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COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS VIA CONE-VALUED MEASURE OF NONCOMPACTNESS

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Abstract. We obtain common fixed point theorems for a pair of condensing multivalued mappings with respect to a cone-valued measure of noncompactness under a semi-weakly isotone condition, and we apply it to the system of multivalued differential equations with deviating argument of the form

$$x'(t) \in f[t, x(t), x(h(t))] \text{ and } \tag{0.1}$$

$$x'(t) \in g[t, x(t), x(h(t))], x(0) = x_0, t \in [0, b]. \tag{0.2}$$

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1. Introduction

The study of fixed points for single/multi-valued mappings using spaces endowed with a partial ordering has interested several mathematicians; see [2, 5, 7, 10, 11, 12, 15, 16, 18] and the references therein. S. Heikkilä and S. Hu in [10], S. Heikkilä in [9] obtained some fixed point theorems for a multivalued mapping T, defined on an ordered topological space, under the following condition

$$x_1 \le y_1 \in Tx_1 \text{ and } x_1 \le x_2 \text{ imply } y_1 \le y_2 \text{ for some } y_2 \in Tx_2. \tag{1.3}$$

In a Banach space X with a partial ordering $\le$, and a binary relation $\preceq$ on the family of all nonempty subsets of X defined by $A \preceq B$ if and only if $a \le b$ for all $(a, b) \in A \times B$, B.C. Dhage, Donal O'Regan and R. P Agarwal [7] suggested a weakly isotone condition for a pair multivalued mappings and under this condition they proved common fixed point results of two condensing multivalued mappings with respect to a real-valued measure of noncompactness. One of these results (see Theorem 2.4 in Section 2) was applied to integral inclusions by D. Turkoglu and I. Altun

[18]. The weakly isotone notion was generalized by H. K. Nashine and Z. Kadelburg [16] (see Definition 2.3 in Section 2). In that work, the authors established the existence of common fixed points for a pair of weakly isotone increasing multivalued mappings under a general weakly contractive condition using altering distance functions. Recently, this topic has been interested by many mathematicians, see for example [3, 6, 13, 17, 19].

In this paper, we first prove common fixed point theorems for a pair of condensing-contractive multivalued mappings with respect to a cone-valued measure of noncompactness (c-MNC for brevity) satisfying a semi-weakly isotone condition (defined in Definition 2.5) which is weaker than the weakly isotone condition (see Example 2.6 in Section 2). Next, in Section 4, we prove the existence of a local solution for a system of multivalued differential equations with deviating argument (0.1)-(0.2) to illustrate the advantage of using the c-MNC.

2. Preliminaries

Let X be a real Banach space and P be a cone, that is, P is a nonempty closed convex subset of X such that $\lambda P \subset P$ for all $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. In X we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, and then the triplet $(X, P, \preceq)$ is called an ordered Banach space. The cone P is said to be normal if there exists a number $N > 0$ such that $0 \preceq u \preceq v$ implies $\|u\| \leq N\|v\|$ for all $u, v \in X$.

The following notations will be used throughout the paper. Denote by 2^X (resp. $c(X), b(X), cb(X), cc(X)$) the class of all nonempty (resp. nonempty closed, nonempty bounded, nonempty closed bounded, nonempty closed convex) subsets of X. Let $S, T : X \to 2^X$ be two multivalued mappings, and we say that $x \in X$ is a common fixed point of a pair (S,T) if $x \in Sx \cap Tx$.

We need the following lemma and definition.

Lemma 2.1. [8] *Let X be an ordered Banach space with a normal cone. Assume that $\{x_n\}$ is a monotone sequence which contains a subsequence $\{x_{\sigma(n)}\}$ converging weakly to some $x \in X$. Then $\{x_n\}$ converges to x.*

Definition 2.2. [16] Let $(X, P, \preceq)$ be an ordered Banach space. For every $(A, B) \in 2^X \times 2^X$ we define

1. $A \preceq_1 B$ iff $\forall a \in A, \exists b \in B, a \preceq b$,
2. $A \preceq_2 B$ iff $\forall b \in B, \exists a \in A, a \preceq b$,
3. $A \preceq_3 B$ iff $a \preceq b \ \forall (a, b) \in A \times B$.

Let us recall the notion of weakly isotone increasing maps and one of the results in [7].

Definition 2.3. [7, 16] Let $(X, \preceq)$ be a partially ordered set.
Two maps $S, T : X \to 2^X$ are said to be *weakly isotone increasing* if for any $x \in X$ we have $Sx \preceq_1 Ty$ for all $y \in Sx$ and $Tx \preceq_1 Sy$ for all $y \in Tx$.

The following theorem is given by B. C. Dhage, D. O'Regan and R. P. Agarwal [7].

Theorem 2.4. [7, Theorem 3.1] *Let B be a closed subset of an ordered Banach space X and let $S, T : B \to cb(B)$ be two closed (i.e. have closed graph) weakly isotone mappings (w.r.t. the relation $\preceq_3$) satisfying for any $a \in B$ and for any countable subset A of B the condition*

$$A \subset \{a\} \cup S(A) \cup T(A)$$

implies A is relatively compact, called condition (C_*); here $T(A) = \cup_{x \in A} Tx$. Then S and T have a common fixed point.

Let $(X, P, \preceq)$ be an ordered Banach space, $a \in X$. Denote $(a] = \{x \in X : x \leq a\}$ and $[a) = \{x \in X : a \leq x\}$. We use the following definitions in our main results.

Definition 2.5. Let D be a nonempty subset of X, and $S, T : D \to 2^D$ be two multivalued mappings. The pair (S, T) is said to be

1. *semi-weakly isotone increasing* if for any $x \in D$ we have

$$\{y\} \preceq_1 Ty \text{ for all } y \in Sx \cap [x) \text{ and } \{y\} \preceq_1 Sy \text{ for all } y \in Tx \cap [x);$$

2. *semi-weakly isotone decreasing* if for any $x \in D$ we have

$$Ty \preceq_2 \{y\} \text{ for all } y \in Sx \cap (x] \text{ and } Sy \preceq_2 \{y\} \text{ for all } y \in Ty \cap (x];$$

3. *semi-weakly isotone* if it is either semi-weakly isotone increasing or semi-weakly isotone decreasing.

Clearly, the pair (S, T) satisfying a weakly isotone condition implies that it is semi-weakly isotone. The following example shows that the backward direction of the assertion is not true.

Example 2.6. In the Banach space $X = \mathbb{R} \times \mathbb{R}$ with a partial ordering $\preceq$ by the cone $P = \{(\lambda, \lambda) : \lambda \geq 0\}$ and we consider two maps $S, T : X \to 2^X$ defined by, for $x \in X$, $Sx = \{x + (0, \lambda) : \lambda \geq 0\}$ and $Tx = \{x + (\lambda, \lambda) : \lambda \geq 0\}$. Clearly, $x \in Sx$ and $x \in Tx$ for all $x \in X$. Hence, (S, T) is semi-weakly isotone increasing in the sense of Definition 2.5. Now, choose $x = (1, 1)$, $y = (1, 1 + 1) \in Sx, a = x$, then $x \leq a \in Sx$. Suppose $b \in Ty$, then $b = (1 + \lambda, 2 + \lambda)$ for some $\lambda \geq 0$. It follows that $b - a \notin P$, i.e. $a \npreceq b$, and so $Sx \npreceq_1 Ty$. We deduce that the pair (S, T) is not weakly isotone increasing in the sense of Definition 2.3.

We use some definitions and statements from [1, 4, 5].

Definition 2.7. [1, Definition 1.2.1] Let $(Y, K, \leq)$ be an ordered Banach space with a partial ordering by the cone K, X be a Banach space, $\mathfrak{M}$ be a family of bounded subsets of X such that if $\Omega \in \mathfrak{M}$, then $\overline{co}\Omega \in \mathfrak{M}$. A function $\psi : \mathfrak{M} \to K$ is called a *measure of noncompactness* (c-MNC for brevity) if $\psi(\overline{co}\Omega) = \psi(\Omega)$ for all $\Omega \in 2^X$. The c-MNC ψ is called

1. *monotone* if $\Omega_1, \Omega_2 \in \mathfrak{M}$ and $\Omega_1 \subset \Omega_2$ implies $\psi(\Omega_1) \leq \psi(\Omega_2)$;
2. *semi-additive* if $\psi(\cup_{\Omega \in \zeta} \Omega) \leq \sup_{\Omega \in \zeta} \psi(\Omega)$, where $\zeta \subset \mathfrak{M}$;
3. *regular* if for $\Omega \in \mathfrak{M}$, $\psi(\Omega) = 0 \Leftrightarrow \Omega$ is relatively compact;
4. *semi-homogeneous* if $\psi(\lambda\Omega) = |\lambda|\psi(\Omega)$ for $\Omega \in \mathfrak{M}, \lambda\Omega \in \mathfrak{M}$;
5. *algebraic semi-additive* if $\psi(\Omega_1 + \Omega) \leq \psi(\Omega_1) + \psi(\Omega_2)$ for $\Omega_1, \Omega_2 \in \mathfrak{M}, \Omega_1 + \Omega_2 \in \mathfrak{M}$;
6. *invariant under translations* if $\psi(x + \Omega) = \psi(\Omega)$, for $x \in X, \Omega \in \mathfrak{M}, x + \Omega \in \mathfrak{M}$;

7. *continuous* (w.r.t. the Hausdorff metric ρ) if $\forall \varepsilon > 0, \Omega \in \mathfrak{M}, \exists \delta > 0$ we have

$$\|\psi(\Omega_1) - \psi(\Omega)\| < \varepsilon \text{ for all } \Omega_1 \in \mathfrak{M}, \rho(\Omega_1, \Omega) < \delta.$$

Remark 2.8. If the c-MNC ψ satisfies the properties 1, 2, 3 (in Definition 2.7) then we have

1. $\psi(\cup_{\Omega\in\zeta}\Omega) = \sup_{\Omega\in\zeta} \psi(\Omega)$, where $\zeta \subset \mathfrak{M}$ and
2. $\psi(\Omega \cup \{a\}) = \psi(\Omega)$ for all $a \in X$ and $\Omega \subset X$ satisfying $\Omega \in \mathfrak{M}, \{a\} \cup \Omega \in \mathfrak{M}$.

Example 2.9. [1, subsection 1.2.4] Consider a Banach space $(E, |.|)$ and a real-valued measure of noncompactness φ defined on the class of all bounded subsets of E. Let $X = C([a, b]; E)$ be the Banach space of all E-valued continuous functions on $[a, b]$ with the norm $\|x\| = \sup_{t\in[a,b]}|x(t)|$. We denote by $\mathfrak{M}$ the family of all bounded equicontinuous subsets of X. For each $\Omega \in \mathfrak{M}$ we set $\Omega(t) = \{x(t) : x \in \Omega\}$ and define a function $\psi(\Omega) : [a, b] \to \mathbb{R}$ by $\psi(\Omega)(t) = \varphi[\Omega(t)]$. If φ is continuous, then $\psi(\Omega) \in C([a, b], \mathbb{R})$. Consequently, there exists the mapping ψ from $\mathfrak{M}$ into the cone K of nonnegative functions in $C([a, b]; \mathbb{R})$. The map ψ is a c-MNC in the sense of Definition 2.7 and if φ has a property in Definition 2.7 then ψ has the same property.

Definition 2.10. [5, 1, 16] Let X be a Banach space, D be a subset of X and $\mathfrak{M}$ be a subset of $b(D)$. Assume that ψ is a c-MNC defined on $\mathfrak{M}$ with values in some partially ordered set $(K, \leq)$. A map $T : D \to 2^D$ is said to be

1. ψ-*condensing* if for $\Omega \in 2^D \cap \mathfrak{M}$ such that $T(\Omega) \in \mathfrak{M}$ and $\psi(\Omega) \leq \psi(T(\Omega))$, then Ω is relatively compact;
2. ψ-*contractive* if for $\Omega \in 2^D \cap \mathfrak{M}$ satisfying $T(\Omega) \in \mathfrak{M}$, then $\psi(T(\Omega)) \leq \psi(\Omega)$.

We need the following definitions and notations in the following.

Definition 2.11. Let X be a Banach space, $D \in 2^X$. A map $T : D \to 2^X$ is said to be upper semicontinuous (u.s.c. for brevity) if the set $\{x \in D : F(x) \subset W\}$ is open in D for every open subset W in X.

Definition 2.12. Let X be an ordered Banach space, $D \in c(X)$. A map $T : D \to 2^X$ is said to be monotone-closed if for each monotone sequence $\{x_n\}$ with $x_n \to x$ and for each sequence $\{y_n\}$ with $y_n \in Tx_n$ such that $y_n \to y$, we have $y \in Tx$.

Let E be a Banach space with the norm $|.|$, and J be a closed bounded interval in $\mathbb{R}$. Denote by $C(J, E)$ the space of all E-valued continuous functions on J with the norm $\|x\| = \sup_{t\in J}|x(t)|$, by $L^1(J, E)$ the class of all E-valued Bochner integrable functions on J, and by $L^1(J, \mathbb{R})$ the class of all real-valued measurable functions whose absolute value is Lebesgue integrable on J.

Definition 2.13. Let D be a bounded subset of E. A map $F : J \times D \to 2^E$ is said to be Carathéodory if

(i) for every $y \in E$ the function $t \mapsto \inf\{|z - y| : z \in F(t, x)\}$ is a measurable function for all $x \in D$, and
(ii) the map $x \mapsto F(t, x)$ is an u.s.c. almost everywhere for $t \in J$

Definition 2.14. Let D be a bounded subset of E. A map $F : J \times D \to 2^E$ is said to be L^1-Carathéodory if

(i) F is Carathéodory, and
(ii) for every real number $r > 0$, there exists a function $\varphi_r \in L^1(J, \mathbb{R})$ such that $\sup_{|x|\leq r} |||F(t,x)||| \leq \varphi_r(t)$, a.e. $t \in J$; where

$$|||F(t,x)||| = \sup\{|u| : u \in F(t, x(t))\}.$$

Remark 2.15. *Let $F : J \times E \to c(E)$ be a L^1-Carathéodory and $x \in L^1(J, E)$. Then, the set $S^1_F(x) := \{u \in L^1(J, E) : u(t) \in F(t, x(t))$ a.e.$t \in J\}$ is nonempty.*

We also need the following lemma. due to Lasota and Opial [14].

Lemma 2.16. [14] *Let $F : J\times E \to cc(E)$ be a Carathéodory function with $S^1_F(x) \neq \emptyset$ for all $x \in L^1(J, E)$ and $\mathfrak{L} : L^1(J, E) \to C(J, E)$ be a continuous linear mapping. Then the graph of the operator $\mathfrak{L}\circ S^1_F : L^1(J, E) \to 2^{C(J,E)}$, defined by $\mathfrak{L}\circ S^1_F(x) = \mathfrak{L}(S^1_F(x))$, is a closed subset in $C(J, E) \times C(J, E)$.*

3. Main results

In this section, we prove common fixed point theorems for a pair of c-MNC-condensing multivalued mappings under a semi-weaky isotone condition.

Theorem 3.1. *Let $(X, P, \preceq)$ be an ordered Banach space with the normal cone P, D be a nonempty subset of X, and ψ be a monotonic and semi-additive c-MNC defined on 2^D such that for any countable subset A of X we have*

$$\psi(A \cup \{a\}) = \psi(A) \text{ for all } a \in X. \tag{3.4}$$

Assume that $S, T : D \to 2^D$ are two monotone-closed mappings satisfying the following conditions:

1. *S is ψ-condensing and T is ψ-contractive,*
2. *(S, T) is semi-weakly isotone increasing (resp. decreasing).*
3. *There exists $x_0 \in D$ such that $\{x_0\} \preceq_1 Sx_0$ (resp. $Sx_0 \preceq_2 \{x_0\}$).*

Then S, T have a common fixed point.

Proof. Assume that (S, T) is semi-weakly isotone increasing. Since $\{x_0\} \preceq_1 Sx_0$, we can choose $x_1 \in Sx_0$ such that $x_0 \preceq x_1$. Since $\{x_1\} \preceq_1 Tx_1$ we can choose an element $x_2 \in Tx_1$ such that $x_1 \preceq x_2$. Repeating the argument above for the pair x_1, x_2, and so on, we can construct an increasing sequence $\{x_n\}$ satisfying

$$x_{2n+1} \in Sx_{2n} \text{ and } x_{2n+2} \in Tx_{2n+1} \text{ for all } n = 0, 1, 2, ... \tag{3.5}$$

Now, define $A_j = \{x_{np+j} : n \in \mathbb{N}\}, j = 0, 1$, and $A_2 = A_0\backslash \{x_0\}$ we have

$$A_1 \subset S(A_0) \text{ and } A_2 \subset T(A_1). \tag{3.6}$$

Set $A = (\cup^2_{k=1} A_k) \cup \{x_0\}$. Since T is ψ-contractive and from the monotonic property of ψ, it follows from (3.6) that

$$\psi(A_2) \leq \psi(T(A_1)) \leq \psi(A_1) \leq \psi(S(A_0)). \tag{3.7}$$

Therefore, we have

$$\begin{aligned}\psi(A) &= \psi((\cup_{k=1}^{2} A_k) \cup \{x_0\}) \\ &= \psi(\cup_{k=1}^{2} A_k) \\ &\le \sup\{\psi(A_k) : k = 1, 2\} \\ &\le \psi(S(A_0)) \\ &\le \psi(S(A)).\end{aligned}$$

Since S is ψ-condensing we deduce that A is relatively compact. It follows from the normality of P that $\{x_n\}$ is convergent to some x_*. Since $x_{2n+1} \in Sx_{2n}$ and S has closed graph, we obtain $x_* \in S(x_*)$. Similarly $x_* \in T(x_*)$, consequently x_* is a common fixed point for S, T. The case (S, T) is semi-weakly isotone decreasing is similar, which ends the proof. □

Theorem 3.2. *Let $(X, P, \preceq)$ be an ordered Banach space with the normal cone P, D be a nonempty subset of X, $(E, K, \le)$ be an ordered Banach space with the cone K, and $\psi : 2^D \to K$ be a regular monotonic semi-additive c-MNC. Let $S, T : D \to 2^D$ be two monotone-closed mappings satisfying the following conditions:*
(C_1) there exists a increasing mapping $\gamma : K \to K$ such that

$$\psi(S(\Omega)) \le \gamma(\psi(\Omega)) \text{ and } \psi(T(\Omega)) \le \gamma(\psi(\Omega)) \text{ for all } \Omega \in 2^D, \tag{3.8}$$

(C_2) if $u \in K$ and $u \le \gamma(u)$ then $u = 0$,
(C_3) (S, T) is semi-weakly isotone increasing (resp. decreasing), and
(C_4) there exists $x_0 \in D$ satisfying $\{x_0\} \preceq_1 Sx_0$ (resp. $Sx_0 \preceq_2 \{x_0\}$).
Then S, T have a common fixed point.

Proof. Assume that (S, T) is semi-weakly isotone increasing. We can construct an increasing sequece $\{x_n\}$, and the sets A_0, A_1, A_2, A similar to the proof in Theorem 3.1. From (3.6) and hypothesis (C_1) we have

$$\psi(A_2) \le \psi(T(A_1)) \le \gamma(\psi(A_1)) \le \gamma(\psi(A)) \tag{3.9}$$

and

$$\psi(A_1) \le \psi(S(A_0)) \le \gamma(\psi(A_0)) \le \gamma(\psi(A)) \tag{3.10}$$

From (3.9), (3.10) and the properties (monotonic, semi-additive) of ψ we obtain $\psi(A) \le \gamma(\psi(A))$. This implies $\psi(A) = 0$ by using hypothesis (C_2). It follows from the regularity of the c-MNC ψ that A is relatively compact. The rest of this proof is argued similarly as in the roof of Theorem 3.1. □

Remark 3.3. If S, T, ψ satisfy conditions (C_1) and (C_2), then condition (C_*) holds.

Corollary 3.4. *Suppose that the mappings S, T and the measure ψ satisfy conditions $(C_1), (C_3), (C_4)$ and*
(C_2') the mapping γ is increasing satisfying $\lim_{n\to\infty} \gamma^n(u) = 0$ for all $u \in K$.
Then S, T have a common fixed point.

Proof. Let us prove that (C_2') implies (C_2). In fact, assume that $u \in K$ and $u \le \gamma(u)$. It follows from the monotonicity of γ that $u \le \gamma^n(u)$, so $u = 0$ by using (C_2'). □

4. Application

Let $(E, P_E, \leqslant)$ be an ordered Banach space with norm $|.|$ and the normal cone P_E. In E, we denote by $B(x_0, r)$ the closed ball centered at x_0 of radius r. Let $J = [0, b]$ be a closed and bounded interval in $\mathbb{R}$, and we consider the following multivalued differential equations

$$x'(t) \in f[t, x(t), x(h(t))] \text{ and } x'(t) \in g[t, x(t), x(h(t))], \tag{4.11}$$

for $t \in J$ and $x(0) = x_0$, where $h : J \to \mathbb{R}$ is continuous, and $f, g : J \times B(x_0, r) \times B(x_0, r) \to 2^E$. By a common local solution for the system of equations (4.11), we mean a differentiable function x defined on $[0, b_1]$ such that

$$x'(t) = v_1(t), x'(t) = v_2(t), x(0) = x_0, \tag{4.12}$$

for some v_1, v_2 which are E-valued Bochner integrable functions on $[0, b_1]$ satisfying

$$v_1(t) \in f[t, x(t), x(h(t))] \text{ and } v_2(t) \in g[t, x(t), x(h(t))] \text{ for all } t \in [0, b_1];$$

where $0 < b_1 \leq b$.

Assume that φ is a real-valued measure of noncompactness, defined for all bounded subsets of E, satisfying properties 1-7 in Definition 2.7, for example, φ is either the Hausdorff MNC or the Kuratowski MNC. Let $F, G : J \times B(x_0, r) \to 2^E$ be two multivalued mappings defined by

$$F(t, x) = f(t, x(t), x(h(t))) \text{ and } G(t, x) = g(t, x(t), x(h(t))). \tag{4.13}$$

We consider the equations in (4.11) under the following assumptions.

(H_1) $f, g : J \times D \to cc(E)$ are Carathéodory, where $D = B(x_0, r) \times B(x_0, r)$,

(H_2) there exists a number $M > 0$ such that $|||F(t, x)||| \leq M$ and $|||G(t, x)||| \leq M$ for all $x \in B(x_0, r), t \in J$,

(H_3) $\exists m, l > 0, \exists \alpha \in (0, 1]$:

$$\varphi[f(t, \Omega_1, \Omega_2)] \leq l\varphi(\Omega_1) + m[\varphi(\Omega_2)]^\alpha, \text{ and}$$
$$\varphi[g(t, \Omega_1, \Omega_2)] \leq l\varphi(\Omega_1) + m[\varphi(\Omega_2)]^\alpha$$

for all $(t, \Omega_1, \Omega_2) \in J \times B(x_0, r) \times B(x_0, r)$ and

(H_4) $0 \leq h(t) \leq t^{1/\alpha}$ for all $t \in J$,

(H_5) $0 \in F(t, a_0(t))$, where $a_0(t) = x_0$ for all $t \in J$, and

(H_6) for each $x \in C(J, E)$ we have $F(t, x) \leqslant_1 G(t, y)$ for all $y \in S^1_F(x)$ and $G(t, x) \leqslant_1 F(t, y)$ for all $y \in S^1_G(x)$.

Theorem 4.1. *Assume (H_1)–(H_6) hold. Then there exists a number $b_1 \in (0, b]$ such that the system of multivalued differential equations (4.11) has a solution on $[0, b_1]$.*

Proof. First we observe that if Ω is an equicontinuous subset of $C(J, E)$, by using properties 4, 5, 7 of the measure φ and that the value $\int_0^t v(s)ds$ can be uniformly approximated by integral sums we deduce that

$$\varphi\left(\left\{\int_0^t v(s)ds : v \in \Omega\right\}\right) \leq \int_0^t \varphi(\Omega)ds. \tag{4.14}$$

Since $\alpha \le 1$ we can choose $b_1 \in (0, b]$ small enough so that $b_1^\alpha \le b_1$ and $M \le \frac{r}{b_1}$. We shall prove that the two maps $S, T : C(J, E) \to 2^{C(J,E)}$, defined by

$$Sx(t) = \left\{ x_0 + \int_0^t v(s)ds : v \in S_F^1(x) \right\} \text{ and}$$
$$Tx(t) = \left\{ x_0 + \int_0^t v(s)ds : v \in S_G^1(x) \right\},$$

have a common fixed point in the set

$$D = \left\{ x \in C([0, b_1], E) : x(0) = x_0, x \text{ is Lipschitz with constant } \frac{r}{b_1} \right\}.$$

For any $x \in D$ and $u \in Sx$, there exists $v \in S_F^1(x)$ so that $u(t) = x_0 + \int_0^t v(s)ds$. It follows from ($\mathrm{H}_2$) that

$$|u(t) - u(t')| \le \int_{\min\{t,t'\}}^{\max\{t,t'\}} |v(s)|ds \le M|t - t'| \le \frac{r}{b_1}|t - t'|$$

for all $t, t' \in [0, b_1]$. Therefore $u \in D$, so consequently $Sx \in 2^D$. Similarly $Tx \in 2^D$. Let $Y = C([0, b_1], \mathbb{R})$, $K \subset Y$ be the cone of nonnegative functions, and ψ be the c-MNC, defined on the family $\mathfrak{M}$ of all bounded equicontinuous subsets of $C(J, E)$, which is introduced in Example 2.9. Let us define the operators $B : Y \to Y$, $C : K \to K$ by

$$Bu(t) = l \int_0^t u(s)ds; \; Cu(t) = \int_0^t (u[h(s)])^\alpha ds$$

Clearly, B is positive linear with spectral radius $r(B) = 0$ and C is increasing. For $\Omega \in 2^D \cap \mathfrak{M}$, using ($\mathrm{H}_3$) and (4.14) we have

$$\begin{aligned}
\varphi[S(\Omega)(t)] &= \varphi \left\{ \cup_{x \in \Omega} \left(\{x_0\} + \int_0^t F(s, x)ds \right) \right\} \\
&= \sup \left\{ \varphi \left(\{x_0\} + \int_0^t F(s, x)ds \right) : x \in \Omega \right\} \\
&= \sup \left\{ \varphi \left(\int_0^t F(s, x)ds \right) : x \in \Omega \right\} \\
&\le \varphi \left(\int_0^t F(s, \Omega)ds \right) \\
&\le \int_0^t \varphi \left(F(s, \Omega) \right) ds \\
&\le l \int_0^t \varphi \left[\Omega(s) \right] ds + m \int_0^t \left(\varphi \left[\Omega \left(h(s) \right) \right] \right)^\alpha ds.
\end{aligned}$$

Consequently, $\psi(S(\Omega)) \le \gamma(\psi(\Omega))$, where $\gamma = B + mC$.
Similarly, we have $\psi(T(\Omega)) \le \gamma(\psi(\Omega))$. Now we show condition (C_2) holds.

Consider an element $u \in K$ satisfying $u \leq \gamma(u)$, that is,

$$u(t) \leq l \int_0^t u(s)ds + m \int_0^t (u\,[h(s)])^{\alpha}\, ds. \tag{4.15}$$

Using the Gronwall inequality from (4.15), where $\phi(t) = m \int_0^t (u\,[h(s)])^{\alpha}\, ds$ is a non-decreasing function, we obtain

$$u(t) \leq e^{lt} m \int_0^t (u\,[h(s)])^{\alpha}\, ds \leq kC(u)(t) \tag{4.16}$$

for some $k > 0$. From (4.16) we can prove by induction that

$$\begin{aligned} u(t) &\leq (kC)^n(u)(t) \\ &\leq k^{1+\alpha+\ldots+\alpha^n} \|u\|^{\alpha^n} t^n \left[2^{\alpha^{n-2}} 3^{\alpha^{n-3}} \ldots (n-1)^{\alpha} n\right]^{-1}. \end{aligned}$$

This implies $u = 0$. We denote by $\preceq$ the partial ordering in $C(J, E)$ with respect to the cone P defined by $x \in P$ iff $x(t) \in P_E$ for all $t \in J$. Clearly, P is normal. Next, we claim that the pair (S, T) is semi-isotone increasing. Suppose that $x \in D$, and $y \in Sx$, then

$$y(t) = x_0 + \int_0^t v(s)ds$$

for some $v \in S_F^1(x)$. From (H_7) we can choose $w \in S_G^1(y)$ satisfying

$$\int_0^t v(s)ds \leqslant \int_0^t w(s)ds.$$

This implies

$$y(t) \leqslant x_0 + \int_0^t w(s)ds,$$

thus $\{y\} \preceq_1 Ty$. Similarly, $\{y\} \preceq_1 Sy$ for all $y \in Tx$. From (H_6) we have $a_0 \in D$ and $\{a_0\} \preceq_1 Sa_0$. Finally, we prove that $S, T : D \to 2^D$ are monotone-closed. Let $\{x_n\}$ be a sequence in D satisfying $x_n \to x$, and $\{y_n\}$ be a sequence such that $y_n \in Sx_n$ and $y_n \to y$. From $y_n \in Sx_n$ it follows that

$$y_n(t) = x_0 + \int_0^t u_n(s)ds,\ t \in [0, b_1] \text{ for some } u_n \in S_F^1(x_n) \tag{4.17}$$

Denote

$$z_n(t) = \int_0^t u_n(s)ds.$$

From (4.17) it follows that $\lim z_n(t) = y(t) - x_0$ (in E). From inequality

$$|z_n(t) - (y(t) - x_0)| = |y_n(t) - y(t)| \leq \|y_n - y\|$$

for all $t \in [0, b_1]$ we obtain $z_n \to y - a_0$.
Consider the continuous linear $\mathfrak{L} : L^1([0, b_1], E) \to C([0, b_1], E)$ defined by

$$\mathfrak{L}u(t) = \int_0^t u(s)ds,\ t \in [0, b_1].$$

It follows from Lemma 2.16 that $\mathfrak{L} \circ S_F^1(C(J,E))$ is a closed subset in $C([0,b_1],E) \times C([0,b_1],E)$. Moreover, since $z_n \in \mathfrak{L} \circ S_F^1(x_n)$ we have $y - a_0 \in \mathfrak{L} \circ S_F^1(x)$. This implies $y \in Sx$, hence S is monotone-closed. Similarly, T also is too. Therefore, the conditions of Theorem 3.2 are satisfied for the pair (S,T). This completes the proof. □

Remark 4.2. *In this section, we illustrated the advantage of using the c-MNC, that is, T is a multivalued mapping and ψ is a c-NMC satisfying $\psi(T(\Omega)) \leq \gamma(\psi(\Omega))$, where $\psi(\Omega)$ and $\psi(T(\Omega))$ are elements of an ordered cone P and $\gamma : P \to P$ is an increasing operator. Then from the relation $\psi(\Omega)) \leq \psi(T(\Omega))$ it follows that $\psi(\Omega)) \leq \gamma(\psi(\Omega))$ and using some analysis we prove $\psi(\Omega) = 0$.*

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