu_n(t).$$

Since $B : U \to X$ is a bounded and linear operator, from (4.5) and (H2), there exist constants c_2, c_3 such that

$$\|f(\cdot, y_n)\|_{L^2(Q)} \leq c_2, \quad \|\omega_n\|_{L^2(Q)} \leq c_3.$$

For each $\varepsilon \in (0, t_1)$, and $\delta > 0$, set

$$\begin{aligned} G_{\varepsilon,\delta} y_n = &\int_{t_1}^{t_2} \int_0^{t-\epsilon} (t-s)^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi_\alpha(\theta) T((t-s)^\alpha \theta)\omega_n(s) d\theta ds dt \\ &+ \int_{t_1}^{t_2} t^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi_\alpha(\theta) T(t^\alpha\theta) y^0 d\theta dt := I_1 + I_2. \end{aligned}$$

Since $T(t)$ is a C_0-semigroup, $s \in (0, t-\varepsilon),\ \theta \in (\delta, \infty)$, we have

$$(t-s)^\alpha\theta - \varepsilon^\alpha\delta > 0.$$

Therefore,

$$\begin{aligned} I_1 = &T(\varepsilon^\alpha\delta) \int_{t_1}^{t_2} \int_0^{t-\varepsilon} (t-s)^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi_\alpha(\theta) T((t-s)^\alpha\theta - \varepsilon^\alpha\delta)\omega_n(s) d\theta ds dt \\ := &T(\varepsilon^\alpha\delta) E_1. \end{aligned}$$

From (3.3), we have

$$\lim_{\delta\to 0} \int_\delta^\infty \theta\Phi_\alpha(\theta) d\theta = \frac{1}{\Gamma(\alpha+1)}.$$

It follows that (since $\Phi_\alpha(\theta) \geq 0$ for all $\theta \geq 0$ [3])

$$\int_\delta^\infty \theta\Phi_\alpha(\theta) d\theta \leq \frac{1}{\Gamma(\alpha+1)},$$

for δ being sufficiently small. Because $T(t)$ is a contractive semigroup, i.e., $\|T(t)\| \leq 1$, using the Hölder inequality, we obtain

$$\begin{aligned} \|E_1\| &= \left\| \int_{t_1}^{t_2} \int_0^{t-\varepsilon} (t-s)^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi(\theta) T((t-s)^\alpha\theta - \varepsilon^\alpha\delta)\omega_n(s) d\theta ds dt \right\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \int_0^{t-\varepsilon} (t-s)^{\alpha-1} \|\omega_n(s)\| ds dt \\ &\leq \frac{(t_2-t_1)}{\Gamma(\alpha)} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \|\omega_n\|_{L^2(Q)}. \end{aligned}$$

We calculate

$$I_2 = T(t_1^\alpha \delta) \int_{t_1}^{t_2} t^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi_\alpha(\theta) T(t^\alpha\theta - t_1^\alpha\delta) y^0 d\theta dt := T(t_1^\alpha\delta) E_2,$$

where

$$\|E_2\| = \left\| \int_{t_1}^{t_2} t^{\alpha-1} \int_\delta^\infty \alpha\theta\Phi_\alpha(\theta) T(t^\alpha\theta - t_1^\alpha\delta) y^0 d\theta dt \right\| \le \frac{(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} \|y^0\|.$$

From the boundness of E_1, E_2 and the compactness of $T(\varepsilon^\alpha\delta), T(t_1^\alpha\delta)$, we deduce $G_{\varepsilon,\delta} y_n$ is relatively compact in X for each $\varepsilon \in (0,t)$ and $\delta > 0$.
Moreover, we have

$$\begin{aligned} &\|Gy_n - G_{\varepsilon,\delta} y_n\| \\ \le\ & \alpha \left\| \int_{t_1}^{t_2} \int_0^{t-\varepsilon} (t-s)^{\alpha-1} \int_0^\delta \theta\Phi_\alpha(\theta) T((t-s)^\alpha\theta) \omega_n(s) d\theta ds dt \right\| \\ & + \alpha \left\| \int_{t_1}^{t_2} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \int_0^\infty \theta\Phi_\alpha(\theta) T((t-s)^\alpha\theta) \omega_n(s) d\theta ds dt \right\| \\ & + \alpha \left\| \int_{t_1}^{t_2} t^{\alpha-1} \int_0^\delta \theta\Phi_\alpha(\theta) T(t^\alpha\theta) y^0 d\theta dt \right\| := I_3 + I_4 + I_5. \end{aligned}$$

By the assumption (H1) and the Hölder inequality, we have

$$I_3 \le \alpha(t_2 - t_1) \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \|\omega_n\|_{L^2(Q)} \int_0^\delta \theta\Phi_\alpha(\theta) d\theta,$$

$$I_4 \le \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \|\omega_n(s)\| ds dt \le \frac{t_2 - t_1}{\Gamma(\alpha)} \sqrt{\frac{\varepsilon^{2\alpha-1}}{2\alpha-1}} \|\omega_n\|_{L^2(Q)},$$

$$I_5 \le (t_2^\alpha - t_1^\alpha) \|y^0\| \int_0^\delta \theta\Phi_\alpha(\theta) d\theta.$$

Because of $\|\omega_n\|_{L^2(Q)} \le c_3$, we deduce that I_3, I_4 and I_5 tend to zero as $\varepsilon \to 0$ and $\delta \to 0$. Therefore,

$$G_{\varepsilon,\delta} y_n \to G y_n, \text{ as } \varepsilon, \delta \to 0.$$

Since $G_{\varepsilon,\delta} y_n$ is relatively compact in X, the sequence Gy_n is relatively compact in X too.
Next, we aim to prove that

$$\int_0^{T-h} \|y_n(t+h) - y_n(t)\|^2 dt \to 0, \quad \text{as } h \to 0^+. \tag{4.6}$$

Similarly to (3.5), we have

$$\begin{aligned}
&\|y_n(t+h)-y_n(t)\| \\
\le\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(y_n(s+h))-f(y_n(s))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{-h}^0 (t-s)^{\alpha-1}\|f(y_n(s+h))\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t \left\|(t+h-s)^{\alpha-1}T_\alpha(h)Bu_n(s)-(t-s)^{\alpha-1}Bu_n(s)\right\|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_t^{t+h} (t+h-s)^{\alpha-1}\|Bu_n(s)\|ds \\
&+\frac{1}{\Gamma(\alpha)}\|(t+h)^{\alpha-1}T_\alpha(h)y^0-t^{\alpha-1}y^0\| \\
:=\ & \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_n(s+h)-y_n(s)\|ds+I_6+I_7+I_8+I_9.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|y_n(t+h)-y_n(t)\|^2 \\
\le& 5\left(\frac{L}{\Gamma(\alpha)}\right)^2\frac{t^{2\alpha-1}}{2\alpha-1}\int_0^t \|y_n(s+h)-y_n(s)\|^2ds+5(I_6^2+I_7^2+I_8^2+I_9^2).
\end{aligned}\tag{4.7}$$

According to the Gronwall's inequality of differential form, we deduce that

$$\int_0^{T-h}\|y_n(s+h)-y_n(s)\|^2ds\le 5e^{5\int_0^{T-h}\left(\frac{L}{\Gamma(\alpha)}\right)^2\frac{t^{2\alpha-1}}{(2\alpha-1)}dt}\int_0^{T-h}(I_6^2+I_7^2+I_8^2+I_9^2)dt.$$

Since

$$\int_0^{T-h}\left(\frac{L}{\Gamma(\alpha)}\right)^2\frac{s^{2\alpha-1}}{(2\alpha-1)}ds\le\frac{(LT^\alpha)^2}{(\Gamma(\alpha))^2(2\alpha)(2\alpha-1)},$$

we have

$$\int_0^{T-h}\|y_n(s+h)-y_n(s)\|^2ds\le 5e^{\frac{5(LT^\alpha)^2}{(\Gamma(\alpha))^2(2\alpha)(2\alpha-1)}}\int_0^{T-h}(I_6^2+I_7^2+I_8^2+I_9^2)ds.\tag{4.8}$$

Using the Hölder inequality and the fact that $\|f(y_n)\|_{L^2(Q)}\le c_2$, we get

$$I_6^2\le\left(\frac{c_2}{\Gamma(\alpha)}\right)^2\int_{-h}^0(t-s)^{2\alpha-2}ds\le\left(\frac{c_2}{\Gamma(\alpha)}\right)^2\int_0^h t^{2\alpha-2}ds=\left(\frac{c_2}{\Gamma(\alpha)}\right)^2 t^{2\alpha-2}h.$$

Then we have

$$\int_0^{T-h}I_6^2dt\le\left(\frac{c_2}{\Gamma(\alpha)}\right)^2\int_0^T t^{2\alpha-2}hdt=\frac{c_2^2T^{2\alpha-1}h}{(2\alpha-1)\Gamma^2(\alpha)}\to 0\text{ as }h\to 0^+.\tag{4.9}$$

On the other hand, we deduce that

$$\begin{aligned}
&\lim_{h\to 0^+}\int_0^{T-h} I_7^2 dt \\
&\leq \lim_{h\to 0^+}\left(\frac{1}{\Gamma(\alpha)}\right)^2\int_0^{T-h}\left(\int_0^t ((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1})\|Bu_n(s)\|ds\right)^2 dt \\
&\leq \lim_{h\to 0^+}\left(\frac{1}{\Gamma(\alpha)}\right)^2\|Bu_n\|^2_{L^2(Q)}\int_0^{T-h}\int_0^t ((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1})^2 dsdt.
\end{aligned}$$

Now for each fixed $t\in(0,T-h)$, as $T^{2\alpha-2}$ is decreasing on $(0,+\infty)$, we have

$$((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1})^2 \leq (t-s)^{2\alpha-2}\in L(J),$$

and

$$\left((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)^2 \to 0, \quad \text{as } h\to 0^+,$$

using the Lebesgue dominated convergence theorem, we have

$$\int_0^t \left((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)^2 ds \to 0 \quad \text{as } h\to 0^+. \tag{4.10}$$

Moreover, we see that

$$\int_0^t \left((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)^2 ds \leq \int_0^t (t-s)^{2\alpha-2}ds = \frac{t^{2\alpha-1}}{2\alpha-1}.$$

Since $t^{2\alpha-1}\in L(J)$, using (4.10) and the Lebesgue dominated convergence theorem once more, we have

$$\lim_{h\to 0^+}\int_0^{T-h}\int_0^t ((t+h-s)^{\alpha-1}-(t-s)^{\alpha-1})^2 dsdt = 0.$$

Thus, by the boundedness of $\|Bu_n\|^2_{L^2(Q)}$, we find

$$\int_0^{T-h} I_7^2 dt \to 0 \text{ as } h\to 0^+. \tag{4.11}$$

Next, by the Hölder inequality, we get

$$\begin{aligned}
\int_0^{T-h} I_8^2 dt &\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\|Bu_n\|^2_{L^2(Q)}\int_0^{T-h}\frac{h^{2\alpha-1}}{2\alpha-1}dt \\
&\leq \left(\frac{1}{\Gamma(\alpha)}\right)^2\|Bu_n\|^2_{L^2(Q)}\frac{Th^{2\alpha-1}}{2\alpha-1} \to 0 \text{ as } h\to 0^+.
\end{aligned}\tag{4.12}$$

We also see that

$$\lim_{h\to 0^+}\int_0^{T-h} I_9^2 dt = \lim_{h\to 0^+}\left(\frac{1}{\Gamma(\alpha)}\right)^2\|y^0\|^2\int_0^{T-h}\left((t+h)^{\alpha-1}-t^{\alpha-1}\right)^2 dt = 0. \tag{4.13}$$

In conclusion, from(4.8), (4.9), (4.11), (4.12) and (4.13), we deduce that (4.6) holds.

Since Gy_n is relatively compact, it follows from (4.6) that the sequence y_n is relatively compact in $L^2(Q)$, Passing to a subsequence (denoted by the same notation for simplicity) and recalling $y_n \rightharpoonup \bar{y}$ in $L^2(Q)$, we have

$$y_n \to \bar{y} \ \text{ in } L^2(Q). \tag{4.14}$$

Finally, we aim to show that $\bar{y} = \bar{y}(\bar{u})$ is the mild solution of problem (4.2).
Using (H2), we have

$$f(y_n) \to f(\bar{y}) \ \text{ in } L^2(Q),$$

and

$$\left\| \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(f(y_n(s)) - f(\bar{y}(s)))ds \right\| \leq \frac{L\sqrt{T^{2\alpha-1}}}{\Gamma(\alpha)\sqrt{2\alpha-1}} \|y_n - \bar{y}\|_{L^2(Q)}.$$

Then, if follows from (4.14) that

$$\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(f(y_n(s)) - f(\bar{y}(s)))ds \to 0 \text{ a.e. } t \in J. \tag{4.15}$$

Using $u_n \rightharpoonup \bar{u}$ in $\mathcal{U}$, we have

$$Bu_n \rightharpoonup B\bar{u} \ \text{ in } L^2(Q). \tag{4.16}$$

For each $\upsilon \in L^2(\Omega)$, we set

$$\bar{\upsilon}(x,s) = \begin{cases} (t-s)^{\alpha-1} T_\alpha^*(t-s)\upsilon(x), & 0 < s < t, \\ 0, & t \leq s \leq T. \end{cases}$$

Obviously, $\bar{\upsilon} \in L^2(Q)$. We see that

$$\begin{aligned} &\int_\Omega \upsilon(x) \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\big(Bu_n(s) - B\bar{u}(s)\big)dsdx \\ &= \int_0^t \int_\Omega \upsilon(x)(t-s)^{\alpha-1} T_\alpha(t-s)\big(Bu_n(s) - B\bar{u}(s)\big)dxds \\ &= \int_0^t \big((t-s)^{\alpha-1}\upsilon, T_\alpha(t-s)\big(Bu_n(s) - B\bar{u}(s)\big)\big)_{L^2(\Omega)} ds \\ &= \int_0^t \big((t-s)^{\alpha-1} T_\alpha^*(t-s)\upsilon, Bu_n(s) - B\bar{u}(s)\big)_{L^2(\Omega)} ds \\ &= (\bar{\upsilon}, Bu_n - B\bar{u})_{L^2(Q)}, \end{aligned}$$

which goes to zero from (4.16) as $n \to \infty$. This implies that for a.e. $t \in J$,

$$\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)Bu_n(s)ds \rightharpoonup \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)B\bar{u}(s)ds \ \text{ in } L^2(\Omega). \tag{4.17}$$

From (4.15) and (4.17), it follows

$$y_n(t) \rightharpoonup t^{\alpha-1} T_\alpha(t)y^0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(f(\bar{y}(s)) + B\bar{u}(s))ds \text{ a.e. } t \in J.$$

Recalling (4.14), we deduce that

$$\bar{y}(t) = t^{\alpha-1}T_\alpha(t)y^0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)(f(\bar{y}(s)) + B\bar{u}(s))ds \text{ a.e. } t \in J. \quad (4.18)$$

Thus, $\bar{y}$ is the mild solution of (4.2) corresponding to the fixed control $\bar{u} \in \mathcal{U}_{ad}$. From the weak lower semi-continuity of the functional $\mathcal{J}(u)$, we have

$$\liminf_{n\to\infty} \mathcal{J}(u_n) \geq \mathcal{J}(\bar{u}).$$

Hence, according to (4.3), we have

$$\mathcal{J}(\bar{u}) \leq \inf_{u\in\mathcal{U}_{ad}} \mathcal{J}(u),$$

which implies that

$$\mathcal{J}(\bar{u}) = \inf_{u\in\mathcal{U}_{ad}} \mathcal{J}(u). \quad (4.19)$$

Thus, from (4.18) and (4.19), $(\bar{y}, \bar{u})$ is an optimal pair of (4.1)-(4.2). The proof of Theorem 4.3 is complete. □

To study the necessary optimality conditions of the optimal control problem (4.1)-(4.2), in what follows, we further assume that U is a Hilbert space. Thus, $\mathcal{U} = L^2(J;U)$ is a Hilbert space too. Let $(\cdot,\cdot)_{\mathcal{U}}$ stand for the inner product on $\mathcal{U}$. Moreover, the following additional hypothesis is considered:

(H3) the mapping $f(y)$ is Fréchet differentiable with respect to y.

Theorem 4.4. If $(\bar{y}, \bar{u})$ is an optimal pair of the problem (4.1) subject to (4.2). Then there exist $\bar{p} \in L^2(Q)$ and the triple $(\bar{y}, \bar{u}, \bar{p})$ satisfies the system

$$\begin{cases} -\mathfrak{D}^\alpha \bar{p} - \Delta\bar{p} - f'(\bar{y})\bar{p} = \bar{y} - z_d \ \text{ in } Q \\ \bar{p} = 0 \ \text{ on } \partial\Omega \times (0,T) \\ \bar{p}(T) = 0 \ \text{ in } \Omega, \end{cases} \quad (4.20)$$

and

$$\big(B(v-\bar{u}), \bar{p}\big)_{L^2(Q)} + N\big(\bar{u}, v-\bar{u}\big)_{\mathcal{U}} \geq 0, \ \forall\, v \in \mathcal{U}_{ad}. \quad (4.21)$$

Note that (4.20) and (4.21) are referred to as the first-order necessary optimality conditions to the optimal control problem (4.1)-(4.2) where (4.20) is known as the adjoint problem of (4.2). To prove this theorem, we present several lemmas. The first is the so-called fractional integration by parts formula; see, e.g, [17].

Lemma 4.5. For any $\varphi \in C^\infty(\overline{Q})$, we have

$$\begin{aligned} &\int_0^T \int_\Omega ({}^L D_t^\alpha(y(x,t)) - \Delta y(x,t))\varphi(x,t)dxdt \\ =& \int_\Omega \varphi(x,T)I^{1-\alpha}y(x,T)dx - \int_\Omega \varphi(x,0)I^{1-\alpha}y(x,0^+)dx \\ &+ \int_0^T\int_{\partial\Omega} y\frac{\partial\varphi}{\partial v}d\sigma dt - \int_0^T\int_{\partial\Omega}\frac{\partial y}{\partial v}\varphi d\sigma dt \\ &+ \int_0^T\int_\Omega y(x,t)(-\mathfrak{D}^\alpha\varphi(x,t) - \Delta\varphi(x,t))dxdt. \end{aligned}$$

The second is connected to the solvability of problem (4.20).

Lemma 4.6. Assume that $\frac{1}{2} < \alpha < 1$, $g \in L^2(Q)$, Then, the following problem

$$\begin{cases} -\mathfrak{D}^\alpha \bar{p}(t) - \Delta \bar{p}(t) - f'(\bar{y})\bar{p}(t) = g(t) \ \text{ in } Q \\ \bar{p} = 0 \ \text{ on } \partial\Omega \times (0,T) \\ \bar{p}(T) = 0 \ \text{ in } \Omega, \end{cases} \tag{4.22}$$

has a unique mild solution $\bar{p} \in C(J;X)$ given by

$$\bar{p}(t) = \int_0^{T-t} (T-t-s)^{\alpha-1} T_\alpha (T-t-s)(f'(\bar{y})\bar{p}(T-s) + g(T-s))ds, \tag{4.23}$$

with the bound

$$\|\bar{p}\|_{C(J;X)} \le \frac{1}{\Gamma(\alpha)} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \|g\|_{L^2(Q)} E_\alpha(T^\alpha \|f'(\bar{y})\|). \tag{4.24}$$

Proof. We now define a mapping $\mathfrak{J}_T p(t) = p(T-t)$, $t \in J$. A direct computation gives $\mathfrak{D}^\alpha \mathfrak{J}_T p(t) = -{}^C D_t^\alpha \mathfrak{J}_T p(t)$; see, e.g. the proof in [17, Proposition 3.6]. Making the change of variable $t \to T - t$ in (4.22), we obtain

$$\begin{cases} {}^C D_t^\alpha \mathfrak{J}_T \bar{p}(t) - \Delta \mathfrak{J}_T \bar{p}(t) - f'(\mathfrak{J}_T \bar{y})\mathfrak{J}_T \bar{p}(t) = \mathfrak{J}_T g(t) \ \text{ in } Q \\ \mathfrak{J}_T \bar{p} = 0 \ \text{ on } \partial\Omega \times (0,T) \\ \mathfrak{J}_T \bar{p}(0) = 0 \ \text{ in } \Omega. \end{cases} \tag{4.25}$$

As usual, a mild solution $\mathfrak{J}_T \bar{p} \in C(J;X)$ to (4.25) is defined by

$$\mathfrak{J}_T \bar{p}(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\big(f'(\mathfrak{J}_T \bar{y})\mathfrak{J}_T \bar{p}(s) + \mathfrak{J}_T g(s)\big)ds. \tag{4.26}$$

Replacing $f(t,y)$ in (H2) with $f'(\mathfrak{J}_T \bar{y})y + \mathfrak{J}_T g$, we see that the inequality in (H2) is satisfied with $L = \|f'(\mathfrak{J}_T \bar{y})\|$ because $\bar{y}$, g are fixed, and $f'(\mathfrak{J}_T \bar{y})$ is a linear and continuous mapping. Taking A as the Laplace operator with the homogeneous Dirichlet boundary condition, by a similar proof to Theorem 3.3, we deduce that (4.25) has a unique mild solution $\mathfrak{J}_T \bar{p} \in C(J;X)$ given by (4.26). In fact, here is easier since (4.25) is linear and the initial value is zero. Moreover,

$$\|\mathfrak{J}_T \bar{p}\|_{C(J;X)} \le \frac{1}{\Gamma(\alpha)} \sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}} \|\mathfrak{J}_T g\|_{L^2(Q)} E_\alpha(T^\alpha \|f'(\mathfrak{J}_T \bar{y})\|). \tag{4.27}$$

Making the change of variable $t \to T - t$ from (4.25) to (4.27), we see that this lemma holds. The proof is complete. $\square$

Now we are in a position to prove Theorem 4.4.

Proof. Define the Lagrangian function $\mathcal{L}$ associated with the problem (4.1)-(4.2),

$$\mathcal{L}(y,u,p) := \mathcal{J}(u) - \int_0^T \int_\Omega ({}^L D_t^\alpha y - \Delta y - f(y) - Bu)p dx dt.$$

Since $(\bar{y}, \bar{u})$ is an optimal pair, by the formal Lagrange multiplier method, see, e.g. [26], we obtain the following first-order necessary optimality conditions:

$$D_y\mathcal{L}(\bar{y}, \bar{u}, p)h = 0, \ \forall\, h \in C^{\infty}(\overline{Q}) \text{ with } h|_{\partial\Omega} = 0\,, I^{1-\alpha}h(0^+) = 0 \ \text{ in } \Omega, \tag{4.28}$$

$$D_u\mathcal{L}(\bar{y}, \bar{u}, p)(v - \bar{u}) \geq 0, \quad \text{for all } v \in \mathcal{U}_{ad}. \tag{4.29}$$

The condition (4.28) leads to the adjoint equation. In fact, from (4.28), we have

$$D_y\mathcal{L}(\bar{y}, \bar{u}, p)h = \int_0^T \int_\Omega (\bar{y} - z_d)hdxdt - \int_0^T \int_\Omega ({}^L D_t^\alpha h - \Delta h - f'_y(\bar{y})h)pdxdt.$$

According to Lemma 4.5, we further deduce that

$$\begin{aligned} 0 = &\int_0^T \int_\Omega (\bar{y} - z_d + \mathfrak{D}^\alpha p + \Delta p + f'(\bar{y})p)hdxdt + \int_0^T \int_{\partial\Omega} \frac{\partial h}{\partial v} pd\sigma dt \\ &- \int_\Omega p(x,T) I^{1-\alpha}h(x,T)dx. \end{aligned} \tag{4.30}$$

Note that for all $h \in C_0^\infty(Q)$ the expressions $I^{1-\alpha}h(T)$ and $\frac{\partial h}{\partial v}$ vanish on Ω and $\partial\Omega$, respectively. Consequently,

$$\int_0^T \int_\Omega (\bar{y} - z_d + \mathfrak{D}^\alpha p + \Delta p + f'(\bar{y})p)hdxdt = 0, \ \forall\ h \in C_0^\infty(Q).$$

Recalling that $C_0^\infty(Q)$ is dense in $L^2(Q)$, we have

$$-\mathfrak{D}^\alpha p - \Delta p - f'(\bar{y})p = \bar{y} - z_d \ \text{ in } Q. \tag{4.31}$$

Then, for all $h \in C^\infty(\overline{Q})$ with $h|_{\partial\Omega} = 0$ and $I^{1-\alpha}h(T) = 0$ on Ω , from (4.30) and (4.31) it follows

$$p|_{\partial\Omega} = 0. \tag{4.32}$$

Finally, for all $h \in C^\infty(\overline{Q})$ with $h|_{\partial\Omega} = 0$, from (4.30) to (4.32), we get

$$p(x,T) = 0 \ \text{ in } \Omega. \tag{4.33}$$

Note that (4.31) to (4.33) is referred to as the adjoint problem to (4.2). Now it follows from Lemma 4.6 that problem (4.31) to (4.33) has a unique mild solution $\bar{p} \in C(J;X)$, i.e. $(\bar{y}, \bar{u}, \bar{p})$ satisfies (4.20).
Moreover, from the condition (4.29) with $p = \bar{p}$, we have

$$D_u\mathcal{L}(\bar{y}, \bar{u}, \bar{p})(v - \bar{u}) = (B(v - \bar{u}), \bar{p})_{L^2(Q)} + N(\bar{u}, v - \bar{u})_{\mathcal{U}} \geq 0 \ \forall\ v \in \mathcal{U}_{ad}.$$

Thus, the inequality (4.21) holds. The proof is complete. □

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References

[1] R.P. Agarwal, D. Baleanu, J.J. Nietod, D.F.M. Torrese, Y. Zhou, *A survey on fuzzy fractional differential and optimal control nonlocal evolution equations*, J. Computational and Applied Mathematics, **339**(2017), 3-29.

[2] B. Ahmad, *Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations*, Applied Math. Letters, **23**(2010), 390-394.

[3] E.G. Bajlekova, *Fractional Evolution Equations in Banach spaces*, Technische Universiteit Eindhoven Eindhoven, The Netherlands, 2001.

[4] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press, San Diego, 1993.

[5] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, **92**, Cambridge University Press, Cambridge, 1990.

[6] J.D. Djida, G. Mophou, I. Area, *Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel*, J. Optimization Theory and Applications, **182**(2019) 540-557.

[7] R. Dorville, G.M. Mophou, V.S. Valmorin, *Optimal control of a nonhomogeneous Dirichlet boundary fractional diffusion equation*, Computers and Mathematics with Applications, **62**(2011), 1472-1481.

[8] L.C. Evans, *Partial Differential Equations*, Vol. **19**, Graduate Studies in Mathematics, American Mathematical Society Providence, 2010.

[9] H.B. Gu, J.J. Trujillo, *Existence of mild solution for evolution equation with Hilfer fractional derivative*, Applied Mathematics and Computation, **257**(2015), 344-354.

[10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, No. **840**, 1981.

[11] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[12] J.L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Heidelberg, Berlin, 1971.

[13] Z.H. Liu, X.W. Li, *Approximate controllability of fractional evolution systems with Riemann-Liouville fractional derivatives*, SIAM J. Control and Optimization, **53**(2015), 1920-1933.

[14] Z.H. Liu, X.M. Li, B. Zeng, *Optimal feedback control for fractional neutral dynamical systems*, Optimization, **67**(2018) 549-564.

[15] N.I. Mahmudov, M.A. McKibben, *On the approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative*, J. Function Spaces, (2015), 1-9.

[16] F. Mainardi, P. Paradis, R. Gorenflo, *Probability Distributions Generated by Fractional Diffusion Equations*, (Eds. J. Kertesz, I. Kondor), Econophysics: An Emerging Science, Kluwer, Dordrecht, 2000.

[17] G.M. Mophou, *Optimal control of fractional diffusion equation*, Computers and Mathematics with Applications, **61**(2011), 68-78.

[18] X. Pan, X.W. Li, J. Zhao, *Solvability and optimal controls of semilinear Riemann-Liouville fractional differential equations*, Abstract and Applied Analysis, (2014), 1-11.

[19] Z. Peng, *Optimal obstacle control problems involving nonsmooth cost functionals and quasilinear variational inequalities*, SIAM J. Control and Optimization, **58**(2020), no. 4, 2236-2255.

[20] Z. Peng, K. Kunisch, *Optimal control of elliptic variational-hemivriational inequalities*, J. Optimization Theory and Applications, **178**(2018), 1-25.

[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[22] Ravichandran, N. Valliammal, J.J. Nieto, *New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces*, Journal of the Franklin Institute, **356**(2019), 1535-1565.

[23] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[24] J. Simon, *Compact sets in the space* $L^p(0, T; B)$, Annali di Matematica Pura ed Applicata, **146**(1986), 65-96.
[25] Q. Tang, *On an optimal control problem of time-fractional advection-diffusion equation*, Discrete and Continuous Dynamical Systems-B, **25**(2020), 761-779.
[26] F. Tröltzsch, *Optimal Control of Partial Differential Equations: Theory, Methods and Applications*, American Mathematical Society Providence, vol. **112**, 2010.
[27] J.R. Wang, Y. Zhou, *A class of fractional evolution equations and optimal controls*, Nonlinear Analysis: Real World Applications, **12**(2011), 262-272.
[28] J.R. Wang, Y. Zhou, W. Wei, *Optimal feedback control for semilinear fractional evolution equations in Banach spaces*, Systems and Control Letters, **61**(2012), 472-476.
[29] Y.B. Xiao, M. Sofonea, *On the optimal control of variational-hemivariational inequalities*, J. Mathematical Analysis and Applications, **475**(2019), 364-384.
[30] M. Yang, Q.R. Wang, *Approximate controllability of Riemann-Liouville fractional differential inclusions*, Applied Mathematics and Computation, **274**(2016), 267-281.
[31] H.P. Ye, J.M. Gao, Y.S. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Mathematical Analysis and Applications, **328**(2007), 1075-1081.
[32] Y. Zhou, F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Computers and Mathematics with Applications, **59**(2010), 1063-1077.
[33] Y. Zhou, J.R. Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2016.
[34] Y. Zhou, L. Zhang, X.H. Shen, *Existence of mild solutions for fractional evolution equations*, J. Integral Equations Appl., **25**(2013), 557-586.

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