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STRONG CONVERGENCE OF INERTIAL EXTRAGRADIENT ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. This paper investigates two inertial extragradient algorithms for seeking a common solution to a variational inequality problem involving a monotone and Lipschitz continuous mapping and a fixed point problem with a demicontractive mapping in real Hilbert spaces. Our algorithms need to calculate the projection on the feasible set only once in each iteration. Moreover, they can work well without the prior information of the Lipschitz constant of the operator and do not contain any linesearch process. Strong convergence theorems of the suggested algorithms are established under suitable conditions. Some experiments are presented to illustrate the numerical efficiency of the suggested algorithms and compare them with some existing ones.
Key Words and Phrases: Variational inequality problem, fixed point problem, subgradient extragradient method, Tseng's extragradient method, inertial method, demicontractive mapping.
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1. Introduction

Throughout this paper, one assumes that $\mathcal{H}$ is a real Hilbert space with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ as its inner product and induced norm, respectively. Let $\mathcal{C}\subset\mathcal{H}$ be convex and closed, and let $P_{\mathcal{C}}$ denote the metric (nearest point) projection of $\mathcal{H}$ onto $\mathcal{C}$. Let $A:\mathcal{C}\to\mathcal{H}$ be a nonlinear operator. The variational inequality problem (in short, VIP) is considered as follows:

$$\text{find } x^{\dagger}\in\mathcal{C} \text{ such that } \langle Ax^{\dagger}, x-x^{\dagger}\rangle\geq 0\,,\quad \forall x\in\mathcal{C}\,. \tag{VIP}$$

The symbol Ω represents the solution set of the problem (VIP).

Variational inequality problems provide a useful and indispensable tool for investigating various interesting issues emerging in many areas, such as social, physics, engineering, economics, network analysis, medical imaging, inverse problems, transportation and much more; see, e.g., [1, 8, 12, 7, 19]. Variational inequality theory has been proven to provide a simple, universal, and consistent structure to deal with possible problems. In the past few decades, researchers have shown tremendous interest in exploring different extensions of variational inequality problems. Recently, various forms of computational approaches have been developed and proposed to solve variational inequalities, such as projection-based methods, hybrid steepest descent methods, and Tikhonov regularization methods. For some related results, the readers can refer to [2, 5, 6, 23, 21, 27].

We concentrate primarily on projection-based approaches in this study. The earliest and cheapest projection-type method is called the projected gradient method. This method contains only one iterative process in each iteration, and only needs to calculate one projection on the feasible set. Unfortunately, the convergence condition of this algorithm is very strong, that is, the operator involved is strongly monotone or inverse strongly monotone, which limits the wide use of the algorithm. To prevent the use of such strong assumptions, Korpelevich proposed the extragradient method (EGM) [13], which can guarantee weak convergence under the condition that the operator is only monotone and Lipschitz continuous. Looking back on the extragradient method, it can be seen that EGM needs to evaluate the value of the operator twice and calculate two projections on the feasible set in each iteration. It should be remembered that when the feasible set has a complex structure, it may be very expensive to calculate the projection on the feasible set, which will further affect the efficiency of the iterative method. Next, let us review two notable approaches to overcome this shortcoming. The first one is the Tseng's extragradient method [30] (TEGM for short, it is also known as the forward-backward-forward algorithm), which is a two-step iterative method. In the second step of TEGM, an explicit formula is used to replace the second projection of EGM. So, this method calculates the projection only once on the feasible set in every iteration. The other method is the subgradient extragradient method (SEGM) proposed in [3], which is widely considered as an improvement of EGM. This method replaces the second projection of EGM with the projection on a half-space. It is known that the projection on a half-space can be calculated by an explicit formula. Therefore, SEGM greatly improves the computational efficiency of EGM.

The second problem that we are interested in is the fixed point problem (in short, FPP). One recalls that the fixed point problem is described as follows:

$$\text{find } x^\dagger \in \mathcal{H} \text{ such that } x^\dagger = Tx^\dagger\,, \tag{FPP}$$

where $T : \mathcal{H} \to \mathcal{H}$ is a general operator, and its fixed point set is represented as $\Gamma = \{x : Tx = x\}$. We always suppose that the fixed point set of T is non-empty, i.e., $\Gamma \neq \emptyset$. Iterative approaches of fixed point problems of nonlinear operators have been bustling some fields due to their applications in engineering and science recently. In recent years, iterative methods of fixed-point estimation for nonexpansive operators and demicontractive operators are studied in [9, 11, 15, 16, 25].

In this paper, we are concerned about finding common solutions of variational inequality problems (VIP) and fixed point problems (FPP). More precisely, we consider the following general problem:

$$\text{find } x^\dagger \text{ such that } x^\dagger \in \Omega \cap \Gamma \,, \tag{VIPFPP}$$

where $A : \mathcal{C} \to \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ are two nonlinear operators. The reason for exploring such problems is that they can be applied to mathematical models, and their constraints can be represented as fixed-point problems and/or variational inequality problems. In recent years, researchers have investigated and proposed many efficient iterative approaches to find common solutions for variational inequalities and fixed-point problems. We here list some of the iterative approaches to solve (VIP) and (FPP) which motivate us to introduce our new schemes for solving (VIPFPP). Recently, Kraikaew and Saejung [14] proposed an algorithm called Halpern subgradient extragradient method to solve (VIPFPP) by combining the subgradient extragradient method and the Halpern method. Their algorithm is expressed as follows:

$$\begin{cases} y^k = P_{\mathcal{C}}(x^k - \lambda A x^k)\,, \\ H_k = \{x \in \mathcal{H} : \langle x^k - \lambda A x^k - y^k, x - y^k \rangle \le 0\}\,, \\ z^k = \alpha_k x^0 + (1-\alpha_k) P_{H_k}(x^k - \lambda A y^k)\,, \\ x^{k+1} = \beta_k x^k + (1-\beta_k) T z^k\,, \end{cases} \tag{HSEGM}$$

where x^0 represents the initial point, $\{\alpha_k\} \subset (0,1)$ satisfies that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, step size $\lambda \in (0, 1/L)$, mapping $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous monotone and mapping $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive with $(I - T)$ being demiclosed at zero. Under the assumption of $\Omega \cap \Gamma \neq \emptyset$, they proved that the sequence $\{x^k\}$ formulated by (HSEGM) converges to an element $u \in \Omega \cap \Gamma$ in norm, where $u = P_{\Omega\cap\Gamma} x^0$. However, the Algorithm (HSEGM) converges very slowly because it uses the initial point x^0 in each iteration. Another method used to obtain strong convergence is called the viscosity method. Recently, based on the extragradient-type method and the viscosity method, Thong and Hieu [28] suggested two extragradient-viscosity algorithms in a Hilbert space for solving (VIPFPP). Let $\{x^k\}$ be formulated by:

$$\begin{cases} y^k = P_{\mathcal{C}}(x^k - \lambda_k A x^k)\,, \\ H_k = \{x \in \mathcal{H} : \langle x^k - \lambda_k A x^k - y^k, x - y^k \rangle \le 0\}\,, \\ z^k = P_{H_k}(x^k - \lambda_k A y^k)\,, \\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k)[(1-\beta_k) z^k + \beta_k T z^k]\,, \end{cases} \tag{VSEGM}$$

and

$$\begin{cases} y^k = P_{\mathcal{C}}(x^k - \lambda_k A x^k)\,, \\ z^k = y^k - \lambda_k (A y^k - A x^k)\,, \\ x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k)[(1-\beta_k) z^k + \beta_k T z^k]\,, \end{cases} \tag{VTEGM}$$

where Algorithms (VSEGM) and (VTEGM) update the step size $\{\lambda_k\}$ by following:

$$\lambda_{k+1} = \begin{cases} \min\left\{ \dfrac{\phi \|x^k - y^k\|}{\|A x^k - A y^k\|}, \lambda_k \right\}, & \text{if } A x^k - A y^k \neq 0; \\ \lambda_k, & \text{otherwise,} \end{cases}$$

where $\{\alpha_k\} \subset (0,1)$ satisfies that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$, $\{\beta_k\} \subset (a, 1-\vartheta)$ for some $a > 0$ and $\lambda_0 > 0$, mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, mapping $T : \mathcal{H} \to \mathcal{H}$ is ϑ-demicontractive such that $(I - T)$ is demiclosed at zero and mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contraction with constant $\rho \in [0,1)$. It was proven that, if $\Omega \cap \Gamma \neq \emptyset$, the sequence $\{x^k\}$ formulated by (VSEGM) and (VTEGM) converges strongly to $u \in \Omega \cap \Gamma$, where $u = P_{\Omega\cap\Gamma} \circ f(u)$. Note that (HSEGM) uses a fixed step size, i.e., it needs to know the prior information of Lipschitz constant of the mapping A. However, (VSEGM) and (VTEGM) do not require the prior information of Lipschitz constants of the mapping, which makes them more flexible in practical applications.

It is worth noting that the methods mentioned above need to calculate at least one projection in every iteration. It is known that calculating the value of the projection is equivalent to finding a solution to an optimization problem, which is computationally expensive. A natural problem appears in front of us. Is there a way to prevent calculating projections and solve variational inequalities? Indeed, Yamada [31] proposed the hybrid steepest descent method, which is read as follows:

$$x^{k+1} = (I - \lambda_k \sigma S) T x^k ,$$

where mapping $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive, mapping $S : \mathcal{C} \to \mathcal{H}$ is κ-Lipschitz continuous and η-strong monotone, $0 < \sigma < 2\eta/\kappa^2$ and the sequence $\{\lambda_k\} \subseteq (0,1)$ satisfies some conditions. He proved that the formulated sequence $\{x^k\}$ converges to an element $x^\dagger$ in norm, which is a unique solution of the variational inequality $\langle Sx^\dagger, y - x^\dagger\rangle \geq 0, \forall y \in \Gamma$.

Very recently, Tong and Tian [29] combined the Tseng's extragradient method with the hybrid steepest descent method, and proposed a new method for solving (VIPFPP). In addition, they used an adaptive criterion to update the step size. Indeed, the sequence $\{x^k\}$ is expressed in the following form:

$$\begin{cases} y^k = P_{\mathcal{C}}(x^k - \lambda_k A x^k) , \\ z^k = y^k - \lambda_k (Ay^k - Ax^k) , \\ x^{k+1} = (1 - \sigma\alpha_k S)[(1-\beta_k) z^k + \beta_k T z^k] , \end{cases} \tag{STEGM}$$

where mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and Lipschitz continuous, mapping $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive such that $(I - T)$ is demiclosed at zero, mapping $S : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone and κ-Lipschitz continuous for $\eta > 0$ and $\kappa > 0$. Furthermore, for any $\chi > 0$, $\ell \in (0,1)$, $\phi \in (0,1)$, the sequence $\{\lambda_k\}$ is selected as the maximum $\lambda \in \{\chi, \chi\ell, \chi\ell^2, \ldots\}$ satisfying $\lambda\|Ax^k - Ay^k\| \leq \phi\|x^k - y^k\|$. This update criterion is called the Armijo linesearch rule. Under some suitable conditions, the sequence $\{x^k\}$ formulated by (STEGM) converges to $u \in \Omega \cap \Gamma$ in norm, where $u = P_{\Omega\cap\Gamma}(I - \sigma S)u$. It should be pointed out that using the Armijo-like linesearch rule may require more computation time, because update the step size in each iteration needs to calculate the value of A many times.

On the other hand, problems in practical applications have the characteristics of diversity, complexity and large-scale. How to build fast and stable algorithms becomes particularly important. Recently, many scholars have developed various types of inertial algorithms by employing inertial extrapolation techniques. The inertial method is based on the discrete version of the second-order dissipative dynamical system originally proposed by Polyak [17]. The main feature of the inertial type methods is that they use the previously known sequence information to generate the next iteration point. More precisely, the procedure requires two iteration steps and the second iteration step is implemented through the preceding two iterations. Note that this small change can greatly accelerate the convergence speed of the iterative algorithms. In recent years, this technique has been investigated intensively and implemented successfully to many problems; see, e.g., [4, 10, 20, 22, 24, 32] and the references therein.

Encouraged and influenced by the above work, the purpose of this paper is to develop two inertial extragradient algorithms with a new step size for discovering a common solution of the variational inequality problem containing a monotone and Lipschitz continuous mapping and of the fixed point problem with a demicontractive mapping in real Hilbert spaces. The suggested algorithms need to calculate the projection on the feasible set only once per iteration, which makes them faster. Strong convergence theorems of the algorithms are established without the prior information of the Lipschitz constant of the operator. Lastly, some computational tests appearing in finite and infinite dimensions are proposed to verify our theoretical results. Our algorithms develop and summarize some of the results in the literature [14, 28, 29].

The organizational structure of our paper is built up as follows. Some essential definitions and technical lemmas that need to be used are given in the next section. In Section 3, we propose the algorithms and analyze their convergence. Some numerical experiments to verify our theoretical results are presented in Section 4. At last, the paper ends with a brief summary in Section 5, the final section.

2. Preliminaries

Let $\mathcal{C}$ be a convex and closed set in a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x^k\}$ to a point x are represented by $x^k \rightharpoonup x$ and $x^k \to x$, respectively. Here we state one inequality and one equality that need to be used in the proofs. For any $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$, we have

- $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$.
- $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

For every point $x \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{C}$, which is represented by $P_{\mathcal{C}}(x)$, such that $P_{\mathcal{C}}(x) := \operatorname{argmin}\{\|x-y\|, y \in \mathcal{C}\}$. $P_{\mathcal{C}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{C}$, and it is a nonexpansive mapping. The following two basic projection properties will be used for many times in subsequent proofs.

- $\langle x - P_{\mathcal{C}}(x), y - P_{\mathcal{C}}(x)\rangle \leq 0$, $\forall y \in \mathcal{C}$.
- $\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\|^2 \leq \langle P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y), x - y\rangle$, $\forall y \in \mathcal{H}$.

Definition 2.1. Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator with $\Gamma \neq \emptyset$. Then, $I - T$ is said to be demiclosed at zero if for any $\{x^k\}$ in $\mathcal{H}$, the following implication holds: $x^k \rightharpoonup x$ and $(I-T)x^k \to 0 \Longrightarrow x \in \Gamma$.

Definition 2.2. For any $x, y \in \mathcal{H}$, $z \in \{x : Mx = x\}$, a mapping $M : \mathcal{H} \to \mathcal{H}$ is said to be:

- *L-Lipschitz continuous* with $L > 0$ if
$$\|Mx - My\| \leq L\|x - y\| .$$
If $L = 1$ then the mapping M is called *nonexpansive* and if $L \in (0, 1)$, M is called *contraction*.
- *monotone* if
$$\langle Mx - My, x - y\rangle \geq 0 .$$
- *quasi-nonexpansive* if
$$\|Mx - z\| \leq \|x - z\| .$$
- *ρ-strictly pseudocontractive* with $0 \leq \rho < 1$ if
$$\|Mx - My\|^2 \leq \|x - y\|^2 + \rho\|(I - M)x - (I - M)y\|^2 .$$
- *ϑ-demicontractive* with $0 \leq \vartheta < 1$ if
$$\|Mx - z\|^2 \leq \|x - z\|^2 + \vartheta\|(I - M)x\|^2 , \tag{2.1}$$
or equivalently
$$\langle Mx - z, x - z\rangle \leq \|x - z\|^2 + \frac{\vartheta - 1}{2}\|x - Mx\|^2 . \tag{2.2}$$

Remark 2.1. According to the above definitions, we can easily see the following facts:

- The class of demicontractive mappings includes the class of quasi-nonexpansive mappings.
- Every strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

The following three lemmas are crucial to prove the convergence of our algorithms.

Lemma 2.1. *Suppose that the mapping $S : \mathcal{H} \to \mathcal{H}$ is κ-Lipschitz continuous and η-strongly monotone with $0 < \eta \leq \kappa$. Let the mapping $U : \mathcal{H} \to \mathcal{H}$ be nonexpansive. Take $\sigma > 0$ and $\alpha \in (0, 1]$. The mapping $U^\sigma : \mathcal{H} \to \mathcal{H}$ is defined by*
$$U^\sigma x = (I - \alpha\sigma S)(Ux), \forall x \in \mathcal{H}.$$
Then, U^σ is a contraction mapping provided $\sigma < \frac{2\eta}{\kappa^2}$, i.e.,
$$\|U^\sigma x - U^\sigma y\| \leq (1 - \alpha\gamma)\|x - y\|, \quad \forall x, y \in \mathcal{H} ,$$
where $\gamma = 1 - \sqrt{1 - \sigma(2\eta - \sigma\kappa^2)} \in (0, 1)$.

Proof. Indeed, it follows that
$$\begin{aligned}\|(I - \sigma S)(Ux) - (I - \sigma S)(Uy)\|^2 &= \|Ux - Uy\|^2 + \sigma^2\|S(Ux) - S(Uy)\|^2 \\ &\quad - 2\sigma\langle Ux - Uy, S(Ux) - S(Uy)\rangle \\ &\leq \|Ux - Uy\|^2 + \sigma^2\kappa^2\|Ux - Uy\|^2 - 2\sigma\eta\|Ux - Uy\|^2 \\ &= (1 - \sigma(2\eta - \sigma\kappa^2))\|Ux - Uy\|^2 .\end{aligned}$$

It follows from $0 < \eta \leq \kappa$ that

$$1 - \sigma(2\eta - \sigma\kappa^2) = (\sigma\kappa - \frac{\eta}{\kappa})^2 + 1 - \frac{\eta^2}{\kappa^2} \geq 0\,.$$

Therefore, we get

$$\|(I - \sigma S)(Ux) - (I - \sigma S)(Uy)\| \leq \sqrt{1 - \sigma(2\eta - \sigma\kappa^2)}\|x - y\|\,.$$

From the definition of $U^\sigma x$, one has

$$\begin{aligned}\|U^\sigma x - U^\sigma y\| &= \|(I - \alpha\sigma S)(Ux) - (I - \alpha\sigma S)(Uy)\| \\ &= \|\alpha[(I - \sigma S)(Ux) - (I - \sigma S)(Uy)] + (1 - \alpha)(Ux - Uy)\| \\ &\leq \alpha\|(I - \sigma S)(Ux) - (I - \sigma S)(Uy)\| + (1 - \alpha)\|x - y)\|\,.\end{aligned}$$

Thus, we conclude that

$$\|U^\sigma x - U^\sigma y\| \leq (1 - \alpha\gamma)\|x - y\|\,.$$

where $\gamma = 1 - \sqrt{1 - \sigma(2\eta - \sigma\kappa^2)} \in (0, 1)$ with $0 < \eta \leq \kappa$ and $\sigma < \frac{2\eta}{\kappa^2}$. □

Lemma 2.2 ([14])**.** *Assume that mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous on $\mathcal{C}$. Set $T = P_{\mathcal{C}}(I - \phi A)$, where $\phi > 0$. If $\{x^k\} \subset \mathcal{H}$ satisfies $x^k \rightharpoonup u$ and $x^k - Tx^k \to 0$. Then $u \in \Omega = \Gamma$.*

Lemma 2.3 ([18])**.** *Let $\{a^k\}$ be a nonnegative real number sequence. The sequence $\{\alpha_k\} \subset (0, 1)$ satisfies $\sum_{k=1}^{\infty} \alpha_k = \infty$. Assume that the following inequality holds:*

$$a^{k+1} \leq (1 - \alpha_k)a^k + \alpha_k b^k, \quad \forall k \geq 1,$$

where $\{b^k\}$ is a real number sequence such that $\limsup_{i\to\infty} b^{k_i} \leq 0$ for every subsequence $\{a^{k_i}\}$ of $\{a^k\}$ satisfying $\liminf_{i\to\infty} (a^{k_i+1} - a^{k_i}) \geq 0$. Then $\lim_{k\to\infty} a^k = 0$.

3. Strong convergence of two inertial algorithms

In this section, we present two inertial extragradient methods with a new step size for searching a common solution of variational inequality problems and fixed point problems and analyze their convergence. Our algorithms consist of four methods: the inertial method, the subgradient extragradient method, the Tseng's extragradient method and the hybrid steepest descent method. The advantages of our iterative schemes are that the projection onto the feasible set needs to be computed only once in each iteration and no prior knowledge of the Lipschitz constant of the mapping is required. Assume that the suggested iterative schemes satisfy the following conditions.

- (C1) The mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous on $\mathcal{H}$.
- (C2) The mapping $T : \mathcal{H} \to \mathcal{H}$ is ϑ-demicontractive such that $(I - T)$ is demiclosed at zero.
- (C3) The solution set of (VIPFPP) is non-empty, i.e., $\Omega \cap \Gamma \neq \emptyset$.
- (C4) The mapping $S : \mathcal{H} \to \mathcal{H}$ is η-strongly monotone and κ-Lipschitz continuous, where η and κ are positive numbers.
- (C5) Let $\{\zeta_k\}$ be a positive sequence satisfying $\lim_{k\to\infty} \frac{\zeta_k}{\alpha_k} = 0$, where $\{\alpha_k\} \subset (0, 1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$. Let $\{\beta_k\}$ be a real sequence such that $\beta_k \subset (a, 1 - \vartheta)$ for some $a > 0$.

3.1. **The self-adaptive inertial subgradient extragradient algorithm.** So far, we can state our first self-adaptive iterative algorithm, which is motivated by the inertial subgradient extragradient method and the hybrid steepest descent method. Our Algorithm 3.1 is described as follows.

Algorithm 3.1 The self-adaptive inertial subgradient extragradient algorithm

Initialization: Take $\xi > 0$, $\lambda_1 > 0$, $\phi \in (0,1)$, $\sigma \in (0, \frac{2\eta}{\kappa^2})$. Let $x^0, x^1 \in \mathcal{H}$.
Iterative Steps: Calculate the next iteration point x^{k+1} as follows:
Step 1. Given two previously known iteration points x^{k-1} and x^k $(k \geq 1)$.
Calculate $u^k = x^k + \xi_k(x^k - x^{k-1})$, where

$$\xi_k = \begin{cases} \min\left\{\dfrac{\zeta_k}{\|x^k - x^{k-1}\|}, \xi\right\}, & \text{if } x^k \neq x^{k-1}; \\ \xi, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Calculate $y^k = P_{\mathcal{C}}(u^k - \lambda_k A u^k)$.
Step 3. Calculate $z^k = P_{H_k}(u^k - \lambda_k A y^k)$, where

$$H_k := \{x \in \mathcal{H} \mid \langle u^k - \lambda_k A u^k - y^k, x - y^k \rangle \leq 0\}.$$

Step 4. Calculate $x^{k+1} = (I - \sigma\alpha_k S)q^k$, where $q^k = (1-\beta_k)z^k + \beta_k T z^k$, and update

$$\lambda_{k+1} = \begin{cases} \min\left\{\dfrac{\phi\|u^k - y^k\|}{\|Au^k - Ay^k\|}, \lambda_k\right\}, & \text{if } Au^k - Ay^k \neq 0; \\ \lambda_k, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1. It follows from (3.1) and Condition (C5) that

$$\lim_{k\to\infty} \frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| = 0\,.$$

Indeed, we obtain $\xi_k\|x^k - x^{k-1}\| \leq \zeta_k$ for all k, which together with $\lim_{k\to\infty} \frac{\zeta_k}{\alpha_k} = 0$ implies that

$$\lim_{k\to\infty} \frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| \leq \lim_{k\to\infty} \frac{\zeta_k}{\alpha_k} = 0\,.$$

The following two lemmas are very helpful for the convergence analysis of the algorithms.

Lemma 3.1. *The sequence* $\{\lambda_k\}$ *formulated by* (3.2) *is nonincreasing and satisfies*

$$\lim_{k\to\infty} \lambda_k = \lambda \geq \min\left\{\lambda_1, \frac{\phi}{L}\right\}.$$

Proof. It follows from (3.2) that $\lambda_{k+1} \leq \lambda_k$ for all $k \in \mathbb{N}$. Hence, $\{\lambda_k\}$ is nonincreasing. Furthermore, we get $\|Au^k - Ay^k\| \leq L\|u^k - y^k\|$ since A is L-Lipschitz continuous. Consequently, we can show that

$$\phi\frac{\|u^k - y^k\|}{\|Au^k - Ay^k\|} \geq \frac{\phi}{L}, \quad \text{if } \; Au^k \neq Ay^k\,.$$

In view of (3.2), it follows that

$$\lambda_k \geq \min\left\{\lambda_1, \frac{\phi}{L}\right\}.$$

Thus, from the sequence $\{\lambda_k\}$ is nonincreasing and lower bounded, we get that $\lim_{k\to\infty}\lambda_k = \lambda \geq \min\left\{\lambda_1, \frac{\phi}{L}\right\}$. □

Lemma 3.2 ([26])**.** *Suppose that Conditions (C1) and (C3) hold. Let the sequence $\{z^k\}$ be formulated by Algorithm 3.1. Then, for any $x^\dagger \in \Omega$,*

$$\|z^k - x^\dagger\|^2 \leq \|u^k - x^\dagger\|^2 - \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|y^k - u^k\|^2 - \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|z^k - y^k\|^2 .$$

Theorem 3.1. *Suppose that Conditions (C1)-(C5) hold. Then the iterative sequence $\{x^k\}$ created by Algorithm 3.1 converges to an element $x^\dagger \in \Omega\cap\Gamma$ in norm, where $x^\dagger = P_{\Omega\cap\Gamma}(I - \sigma S)x^\dagger$.*

Proof. According to Lemma 2.1, we get that $(I - \sigma S)$ is a contractive mapping. Therefore, $P_{\Omega\cap\Gamma}(I - \sigma S)$ is also a contraction mapping. By means of the Banach contraction principle, one concludes that there exists a unique point $x^\dagger \in \mathcal{H}$ such that $x^\dagger = P_{\Omega\cap\Gamma}(I - \sigma S)x^\dagger$. Let $x^\dagger \in \Omega\cap\Gamma$.

Claim 1. The sequence $\{x^k\}$ is bounded. On account of Lemma 3.1, we see that

$$\lim_{k\to\infty}\left(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\right) = 1 - \phi > 0.$$

Hence, there exists $k_0 \in \mathbb{N}$ such that

$$1 - \phi\frac{\lambda_k}{\lambda_{k+1}} > 0, \quad \forall k \geq k_0 . \tag{3.3}$$

Combining Lemma 3.2 and (3.3), it follows that

$$\|z^k - x^\dagger\| \leq \|u^k - x^\dagger\|, \quad \forall k \geq k_0 . \tag{3.4}$$

According to the definition of u^k, we can write

$$\begin{aligned}\|u^k - x^\dagger\| &\leq \|x^k - x^\dagger\| + \xi_k\|x^k - x^{k-1}\| \\ &= \|x^k - x^\dagger\| + \alpha_k\cdot\frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| .\end{aligned} \tag{3.5}$$

From Remark 3.1, one sees that $\frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| \to 0$. Therefore, there exists a constant $Q_1 > 0$ such that

$$\frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| \leq Q_1, \quad \forall k \geq 1 . \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we have

$$\|z^k - x^\dagger\| \leq \|u^k - x^\dagger\| \leq \|x^k - x^\dagger\| + \alpha_k Q_1, \quad \forall k \geq k_0 . \tag{3.7}$$

On the other hand, from the definition of q^k, (2.1) and (2.2), we get

$$\begin{aligned}\|q^k - x^\dagger\|^2 &= \|(1-\beta_k)(z^k - x^\dagger) + \beta_k(Tz^k - x^\dagger)\| \\ &= (1-\beta_k)^2\|z^k - x^\dagger\|^2 + \beta_k^2\|Tz^k - x^\dagger\|^2 \\ &\quad + 2(1-\beta_k)\beta_k\langle Tz^k - x^\dagger, z^k - x^\dagger\rangle \\ &\le (1-\beta_k)^2\|z^k - x^\dagger\|^2 + \beta_k^2\|z^k - x^\dagger\|^2 + \beta_k^2\vartheta\|Tz^k - z^k\|^2 \\ &\quad + 2(1-\beta_k)\beta_k\left[\|z^k - x^\dagger\|^2 - \frac{1-\vartheta}{2}\|Tz^k - z^k\|^2\right] \\ &= \|z^k - x^\dagger\|^2 + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2 .\end{aligned} \tag{3.8}$$

In view of $\{\beta_k\} \subset (0, 1-\vartheta)$ and (3.7), we get

$$\|q^k - x^\dagger\| \le \|u^k - x^\dagger\| \le \|x^k - x^\dagger\| + \alpha_k Q_1, \quad \forall k \ge k_0 . \tag{3.9}$$

Therefore, on account of Lemma 2.1 and (3.9), we have

$$\begin{aligned}\|x^{k+1} - x^\dagger\| &= \|(I - \sigma\alpha_k S)q^k - (I - \sigma\alpha_k S)p - \sigma\alpha_k Sx^\dagger\| \\ &\le \|(I - \sigma\alpha_k S)q^k - (I - \sigma\alpha_k S)p\| + \sigma\alpha_k\|Sx^\dagger\| \\ &\le (1-\gamma\alpha_k)\|q^k - x^\dagger\| + \sigma\alpha_k\|Sx^\dagger\| \\ &\le (1-\gamma\alpha_k)\|x^k - x^\dagger\| + \gamma\alpha_k\frac{\sigma}{\gamma}\|Sx^\dagger\| + \gamma\alpha_k\frac{Q_1}{\gamma} \\ &\le \max\left\{\|x^k - x^\dagger\|, \frac{\sigma\|Sx^\dagger\| + Q_1}{\gamma}\right\} \\ &\le \cdots \le \max\left\{\|x^{k_0} - x^\dagger\|, \frac{\sigma\|Sx^\dagger\| + Q_1}{\gamma}\right\},\end{aligned}$$

where $\gamma = 1 - \sqrt{1 - \sigma(2\eta - \sigma\kappa^2)} \in (0,1)$. This means that the sequence $\{x^k\}$ is bounded. Thus, the sequences $\{y^k\}, \{z^k\}, \{q^k\}$ and $\{(I - \sigma S)x^k\}$ are also bounded.
Claim 2.

$$\begin{aligned}&\beta_k[1-\vartheta-\beta_k]\|z^k - Tz^k\|^2 + \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|y^k - u^k\|^2 + \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|z^k - y^k\|^2 \\ &\le \|x^k - x^\dagger\|^2 - \|x^{k+1} - x^\dagger\|^2 + \alpha_k Q_4 , \quad \forall k \ge k_0\end{aligned}$$

for some $Q_4 > 0$. Indeed, on account of Lemma 2.1 and (3.8), it follows that

$$\begin{aligned}\|x^{k+1} - x^\dagger\|^2 &= \|(I - \sigma\alpha_k S)q^k - (I - \sigma\alpha_k S)p - \sigma\alpha_k Sx^\dagger\|^2 \\ &\le \|(I - \sigma\alpha_k S)q^k - (I - \sigma\alpha_k S)p\|^2 - 2\sigma\alpha_k\langle Sx^\dagger, x^{k+1} - x^\dagger\rangle \\ &\le (1-\gamma\alpha_k)^2\|q^k - x^\dagger\|^2 + 2\sigma\alpha_k\langle Sx^\dagger, x^\dagger - x^{k+1}\rangle \\ &\le \|q^k - x^\dagger\|^2 + \alpha_k Q_2 \\ &\le \|z^k - x^\dagger\|^2 + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2 + \alpha_k Q_2\end{aligned} \tag{3.10}$$

for some $Q_2 > 0$. In the light of Lemma 3.2, one has

$$\|x^{k+1} - x^\dagger\|^2 \leq \|u^k - x^\dagger\|^2 - \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\big(\|y^k - u^k\|^2 + \|z^k - y^k\|^2\big) \\ + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2 + \alpha_k Q_2\,. \tag{3.11}$$

In view of (3.7), we have

$$\begin{aligned}\|u^k - x^\dagger\|^2 &\leq (\|x^k - x^\dagger\| + \alpha_k Q_1)^2 \\ &= \|x^k - x^\dagger\|^2 + \alpha_k(2Q_1\|x^k - x^\dagger\| + \alpha_k Q_1^2) \\ &\leq \|x^k - x^\dagger\|^2 + \alpha_k Q_3\end{aligned} \tag{3.12}$$

for some $Q_3 > 0$. From (3.11) and (3.12), we get

$$\|x^{k+1} - x^\dagger\|^2 \leq \|x^k - x^\dagger\|^2 - \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|y^k - u^k\|^2 - \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|z^k - y^k\|^2 \\ + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2 + \alpha_k Q_2 + \alpha_k Q_3\,.$$

which yields

$$\beta_k[1 - \vartheta - \beta_k]\|z^k - Tz^k\|^2 + \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|y^k - u^k\|^2 + \big(1 - \phi\frac{\lambda_k}{\lambda_{k+1}}\big)\|z^k - y^k\|^2 \\ \leq \|x^k - x^\dagger\|^2 - \|x^{k+1} - x^\dagger\|^2 + \alpha_k Q_4\,, \quad \forall k \geq k_0\,,$$

where $Q_4 := Q_2 + Q_3$.

Claim 3.

$$\|x^{k+1} - x^\dagger\|^2 \leq (1 - \gamma\alpha_k)\|x^k - x^\dagger\|^2 \\ + \gamma\alpha_k\Big[\frac{2\sigma}{\gamma}\langle Sx^\dagger, x^\dagger - x^{k+1}\rangle + \frac{3Q\xi_k}{\gamma\alpha_k}\|x^k - x^{k-1}\|\Big], \forall k \geq k_0$$

for some $Q > 0$. Indeed, by the definition of u^k, one obtains

$$\begin{aligned}\|u^k - x^\dagger\|^2 &= \|x^k + \xi_k(x^k - x^{k-1}) - x^\dagger\| \\ &= \|x^k - x^\dagger\|^2 + 2\xi_k\langle x^k - x^\dagger, x^k - x^{k-1}\rangle + \xi_k^2\|x^k - x^{k-1}\|^2 \\ &\leq \|x^k - x^\dagger\|^2 + 3Q\xi_k\|x^k - x^{k-1}\|\,,\end{aligned} \tag{3.13}$$

where $Q := \sup_{k\in\mathbb{N}}\big\{\|x^k - x^\dagger\|, \xi\|x^k - x^{k-1}\|\big\} > 0$. Using (3.9) and (3.10), we obtain

$$\|x^{k+1} - x^\dagger\|^2 \leq (1 - \gamma\alpha_k)\|u^k - x^\dagger\|^2 + 2\sigma\alpha_k\langle Sx^\dagger, x^\dagger - x^{k+1}\rangle\,. \tag{3.14}$$

Substituting (3.13) into (3.14), it follows that

$$\|x^{k+1} - x^\dagger\|^2 \leq (1 - \gamma\alpha_k)\|x^k - x^\dagger\|^2 \\ + \gamma\alpha_k\Big[\frac{2\sigma}{\gamma}\langle Sx^\dagger, x^\dagger - x^{k+1}\rangle + \frac{3Q\xi_k}{\gamma\alpha_k}\|x^k - x^{k-1}\|\Big], \forall k \geq k_0\,.$$

Claim 4. The sequence $\big\{\|x^k - x^\dagger\|^2\big\}$ converges to zero. From Lemma 2.3, we need to show that $\limsup_{i\to\infty}\langle Sx^\dagger, x^\dagger - x^{k_i+1}\rangle \leq 0$ for every subsequence $\{\|x^{k_i} - x^\dagger\|\}$ of $\{\|x^k - x^\dagger\|\}$ satisfying

$$\liminf_{i\to\infty}(\|x^{k_i+1} - x^\dagger\| - \|x^{k_i} - x^\dagger\|) \geq 0\,.$$

For this purpose, one assumes that $\{\|x^{k_i}-x^\dagger\|\}$ is a subsequence of $\{\|x^k-x^\dagger\|\}$ such that $\liminf_{i\to\infty}(\|x^{k_i+1}-x^\dagger\|-\|x^{k_i}-x^\dagger\|)\geq 0$. We obtain

$$\begin{aligned}&\liminf_{i\to\infty}\left(\|x^{k_i+1}-x^\dagger\|^2-\|x^{k_i}-x^\dagger\|^2\right)\\ &=\liminf_{i\to\infty}\left[\left(\|x^{k_i+1}-x^\dagger\|-\|x^{k_i}-x^\dagger\|\right)\left(\|x^{k_i+1}-x^\dagger\|+\|x^{k_i}-x^\dagger\|\right)\right]\geq 0\,.\end{aligned}$$

From Claim 2 and Condition (C5), it follows that

$$\begin{aligned}&\limsup_{i\to\infty}\Big[\big(1-\phi\frac{\lambda_{k_i}}{\lambda_{k_i+1}}\big)\|y^{k_i}-u^{k_i}\|^2+\big(1-\phi\frac{\lambda_{k_i}}{\lambda_{k_i+1}}\big)\|z^{k_i}-y^{k_i}\|^2\\ &\quad+\beta_{k_i}(1-\vartheta-\beta_{k_i})\|Tz^{k_i}-z^{k_i}\|^2\Big]\\ &\leq\limsup_{i\to\infty}\left[\|x^{k_i}-x^\dagger\|^2-\|x^{k_i+1}-x^\dagger\|^2+\alpha_{k_i}Q_4\right]\\ &\leq\limsup_{i\to\infty}\left[\|x^{k_i}-x^\dagger\|^2-\|x^{k_i+1}-x^\dagger\|^2\right]+\limsup_{i\to\infty}\alpha_{k_i}Q_4\\ &=-\liminf_{i\to\infty}\left[\|x^{k_i+1}-x^\dagger\|^2-\|x^{k_i}-x^\dagger\|^2\right]\\ &\leq 0\,.\end{aligned}$$

Thus, we obtain the following results:

$$\lim_{i\to\infty}\|y^{k_i}-u^{k_i}\|=0,\ \lim_{i\to\infty}\|z^{k_i}-y^{k_i}\|=0 \text{ and } \lim_{i\to\infty}\|Tz^{k_i}-z^{k_i}\|=0\,. \tag{3.15}$$

Therefore, we have

$$\lim_{i\to\infty}\|z^{k_i}-u^{k_i}\|\leq\lim_{i\to\infty}\|z^{k_i}-y^{k_i}\|+\lim_{i\to\infty}\|y^{k_i}-u^{k_i}\|=0\,, \tag{3.16}$$

and

$$\lim_{i\to\infty}\|x^{k_i}-u^{k_i}\|=\lim_{i\to\infty}\xi_{k_i}\|x^{k_i}-x^{k_i-1}\|=\lim_{i\to\infty}\alpha_{k_i}\cdot\frac{\xi_{k_i}}{\alpha_{k_i}}\|x^{k_i}-x^{k_i-1}\|=0\,. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\lim_{i\to\infty}\|z^{k_i}-x^{k_i}\|\leq\lim_{i\to\infty}\|z^{k_i}-u^{k_i}\|+\lim_{i\to\infty}\|u^{k_i}-x^{k_i}\|=0\,. \tag{3.18}$$

From $q^{k_i}=(1-\beta_{k_i})z^{k_i}+\beta_{k_i}Tz^{k_i}$, one sees that

$$\|q^{k_i}-z^{k_i}\|\leq\beta_{k_i}\|Tz^{k_i}-z^{k_i}\|\leq(1-\vartheta)\|Tz^{k_i}-z^{k_i}\|\,.$$

In view of (3.15), we get

$$\lim_{i\to\infty}\|q^{k_i}-z^{k_i}\|=0\,. \tag{3.19}$$

Moreover,

$$\|x^{k_i+1}-q^{k_i}\|=\sigma\alpha_{k_i}\|Sq^{k_i}\|\to 0\,. \tag{3.20}$$

Combining (3.18), (3.19) and (3.20), we obtain

$$\|x^{k_i+1}-x^{k_i}\|\leq\|x^{k_i+1}-q^{k_i}\|+\|q^{k_i}-z^{k_i}\|+\|z^{k_i}-x^{k_i}\|\to 0\,. \tag{3.21}$$

It follows from $\{x^{k_i}\}$ is bounded that there is a subsequence $\{x^{k_{i_j}}\}$ of $\{x^{k_i}\}$ such that $x^{k_{i_j}}\rightharpoonup z$, where $z\in\mathcal{H}$. From (3.17), we get $u^{k_i}\rightharpoonup z$ as $k\to\infty$. This together with $\lim_{i\to\infty}\|u^{k_i}-y^{k_i}\|=0$ and Lemma 2.2 implies that $z\in\Omega\cap\Gamma$. According to the definition of $x^\dagger = P_{\Omega\cap\Gamma}(I-\sigma S)x^\dagger$, using the property of projection, one has $\langle (I-\sigma S)x^\dagger - x^\dagger, z - x^\dagger\rangle \le 0$. Thus, we get

$$\limsup_{i\to\infty}\langle Sx^\dagger, x^\dagger - x^{k_i}\rangle = \lim_{j\to\infty}\langle Sx^\dagger, x^\dagger - x^{k_{i_j}}\rangle = \langle Sx^\dagger, x^\dagger - z\rangle \le 0\,. \tag{3.22}$$

Combining (3.21) and (3.22), we obtain

$$\limsup_{i\to\infty}\langle Sx^\dagger, x^\dagger - x^{k_i+1}\rangle = \limsup_{i\to\infty}\langle Sx^\dagger, x^\dagger - x^{k_i}\rangle \le 0\,. \tag{3.23}$$

Hence, combining (3.23), $\lim_{k\to\infty}\frac{\xi_k}{\alpha_k}\|x^k - x^{k-1}\| = 0$, Claim 3 and Lemma 2.3, it follows that $\lim_{k\to\infty}\|x^k - x^\dagger\| = 0$, namely, $x^k \to x^\dagger$. We have thus proved the theorem. □

Next, we state a particular situation of Algorithm 3.1. When $S(x) = x - x^0$ (x^0 is an initial point) in Theorem 3.1. It can be easily checked that mapping $S : \mathcal{H} \to \mathcal{H}$ is strongly monotone and Lipschitz continuous with modulus $\eta = \kappa = 1$. In this situation, by selecting $\sigma = 1$, we obtain a new self-adaptive inertial Halpern subgradient extragradient algorithm to solve (VIPFPP). More specifically, we have the following result.

Corollary 3.1. *Suppose that mapping $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous monotone and mapping $T : \mathcal{H} \to \mathcal{H}$ is ϑ-demicontractive such that $(I-T)$ is demiclosed at zero. Take $\xi > 0$, $\lambda_1 > 0$, $\phi \in (0,1)$. Let sequence $\{\zeta_k\}$ be positive numbers such that $\lim_{k\to\infty}\frac{\zeta_k}{\alpha_k} = 0$, where $\{\alpha_k\} \subset (0,1)$ satisfies $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_{k=0}^{\infty}\alpha_k = \infty$. Let $\{\beta_k\}$ be a real sequence such that $\beta_k \subset (a, 1-\vartheta)$ for some $a > 0$. With two start points $x^0, x^1 \in \mathcal{H}$, the sequence $\{x^k\}$ is defined by*

$$\begin{cases} u^k = x^k + \xi_k(x^k - x^{k-1})\,, \\ y^k = P_{\mathcal{C}}(u^k - \lambda_k A u^k)\,, \\ z^k = P_{H_k}(u^k - \lambda_k A y^k)\,, \\ H_k := \{x \in \mathcal{H} \mid \langle u^k - \lambda_k A u^k - y^k, x - y^k\rangle \le 0\}\,, \\ x^{k+1} = \alpha_k x^0 + (1-\alpha_k)[(1-\beta_k)z^k + \beta_k T z^k]\,, \end{cases} \tag{3.24}$$

where ξ_k and λ_k are defined in (3.1) *and* (3.2)*, respectively. Then the iterative sequence $\{x^k\}$ formulated by* (3.24) *converges to $x^\dagger \in \Omega\cap\Gamma$ in norm, where $x^\dagger = P_{\Omega\cap\Gamma}x^0$.*

3.2. **The self-adaptive inertial Tseng's extragradient algorithm.** Next, we introduce a new self-adaptive inertial Tseng's extragradient algorithm to solve (VIPFPP). The advantages of this algorithm are that only one projection needs to be calculated in each iteration, and it can work without the prior information of the Lipschitz constant of the mapping. The Algorithm 3.2 is read as follows.

The following lemma is very useful for studying the convergence of the Algorithm 3.2.

Algorithm 3.2 The self-adaptive inertial Tseng's extragradient algorithm

Initialization: Take $\xi > 0$, $\lambda_1 > 0$, $\phi \in (0,1)$, $\sigma \in (0, \frac{2\eta}{\kappa^2})$. Let $x^0, x^1 \in \mathcal{H}$.
Iterative Steps: Calculate the next iteration point x^{k+1} as follows:
Step 1. Given two previously known iteration points x^{k-1} and x^k $(k \geq 1)$.
Calculate $u^k = x^k + \xi_k(x^k - x^{k-1})$, where the inertial parameter ξ_k is defined in (3.1).
Step 2. Calculate $y^k = P_{\mathcal{C}}(u^k - \lambda_k A u^k)$.
Step 3. Calculate $z^k = y^k - \lambda_k(Ay^k - Au^k)$.
Step 4. Calculate $x^{k+1} = (I - \sigma\alpha_k S)q^k$, where $q^k = (1-\beta_k)z^k + \beta_k T z^k$, and update the step size λ_{k+1} by (3.2).
Set $n := n+1$ and go to **Step 1**.

Lemma 3.3 ([26])**.** *Suppose that Conditions (C1) and (C3) hold. Let the sequence $\{z^k\}$ be formulated by Algorithm 3.2. Then, it follows that*

$$\|z^k - x^\dagger\|^2 \leq \|u^k - x^\dagger\|^2 - \big(1 - \phi^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\big)\|u^k - y^k\|^2, \quad \forall x^\dagger \in \Omega\,,$$

and

$$\|z^k - y^k\| \leq \phi \frac{\lambda_k}{\lambda_{k+1}} \|u^k - y^k\|\,.$$

Theorem 3.2. *Suppose that Conditions (C1)-(C5) hold. Then the iterative sequence $\{x^k\}$ generated by Algorithm 3.2 converges to an element $x^\dagger \in \Omega \cap \Gamma$ in norm, where $x^\dagger = P_{\Omega\cap\Gamma}(I - \sigma S)x^\dagger$.*

Proof. **Claim 1.** The sequence $\{x^k\}$ is bounded. By Lemma 3.1, there exists a constant $k_0 \in \mathbb{N}$ such that $1 - \phi^2 \frac{\lambda_k^2}{\lambda_{k+1}^2} > 0, \forall k \geq k_0$. Thanks to Lemma 3.3, one sees that

$$\|z^k - x^\dagger\| \leq \|u^k - x^\dagger\|, \quad \forall k \geq k_0\,.$$

Using the same arguments as in Claim 1 of Theorem 3.1, we get that $\{x^k\}$ is bounded. So the sequences $\{y^k\}, \{z^k\}, \{q^k\}$ and $\big\{(I - \sigma S)x^k\big\}$ are also bounded.
Claim 2.

$$\begin{aligned}&\beta_k[1 - \vartheta - \beta_k]\|z^k - Tz^k\|^2 + \big(1 - \phi^2 \frac{\lambda_{n^2}}{\lambda_{k+1}^2}\big)\|y^k - u^k\|^2 \\ &\leq \|x^k - x^\dagger\|^2 - \|x^{k+1} - x^\dagger\|^2 + \alpha_k Q_4\,, \quad \forall k \geq k_0\end{aligned}$$

for some $Q_4 > 0$. From (3.10), (3.12) and Lemma 3.3, we can show that

$$\begin{aligned}\|x^{k+1} - x^\dagger\|^2 &\leq \|z^k - x^\dagger\|^2 + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2 + \alpha_k Q_2 \\ &\leq \|x^k - x^\dagger\|^2 - \big(1 - \phi^2 \frac{\lambda_k^2}{\lambda_{k+1}^2}\big)\|y^k - u^k\|^2 + \alpha_k Q_4 \\ &\quad + \beta_k[\beta_k - (1-\vartheta)]\|Tz^k - z^k\|^2\,, \quad \forall k \geq k_0\,,\end{aligned}$$

where $Q_4 := Q_2 + Q_3$, both Q_2 and Q_3 are defined in Claim 2 of Theorem 3.1.

Claim 3.

$$\|x^{k+1} - x^\dagger\|^2 \le (1 - \gamma\alpha_k)\|x^k - x^\dagger\|^2 + \gamma\alpha_k\Big[\frac{2\sigma}{\gamma}\langle Sx^\dagger, x^\dagger - x^{k+1}\rangle + \frac{3Q\xi_k}{\gamma\alpha_k}\|x^k - x^{k-1}\|\Big],\ \forall k \ge k_0\,.$$

This result can be obtained using the same arguments as in Claim 3 of Theorem 3.1.
Claim 4. The sequence $\{\|x^k - x^\dagger\|^2\}$ converges to zero. The proof is similar to Claim 4 in Theorem 3.1. We leave it for the reader to check. □

Now, we give a special case of Algorithm 3.2. When $S(x) = x - f(x)$ in Theorem 3.2, where mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contraction. It can be easily verified that mapping $S : \mathcal{H} \to \mathcal{H}$ is $(1+\rho)$-Lipschitz continuous and $(1-\rho)$-strongly monotone. In this situation, by picking $\sigma = 1$, we get a new self-adaptive inertial viscosity-type Tseng's extragradient algorithm for solving (VIPFPP). Similar to Corollary 3.1, we can get the following result immediately.

Corollary 3.2. *Suppose that mapping $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous monotone, mapping $T : \mathcal{H} \to \mathcal{H}$ is ϑ-demicontractive such that $(I - T)$ is demiclosed at zero and mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contractive with $\rho \in [0, \sqrt{5} - 2)$. Take $\xi > 0$, $\lambda_1 > 0$, $\phi \in (0, 1)$. Let sequence $\{\zeta_k\}$ be positive numbers such that $\lim_{k\to\infty} \frac{\zeta_k}{\alpha_k} = 0$, where $\{\alpha_k\} \subset (0, 1)$ satisfies $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$. Let $\{\beta_k\}$ be a real sequence such that $\beta_k \subset (a, 1-\vartheta)$ for some $a > 0$. Let $x^0, x^1 \in \mathcal{H}$ and $\{x^k\}$ be defined by*

$$\begin{cases} u^k = x^k + \xi_k(x^k - x^{k-1})\,, \\ y^k = P_{\mathcal{C}}(u^k - \lambda_k A u^k)\,, \\ z^k = y^k - \lambda_k(Ay^k - Au^k)\,, \\ q^k = (1-\beta_k)z^k + \beta_k T z^k\,, \\ x^{k+1} = (1-\alpha_k)q^k + \alpha_k f(q^k)\,, \end{cases} \tag{3.25}$$

where ξ_k and λ_k are defined in (3.1) and (3.2), respectively. Then the iterative sequence $\{x^k\}$ created by (3.25) converges to $x^\dagger \in \Omega\cap\Gamma$ in norm, where $x^\dagger = P_{\Omega\cap\Gamma}\circ f(p)$.

Remark 3.2. (1) Set $S(x) = x - f(x)$ in Theorem 3.1 and select $S(x) = x - x^0$ in Theorem 3.2. We can get two new algorithms to seek the common solution of problem (VIP) and problem (FPP). Note that these algorithms own strong convergence results in Hilbert spaces. Furthermore, they can work without the prior information of the Lipschitz constant of the operator.

(2) The algorithms proposed in this paper improve and extend some recent results in the literature [14, 28, 29]. Our iterative schemes embed inertial terms and use a new iteration step size, which makes them faster and more flexible. In addition, it is worth noting that the mapping T in Algorithms (HSEGM) and (STEGM) is quasi-nonexpansive, but ours is a demicontractive mapping. Therefore, our algorithms have a wider range of applications.

4. Numerical examples

In this section, we provide some computational tests to illustrate the numerical behavior of the proposed Algorithm 3.1 (in short, iSSEGM), Algorithm 3.2 (in short, iSTEGM) and compare them with some existed strongly convergent methods, including the Halpern subgradient extragradient method (HSEGM) [14], the viscosity-type subgradient extragradient method (VSEGM) [28], the viscosity-type Tseng's extragradient method (VTEGM) [28], and the self-adaptive Tseng's extragradient method (STEGM) [29]. All the programs were implemented in Matlab 2018a on a Intel(R) Core(TM) i5-8250T CPU @ 1.60GHz computer with RAM 8.00 GB.

Our parameters are set as follows. In all algorithms, we set

$$\alpha_k = 1/(k+1) \quad \text{and} \quad \beta_k = k/(2k+1).$$

For the proposed algorithms and the Algorithms (VSEGM) and (VTEGM), we choose

$$\lambda_1 = 0.9, \quad \phi = 0.5.$$

Set $f(x) = 0.5x$ in the Algorithms (VSEGM) and (VTEGM). Take

$$\sigma = 0.5,\ \xi = 0.4 \text{ and } \zeta_k = 1/(k+1)^2$$

for our suggested algorithms. For the Algorithm (STEGM), we select

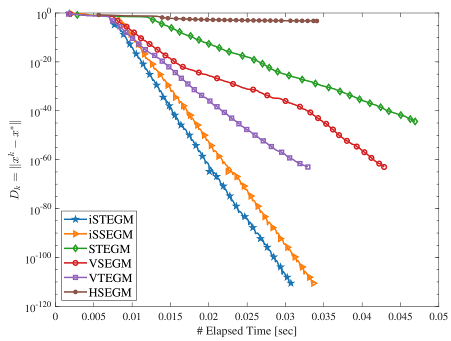
$$\chi = 0.5,\ \ell = 0.5,\ \phi = 0.4 \text{ and } \sigma = 0.5.$$

Pick the step size $\lambda = 0.99/L$ for the Algorithm (HSEGM). In our numerical examples, when the number of iterations is the same, we use the runtime in seconds to measure the computational performance of all algorithms. In addition, the solution x^* of the problems are known. Thus, we use the function $D_k = \|x^k - x^*\|$ to measure the k-th iteration error. It should be noted that $D_k \to 0$ means that the sequence $\{x^k\}$ converges to x^*.

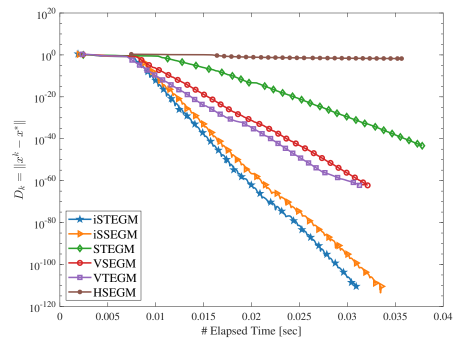
Example 4.1. In first example, let the nonlinear mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows:

$$A(x,y) = (x + y + \sin x; -x + y + \sin y)\,.$$

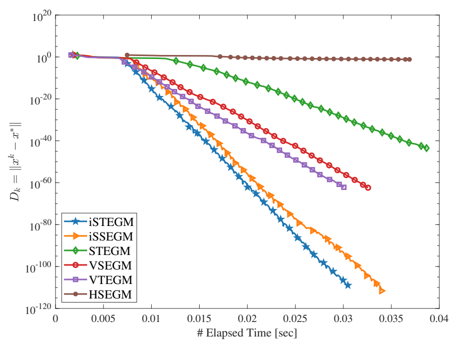
It is easy to verify that mapping A is Lipschitz continuous monotone with modulus $L = 3$. Assume that the feasible set $\mathcal{C}$ is a two-dimensional box with lower bounds $l_i = [-1;-1]$ and upper bounds $u_i = [1;1]$. Then the projection of a point $x_i \in \mathbb{R}^2$ on this box can be calculated explicitly by $P_{\mathcal{C}}(x)_i = \min\{u_i, \max\{l_i, x_i\}\}$. Moreover, the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $Tx = \|D\|_2^{-1} Dx$, where $D = [1,0;0,2]$ and $\|D\|_2$ is defined as $\|D\|_2 = \sqrt{\lambda_{\max}(D^{\mathsf{T}} D)}$ (i.e., the square root of the largest eigenvalue of the matrix $D^{\mathsf{T}} D$, where D^{T} denotes the conjugate transpose of D). The mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ is selected as $Sx = 0.5x$. It can be easily seen that mapping T is 0-demicontractive and mapping S is Lipschitz continuous and strongly monotone. We can easily find that the solution of the problem is $x^* = (0,0)^{\mathsf{T}}$. In order to verify the effectiveness of the suggested algorithms, we select four different initial values $x^0 = x^1$ in Matlab, namely, (Case I): $x^1 =$ *rand(2,1)*, (Case II): $x^1 =$ *5rand(2,1)*, (Case III): $x^1 =$ *10rand(2,1)*, (Case IV): $x^1 =$ *20rand(2,1)*, and use the maximum iteration 400 as a common stopping criterion for all algorithms. The numerical results of all the algorithms with four different initial values are plotted in Fig. 1.



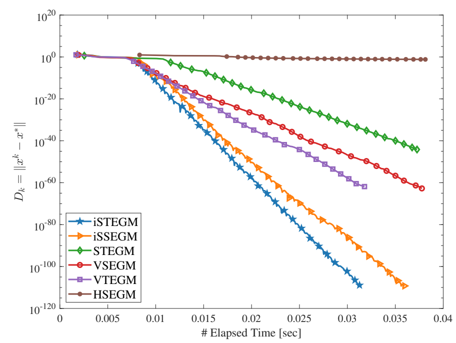
(a) x^1 = *rand(2,1)*



(b) x^1 = *5rand(2,1)*



(c) x^1 = *10rand(2,1)*



(d) x^1 = *20rand(2,1)*

FIGURE 1. Numerical results for Example 4.1

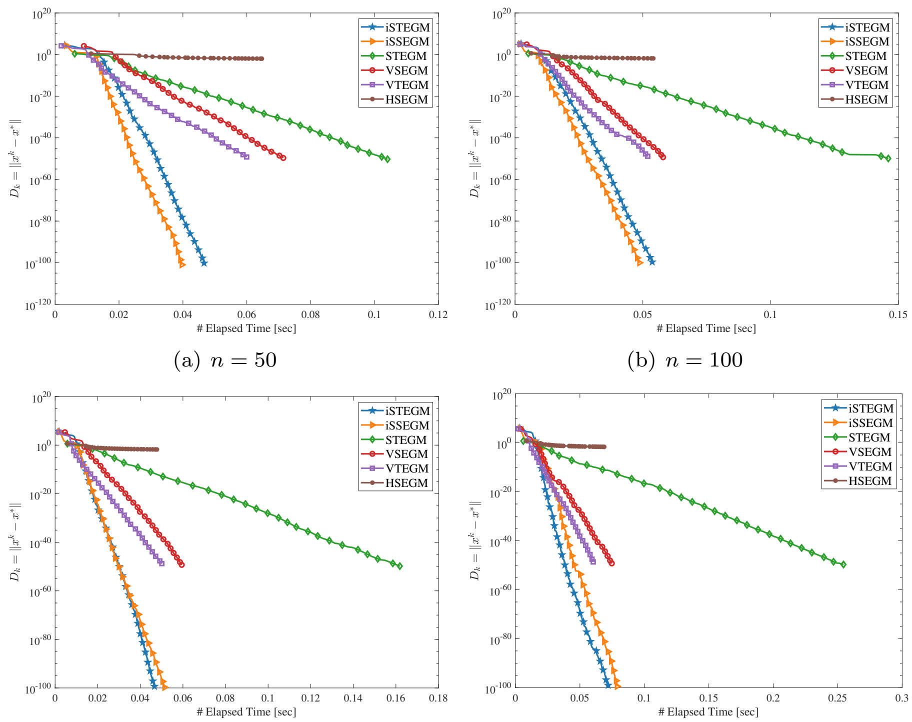
Example 4.2. In the second example, we consider the form of linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ $(n = 50, 100, 150, 200)$ as follows:

$$A(x) = Gx + g,$$

where $g \in \mathbb{R}^n$ and $G = BB^\mathsf{T} + M + E$, matrix $B \in \mathbb{R}^{n\times n}$, matrix $M \in \mathbb{R}^{n\times n}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{n\times n}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set as

$$\mathcal{C} = \{x \in \mathbb{R}^n : -2 \le x_i \le 5,\, i = 1, \ldots, n\}.$$

It can be easily checked that mapping A is Lipschitz continuous monotone and its Lipschitz constant $L = \|G\|$. In this numerical example, both entries B, E are randomly created in $[0, 2]$, M is generated randomly in $[-2, 2]$ and $g = \mathbf{0}$. Let $T : \mathcal{H} \to \mathcal{H}$ and $S : \mathcal{H} \to \mathcal{H}$ be provided by $Tx = 0.5x$ and $Sx = 0.5x$, respectively. We obtain the solution to the problem is $x^* = \{\mathbf{0}\}$. The maximum iteration 400 as a common stopping criterion and the initial values $x^0 = x^1$ are randomly generated by *rand(n,1)* in MATLAB. The numerical results of all the algorithms with four different dimensions are described in Fig. 2.



(a) $n = 50$ (b) $n = 100$

(c) $n = 150$ (d) $n = 200$

FIGURE 2. Numerical results for Example 4.2

Example 4.3. In the last example, we focus on a case in a Hilbert space

$$\mathcal{H} = L^2([0,1]).$$

Its inner product and induced norm are defined as

$$\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t \quad \text{and} \quad \|x\| := \left(\int_0^1 |x(t)|^2 \mathrm{d}t\right)^{1/2},$$

respectively. Let the feasible set be the unit ball $\mathcal{C} := \{x \in \mathcal{H} : \|x\| \leq 1\}$. Let the operator $A : \mathcal{C} \to \mathcal{H}$ be generated as follows:

$$(Ax)(t) = \max\{0, x(t)\} = \frac{x(t) + |x(t)|}{2}.$$

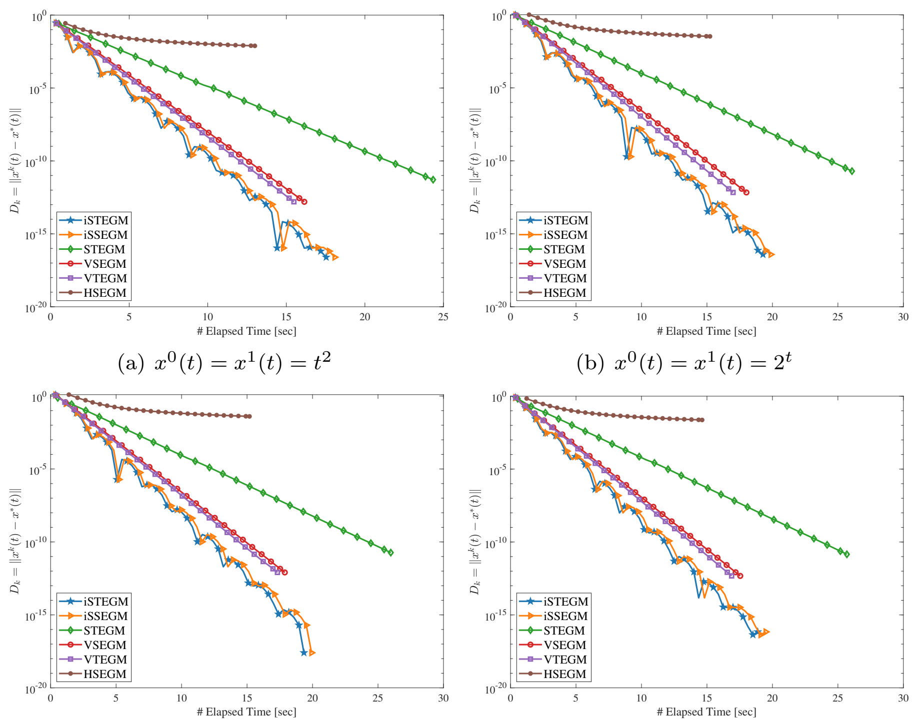
It can be easily verified that operator A is monotone and Lipschitz continuous with modulus $L = 1$. Moreover, the projection onto the feasible set $\mathcal{C}$ is explicit, and we can use the following formula to calculate the projection:

$$P_{\mathcal{C}}(x) = \begin{cases} \dfrac{x}{\|x\|}, & \text{if } \|x\| > 1; \\ x, & \text{if } \|x\| \leq 1. \end{cases}$$

We choose the mapping $T : L^2([0,1]) \to L^2([0,1])$ is of the form

$$(Tx)(t) = \int_0^1 tx(s)\mathrm{d}s,\ t \in [0,1].$$

A simple computation indicates that T is 0-demicontractive and demiclosed at zero. Let mapping $S : \mathcal{H} \to \mathcal{H}$ be taken as $(Sx)(t) = 0.5x(t), t \in [0,1]$. It can be easily proved that the mapping S is strongly monotone and Lipschitz continuous. The solution to this problem is $x^*(t) = 0$, and the maximum iteration 50 is used as the stopping criterion. With four types of starting points: (Case I): $x^0(t) = x^1(t) = t^2$, (Case II): $x^0(t) = x^1(t) = 2^t$, (Case III): $x^0(t) = x^1(t) = e^t$ and (Case IV): $x^0(t) = x^1(t) = \log(t)$. The numerical behaviors of $D_k = \|x^k(t) - x^*(t)\|$ formulated by all the algorithms are shown in Fig. 3.



(a) $x^0(t) = x^1(t) = t^2$

(b) $x^0(t) = x^1(t) = 2^t$

(c) $x^0(t) = x^1(t) = e^t$

(d) $x^0(t) = x^1(t) = \log(t)$

FIGURE 3. Numerical results for Example 4.3

Remark 4.1. We have the following observations from Examples 4.1-4.3.

(1) From Figs. 1-3, it is known that the proposed methods outperform some existing algorithms in the literature. These results are independent of the selection of initial values and the size of dimensions. Note that our algorithms converge quickly, and there are some oscillations due to inertial effects.

(2) The maximum number of iterations we choose is only 400. It should be noted that the iteration error of Algorithm (HSEGM) is very big. In actual applications, it may require more iterations to meet the accuracy requirements. Furthermore, we point out that since the Algorithm (STEGM) uses the Armijo-like step size rule, which leads to taking more execution time.
(3) In our future work, we will improve the generality of the operator involved, for example, consider the operator A is pseudo-monotone and uniformly continuous. We will also consider how to reduce the oscillation effect caused by the inertial terms.

5. Conclusions

In this study, we investigated two self-adaptive iterative schemes for seeking a common solution to the variational inequality problem involving a monotone and Lipschitz continuous mapping and the fixed point problem with a demicontractive mapping. We proposed two new inertial extragradient methods with a new step size to compute the approximate solutions of problems in a real Hilbert space. The strong convergence of the suggested methods is established under standard and suitable conditions. Finally, some computational tests are given to explain our convergent results. The algorithms obtained in this paper improved and summarized some of the recent results in the literature.

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