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SOME EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS VIA FIXED POINT THEOREMS

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Abstract. This paper is generally concerned with the existence of solutions for a certain class of fractional differential inclusions with boundary conditions. By means a known fixed point theorem, some existence results are obtained. Utilizing some contractions including $\alpha-\phi-$Geraghty contraction, we examine the existence of solutions for some fractional differential inclusions. An example is given to illustrate the results.

Key Words and Phrases: Fixed point, fractional differential inclusion, integral boundary value problems, multifunction.
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1. Introduction

Fractional Calculus is a powerful tool which has been recently employed to the most of the sciences including physics, engineering, biology and chemical phenomena (see for example [1]-[3], [9], [12], [18], [23], [25] and the references therein). Moreover, significant progress was made in the field of Fractional Differential Inclusions (FDIs) (see for example [5]-[7], [10], [11], [13]-[17], [19], [20] and [22]).

Presuppose $(\mathcal{Y},d)$ is a b-metric space and $P(\mathcal{Y})$ and $2^{\mathcal{Y}}$ are the class of all subsets and the class of all nonempty subsets of $\mathcal{Y}$, respectively. For a normed space $(\mathcal{Y},\|.\|)$, let

$P_{cl}(\mathcal{Y})=\{\mathcal{Y}\in P(\mathcal{Y}):\mathcal{Y}\text{ is closed}\}$,
$P_{bd}(\mathcal{Y})=\{\mathcal{Y}\in P(\mathcal{Y}):\mathcal{Y}\text{ is bounded}\}$,
$P_{cp}(\mathcal{Y})=\{\mathcal{Y}\in P(\mathcal{Y}):\mathcal{Y}\text{ is compact}\}$ and
$P_{bd,cl}(\mathcal{Y})=\{\mathcal{Y}\in P(\mathcal{Y}):\mathcal{Y}\text{ is bounded and closed}\}$.

We say that $Q:\mathcal{Y}\to 2^{\mathcal{Y}}$ is a multifunction on $\mathcal{Y}$ and if $u\in\mathcal{Y}$, then $u\in Qu$ is a fixed point of Q ([15]). Consider $J = [0,1]$; a multivalued map $U : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $e \in \mathbb{R}$, the function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\varphi(\aleph) = d(e, U(\aleph)) = \inf\{|e-a| : a \in U(\aleph)\}$$

is measurable ([15]). Using some fixed point theorems, we investigate the existence of solutions for two FDIs stated in this article.
Consider the Hausdorff metric $H_d : 2^{\mathcal{Y}} \times 2^{\mathcal{Y}} \to [0,\infty)$ defined by

$$H_d(M,N) = \max\{\sup_{m\in M} d(m,N), \sup_{n\in N} d(M,n)\},$$

where $d(M,n) = \inf_{m\in M} d(m,n)$. Note that $(P_{bd,cl}(\mathcal{Y}), H_d)$ is a metric space and $(P_{cl}(\mathcal{Y}), H_d)$ is a generalized metric space ([15] and [19]).

Let $(\mathcal{Y}, d)$ be a metric space, $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0,\infty)$ be a map and $\top : \mathcal{Y} \to 2^{\mathcal{Y}}$ be a multifunction. We say that $\mathcal{Y}$ has condition (C_α) whenever for each sequence $\{\Im_n\}$ in $\mathcal{Y}$ with $\alpha(\Im_n, \Im_{n+1}) \geq 1$ for all n and $\Im_n \to \Im$, there exists a subsequence $\{\Im_{n_k}\}$ of $\{\Im_n\}$ such that $\alpha(\Im_{n_k}, \Im) \geq 1$ for all k. Also, $\top$ is said to be α-admissible whenever for each $\Im \in \mathcal{Y}$ and $\flat \in \top\Im$ with $\alpha(\Im, \flat) \geq 1$, we have $\alpha(\flat, z) \geq 1$ for all $z \in \top\mathcal{Y}$. Suppose that Φ is a family of nondecreasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(\aleph) < \infty$ for all $\aleph > 0$.

Definition 1.1. Assume $h : [0,\infty) \to \mathbb{R}$ is continuous, the Caputo derivative of order ι is defined by

$${}^cD^\iota h(\mu) = \frac{1}{\Gamma(p-\iota)} \int_0^\mu (\mu - \varpi)^{p-\iota-1} h^{(p)}(\varpi) d\varpi,$$

where $p - 1 < \iota < p$, $p = [\iota] + 1$ and $[\iota]$ is the integer part of ι.

Definition 1.2. ([23, 18]) The Riemann-Lioville derivative of order ι for a continuous function h is defined by

$$D^\iota h(\mu) = \frac{1}{\Gamma(p-\iota)} \left(\frac{d}{d\mu}\right)^p \int_0^\mu \frac{h(\varpi)}{(\mu-\varpi)^{\iota-p-1}} d\varpi, \ \ (p = [\iota] + 1),$$

where the right-hand side defined on $(0,\infty)$.

Let Φ be the set of all increasing and continuous functions $\phi : [0,\infty) \to [0,\infty)$ satisfying: $\phi(\varepsilon\Im) \leq \varepsilon\phi(\Im) \leq \varepsilon w$ for all $\varepsilon > 1$, also $\mathcal{B}$ is the family of all nondecreasing functions $\gamma : [0,\infty) \to [0, \frac{1}{\upsilon^2})$ for some $\upsilon \geq 1$.

Here consider the following definitions that are special cases of definitions which are stated in [2].

Definition 1.3. Let $(\mathcal{Y}, d)$ be a b-metric space (with constant υ) and $S : \mathcal{Y} \to P_{b,cl}(\mathcal{Y})$ be a multivalued mapping. We say that S is an $\alpha - \phi$-Geraghty contraction type mapping whenever there exists $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0,\infty)$ such that

$$\alpha(c,\aleph)\phi(\upsilon^3 d(Sc, S\aleph)) \leq \gamma(\phi(d(c,\aleph)))\phi(d(c,\aleph)), \tag{1.1}$$

for all $c, \aleph \in \mathcal{Y}$, where $\gamma \in \mathcal{B}$ and $\phi \in \Phi$.

Definition 1.4. ([24]) Let $S : \mathcal{Y} \to \mathcal{Y}$ where $\mathcal{Y}$ is nonempty and $\alpha : \mathcal{Y} \times \mathcal{Y} \to [0, \infty)$ be given. Then S is $\alpha-$admissible if for all $c, \aleph \in \mathcal{Y}$,

$$\alpha(c, \aleph) \geq 1 \Longrightarrow \alpha(Sc, S\aleph) \geq 1. \tag{1.2}$$

Definition 1.5. Let $(\mathcal{Y}, d)$ be a b-metric space. $\mathcal{Y}$ is said α-regular, if for every sequence $\{\Im_n\}$ in $\mathcal{Y}$ such that $\alpha(\Im_n, \Im_{n+1}) \geq 1$ for all n and $\Im_n \to \Im \in \mathcal{Y}$ as $n \to \infty$, then there exists a subsequence $\{\Im_{n(k)}\}$ of $\{\Im_n\}$ such that $\alpha(\Im_{n(k)}, \Im) \geq 1$ for all k.

To state and prove our main results we need the following lemmas.

Lemma 1.6. ([2, Corollary 2.5]) *Let $(\mathcal{Y}, d)$ be a complete b-metric space and $\top : \mathcal{Y} \to P_{bd,cl}(\mathcal{Y})$ be an $\alpha - \phi-$Geraghty contraction type multivalued mapping such that $\top$ is α-admissible. Assume that there exists $\Im_0 \in \mathcal{Y}$ such that $\alpha(\Im_0, \top\Im_0) \geq 1$ and $\mathcal{Y}$ is α-regular. Then, $\top$ has a fixed point.*

Lemma 1.7. ([8]) *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\hbar : \overline{U} \to P_{cp,cv}(C)$ is an upper semi-continuous compact map where $P_{cp,cv}(C)$ denotes the family of nonempty, compact and convex subsets of C. Then either $\hbar$ has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\varepsilon \in (0, 1)$ such that $u \in \varepsilon\hbar(u)$.*

2. Main results

In this section we prove the existence of solutions for two fractional boundary value inclusions. First, consider the problem

$$\begin{aligned} &{}^cD^\varsigma \Im(\aleph) \in \hbar(\aleph, \Im(\aleph), {}^c D^\vartheta \Im(\aleph)). \\ &\Im(1) + \Im'(1) = \int_0^\kappa \Im(\upsilon)d\upsilon, \ \Im(0) = 0, \end{aligned} \tag{2.1}$$

where $\aleph \in J$, $\vartheta, \kappa \in (0, 1)$, $\varsigma \in (1, 2]$ with $\varsigma - \vartheta > 1$, ${}^cD^\varsigma$ is the Caputo differentiation and $\hbar : J \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ denotes a compact valued multifunction.

A function $\Im \in C(J, \mathbb{R})$ is a solution of problem (2.1) whenever it satisfies the boundary conditions and there exists a function $v \in L^1(J)$ such that $v(\aleph) \in \hbar(\aleph, \Im(\aleph), {}^c D^\vartheta \Im(\aleph))$ for almost all $\aleph \in J$ and

$$\begin{aligned} \Im(\aleph) &= \frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma-1} v(\upsilon) d\upsilon \\ &\quad + \frac{2\aleph}{(4 - \kappa^2)\Gamma(\varsigma)} \int_0^\kappa \int_0^\upsilon (\upsilon - m)^{\varsigma-1} v(m) dm d\upsilon \\ &\quad - \frac{2\aleph}{(4 - \kappa^2)\Gamma(\varsigma)} \int_0^1 (1 - \upsilon)^{\varsigma-1} v(\upsilon) d\upsilon \\ &\quad - \frac{2\aleph}{(4 - \kappa^2)\Gamma(\varsigma - 1)} \int_0^1 (1 - \upsilon)^{\varsigma-2} v(\upsilon) d\upsilon \\ &= P_v(\aleph) + \int_0^1 G(\aleph, \upsilon) v(\upsilon) d\upsilon. \end{aligned}$$

Before stating the main results, a lemma is introduced here which is needed in the proof of the results.

Lemma 2.1. ([5]) *Let $v \in C(J,\mathbb{R})$. Then, the unique solution of the fractional differential equation ${}^cD^{\varsigma}\Im(\aleph) = v(\aleph)$ with the boundary conditions*

$$\Im(1) + \Im'(1) = \int_0^{\kappa} \Im(\upsilon)d\upsilon \text{ and } \Im(0) = 0,$$

where $\vartheta, \kappa \in (0,1)$, $\varsigma \in (1,2]$ with $\varsigma - \vartheta > 1$, is given by

$$\begin{aligned}\Im(\aleph) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph - s)^{\varsigma-1}v(\upsilon)d\upsilon + \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^{\kappa}\int_0^{\upsilon}(\upsilon - m)^{\varsigma-1}v(m)dmd\upsilon \\ &\quad - \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^1(1-s)^{\varsigma-1}v(\upsilon)d\upsilon - \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)}\int_0^1(1-\upsilon)^{\varsigma-2}v(\upsilon)d\upsilon \\ &= P_v(\aleph) + \int_0^1 G(\aleph,s)v(\upsilon)d\upsilon,\end{aligned}$$

where

$$P_v(\aleph) = \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^{\kappa}\int_0^{\upsilon}(\upsilon - m)^{\varsigma-1}v(m)dmd\upsilon$$

and

$$G(\aleph,\upsilon) = \begin{cases} \frac{(4-\kappa^2)(\aleph-\upsilon)^{(\varsigma-1)}-2\aleph(1-\upsilon)^{\varsigma-1}}{(4-\kappa^2)\Gamma(\varsigma)} - \frac{2t(1-\upsilon)^{\varsigma-2}}{(4-\kappa^2)\Gamma(\varsigma-1)} & 0<\upsilon<\aleph<1, \\ \frac{-2\aleph(1-\upsilon)^{\varsigma-1}}{(4-\kappa^2)\Gamma(\varsigma)} - \frac{2\aleph(1-\upsilon)^{\varsigma-2}}{(4-\kappa^2)\Gamma(\varsigma-1)} & 0<\aleph<\upsilon<1. \end{cases}$$

Now let $\mathcal{Y} = \{\Im : \Im, {}^cD^{\vartheta}\Im \in C(J,\mathbb{R})\}$ and $d : \mathcal{Y}\times\mathcal{Y} \to [0,\infty)$ be given by

$$d(\Im,\flat) = \|\Im - \flat\|^2 = \sup_{\aleph\in J}|\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph\in J}|{}^cD^{\vartheta}\Im(\aleph) -{}^c D^{\vartheta}\flat(\aleph)|^2.$$

Evidently, $(\mathcal{Y}, \|.\|)$ is a complete b-metric space with $\upsilon = 2$ but is not a metric space ([26]). Let $\Im \in \mathcal{X}$ and define the set of selections of $\hbar$ by

$$S_{\hbar,\Im} = \{v \in L^1(J) : v(\aleph) \in \hbar(\aleph, \Im(\aleph), {}^cD^{\vartheta}\Im(\aleph)) \text{ for almost all } \aleph \in J\}.$$

Theorem 2.2. *Suppose that $\hbar : J\times\mathbb{R}\times\mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that $\hbar$ is integrable and bounded and $\hbar(.,\Im,\flat) : J \to P_{cp}(\mathbb{R})$ is measurable for all $\Im,\flat \in \mathbb{R}$. Assume that there exist a function $\xi : \mathbb{R}^4 \to \mathbb{R}$, $\phi \in \Phi$ and $m \in C(J,[0,\infty))$ such that*

$$\begin{aligned}&H_d(\hbar(\aleph,\Im,\flat), \hbar(\aleph,z,\Im)) \\ &\le \frac{m(\aleph)}{2\sqrt{2}}\frac{\phi(|\Im(\aleph) - z(\aleph)|^2 + |\flat(\aleph) - \Im(\aleph)|^2)}{\sqrt{4(\sup_{\aleph\in J}|\Im(\aleph) - z(\aleph)|^2 + \sup_{\aleph\in J}|\flat(\aleph) - \Im(\aleph)|^2) + 1}}\frac{1}{||m||_{\infty}\sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}},\end{aligned}$$

for all $\aleph \in J$, $\upsilon \ge 1$ and $\Im,\flat,z \in \mathbb{R}$, where

$$\Lambda_1 = \frac{2\varsigma^2 + 7\varsigma + 7}{3\Gamma(\varsigma+2)},\ \Lambda_2 = \frac{1}{\Gamma(\varsigma - \vartheta + 1)} + \frac{2(\varsigma^2 + 2\varsigma + 1)}{3\Gamma(\varsigma+2)\Gamma(2-\vartheta)}.$$

and in addition suppose the following three conditions $(i)-(iii)$ hold.
(i) If $\{\Im_n\}$ is a sequence in $\mathcal{Y}$ such that $\Im_n \to \Im$ and

$$\xi((\Im_n(\aleph), {}^cD^{\vartheta}\Im_n(\aleph)), (\Im_{n+1}(\aleph), {}^cD^{\vartheta}\Im_{n+1}(\aleph))) \geq 0,$$

for all $\aleph \in J$, then there exists a subsequence $\{\Im_{n_k}\}$ of $\{\Im_n\}$ such that

$$\xi((\Im_{n_k}(\aleph), {}^cD^{\vartheta}\Im_{n_k}(\aleph)), (\Im(\aleph), {}^cD^{\vartheta}\Im(\aleph))) \geq 0,$$

for all $\aleph \in J$.
(ii) For each $\Im \in \mathcal{Y}$ and $h \in \Omega_{\hbar}(\Im)$ with

$$\xi((\Im(\aleph), {}^cD^{\vartheta}\Im(\aleph)), (h(\aleph), {}^cD^{\vartheta}h(\aleph))) \geq 0,$$

there exists $z \in \Omega_{\hbar}(h)$ such that $\xi((h(\aleph), {}^cD^{\vartheta}h(\aleph)), (z(\aleph), {}^cD^{\vartheta}z(\aleph))) \geq 0$, where the operator $\Omega_{\hbar} : \mathcal{Y} \to P(\mathcal{Y})$ is defined by

$$\Omega_{\hbar}(\Im) = \{h \in \mathcal{Y} : \ \exists v \in S_{\hbar,\Im} \text{ such that } h(\aleph) = P_v(\aleph) + \int_0^1 G(\aleph, \upsilon)v(\upsilon)d\upsilon \ \forall \ \aleph \in J\}.$$

(iii) There exist $\Im_0 \in \mathcal{Y}$ and $h \in \Omega_{\hbar}(\Im_0)$ such that

$$\xi((\Im_0(\aleph), {}^cD^{\vartheta}\Im_0(\aleph)), (h(\aleph), {}^cD^{\vartheta}h(\aleph))) \geq 0,$$

for all $\aleph \in J$.
Then, the boundary value inclusion (2.1) admits a solution.

Proof. We show that the operator $\Omega_{\hbar} : \mathcal{Y} \to P(\mathcal{Y})$ has a fixed point. Note that, the multi-valued map $\aleph \mapsto \hbar(\aleph, \Im(\aleph), {}^cD^{\vartheta}\Im(\aleph))$ is measurable and closed valued for all $\Im \in \mathcal{Y}$. Hence, it has measurable selection and therefore the set $S_{\hbar,\Im}$ is nonempty.

First, we show that $\Omega_{\hbar}(\Im)$ is closed subset of $\mathcal{Y}$ for all $\Im \in \mathcal{Y}$. Let $\Im \in \mathcal{Y}$ and $\{u_n\}_{n\geq 1}$ is a sequence in $\Omega_{\hbar}(\Im)$ with $u_n \to u$. For each n, choose $v_n \in S_{\hbar,\Im}$ such that

$$\begin{aligned} u_n(\aleph) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v_n(\upsilon)d\upsilon + \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^{\kappa}\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v_n(m)dmd\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^1(1-\upsilon)^{\varsigma-1}v_n(\upsilon)d\upsilon - \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)}\int_0^1(1-\upsilon)^{\varsigma-2}v_n(\upsilon)d\upsilon, \end{aligned}$$

for almost all $\aleph \in J$. Since $\hbar$ has compact values, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1(J)$. This subsequence is denoted again by $\{v_n\}_{n\geq 1}$. It is easy to check that $v \in S_{\hbar,\Im}$ and

$$\begin{aligned} &u_n(\aleph) \to u(\aleph) \\ &= \frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon + \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^{\kappa}\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dmd\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)}\int_0^1(1-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon - \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)}\int_0^1(1-\upsilon)^{\varsigma-2}v(\upsilon)d\upsilon, \end{aligned}$$

for all $\aleph \in J$. This implies that $u \in \Omega_{\hbar}(\Im)$. Thus, the multifunction $\Omega_{\hbar}$ has closed values. Since $\hbar$ is a compact multi-valued map, $\Omega_{\hbar}(\Im)$ is bounded set in $\mathcal{Y}$ for all $\Im \in \mathcal{Y}$.

Define $\alpha : C(J) \times C(J) \to [0, \infty)$ by

$$\alpha(\Im, \flat) = \begin{cases} 1 & \xi((\Im(\aleph), {}^c D^{\vartheta}\Im(\aleph)), (\flat(\aleph), {}^c D^{\vartheta}\flat(\aleph))) \geq 0, \quad for\ all\ \aleph \in J, \\ 0 & else \end{cases}$$

and define $\daleth : [0, \infty) \to [0, \frac{1}{4})$ by $\daleth(\varsigma) = \frac{\varsigma}{4\varsigma+1}$ and let $\upsilon = 2$.

It will be shown that $\alpha(\Im, \flat)\phi(8H_d(\Omega_\hbar(\Im), \Omega_\hbar(\flat))) \leq \daleth(\phi(\| \Im - \flat \|))\phi(\|\Im - \flat\|)$ for all $\Im, \flat \in \mathcal{Y}$. Let $\Im, \flat \in \mathcal{Y}$ and $\varrho_1 \in \Omega_\hbar(\flat)$. Choose $v_1 \in S_{\hbar,y}$ such that

$$\varrho_1(\aleph) = \frac{1}{\Gamma(\varsigma)} \int_0^{\aleph} (\aleph - \upsilon)^{\varsigma-1} v_1(\upsilon) d\upsilon + \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^{\kappa} \int_0^{\upsilon} (\upsilon - m)^{\varsigma-1} v_1(m) dm d\upsilon$$
$$- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^1 (1-\upsilon)^{\varsigma-1} v_1(\upsilon) d\upsilon - \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)} \int_0^1 (1-\upsilon)^{\varsigma-2} v_1(\upsilon) d\upsilon,$$

for all $\aleph \in J$. Since

$$\begin{aligned} &H_d(\hbar(\aleph, \Im(\aleph), {}^c D^{\vartheta}\Im(\aleph)), \hbar(\aleph, \flat(\aleph), {}^c D^{\vartheta}\flat(\aleph))) \\ &\quad \leq \frac{m(\aleph)}{2\sqrt{2}} \times \frac{\phi(|\Im(\aleph) - \flat(\aleph)|^2 + |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2)}{\sqrt{4(\sup_{\aleph \in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph \in J} |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2) + 1}} \\ &\quad \times \frac{1}{||m||_\infty \sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}}, \end{aligned}$$

for all $\Im, \flat \in \mathcal{Y}$ with $\xi((\Im(\aleph), {}^c D^{\vartheta}\Im(\aleph)), (\flat(\aleph), {}^c D^{\vartheta}\flat(\aleph))) \geq 0$ for almost $\aleph \in J$, there exists $g \in \hbar(\aleph, \Im(\aleph), {}^c D^{\vartheta}\Im(\aleph))$ such that

$$\begin{aligned} |v_1(\aleph) - g| &\leq \frac{m(\aleph)}{2\sqrt{2}} \\ &\times \frac{\phi(|\Im(\aleph) - \flat(\aleph)|^2 + |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2)}{\sqrt{4(\sup_{\aleph \in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph \in J} |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2) + 1}} \\ &\times \frac{1}{||m||_\infty \sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}}. \end{aligned}$$

Consider the multi-valued map $U : J \to P(\mathbb{R})$ as

$$\begin{aligned} U(\aleph) = \big\{ g \in \mathbb{R} : |v_1(\aleph) - g| &\leq \frac{m(\aleph)}{2\sqrt{2}} \\ &\times \frac{\phi(|\Im(\aleph) - \flat(\aleph)|^2 + |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2)}{\sqrt{4(\sup_{\aleph \in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph \in J} |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2) + 1}} \\ &\times \frac{1}{||m||_\infty \sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}} \big\}, \end{aligned}$$

for all $\aleph \in J$. Since v_1 and

$$\begin{aligned} \varphi = \frac{m(\aleph)}{2\sqrt{2}} &\times \frac{\phi(|\Im(\aleph) - \flat(\aleph)|^2 + |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2)}{\sqrt{4(\sup_{\aleph \in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph \in J} |{}^c D^{\vartheta}\Im(\aleph) - {}^c D^{\vartheta}\flat(\aleph)|^2) + 1}} \\ &\times \frac{1}{||m||_\infty \sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}}, \end{aligned}$$

are measurable, $U(.) \cap \hbar(., \Im(.), {}^c D^\vartheta \Im(.))$ is also measurable. Thus, for each $\aleph \in J$, we can choose $\upsilon_2(\aleph) \in \hbar(\aleph, \Im(\aleph), {}^c D^\vartheta \Im(\aleph))$ such that

$$\begin{aligned}|\upsilon_1(\aleph) - \upsilon_2(\aleph)| \leq \frac{m(\aleph)}{2\sqrt{2}} \\ &\times \frac{\phi(|\Im(\aleph) - \flat(\aleph)|^2 + |{}^c D^\vartheta \Im(\aleph) -{}^c D^\vartheta \flat(\aleph)|^2)}{\sqrt{4(\sup_{\aleph \in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph \in J} |{}^c D^\vartheta \Im(\aleph) -{}^c D^\vartheta \flat(\aleph)|^2) + 1}} \\ &\times \frac{1}{||m||_\infty \sqrt{{\Lambda_1}^2 + {\Lambda_2}^2}}.\end{aligned}$$

Now consider $\varrho_2 \in \Omega_\hbar(\Im)$ which is given by

$$\begin{aligned}\varrho_2(\aleph) = &\frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma-1} \upsilon_2(\upsilon) d\upsilon \\ &+ \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^\kappa \int_0^\upsilon (\upsilon - m)^{\varsigma-1} \upsilon_2(m) dm d\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^1 (1-\upsilon)^{\varsigma-1} \upsilon_2(\upsilon) d\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)} \int_0^1 (1-\upsilon)^{\varsigma-2} \upsilon_2(\upsilon) d\upsilon,\end{aligned}$$

for all $\aleph \in J$. Thus,

$$\begin{aligned}|\varrho_1(\aleph) - \varrho_2(\aleph)| = &\left| \frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma-1} \upsilon_1(\upsilon) d\upsilon \right. \\ &+ \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^\kappa \int_0^\upsilon (\upsilon - m)^{\varsigma-1} \upsilon_1(m) dm d\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^1 (1-\upsilon)^{\varsigma-1} \upsilon_1(\upsilon) d\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)} \int_0^1 (1-\upsilon)^{\varsigma-2} \upsilon_1(\upsilon) d\upsilon \\ &- \frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma-1} \upsilon_2(\upsilon) d\upsilon \\ &- \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^\kappa \int_0^\upsilon (\upsilon - m)^{\varsigma-1} \upsilon_2(m) dm d\upsilon \\ &+ \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma)} \int_0^1 (1-\upsilon)^{\varsigma-1} \upsilon_2(\upsilon) d\upsilon \\ &\left. + \frac{2\aleph}{(4-\kappa^2)\Gamma(\varsigma-1)} \int_0^1 (1-\upsilon)^{\varsigma-2} \upsilon_2(\upsilon) d\upsilon \right| \\ &\leq \frac{\|m\|_\infty}{2\sqrt{2}} \left(\frac{2\varsigma^2 + 7\varsigma + 7}{3\Gamma(\varsigma+2)} \right) \left(\frac{1}{\sqrt{{\Lambda_1}^2 + {\Lambda_2}^2} \|m\|_\infty} \right) \frac{\phi(\|\Im - \flat\|)}{\sqrt{4\|\Im - \flat\| + 1}}.\end{aligned}$$

Hence,

$$|\varrho_1(\aleph)-\varrho_2(\aleph)|^2\leq\frac{1}{8}\left(\frac{2\varsigma^2+7\varsigma+7}{3\Gamma(\varsigma+2)}\right)^2\frac{1}{{\Lambda_1}^2+{\Lambda_2}^2}\frac{(\phi(\|\Im-\flat\|^2))^2}{4\|\Im-\flat\|+1}$$

and

$$\begin{aligned}|^cD^\vartheta\varrho_1(\aleph)-{}^cD^\vartheta\varrho_2(\aleph)|&=\bigg|\frac{1}{\Gamma(\varsigma-\vartheta)}\int_0^\aleph(\aleph-\upsilon)^{\varsigma-\daleth-1}v_1(\upsilon)d\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^\kappa\int_0^\upsilon(\upsilon-m)^{\varsigma-1}v_1(m)dmd\upsilon\\&\quad-\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-1}v_1(\upsilon)d\upsilon\\&\quad-\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma-1)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-2}v_1(\upsilon)d\upsilon\\&\quad-\frac{1}{\Gamma(\varsigma-\vartheta)}\int_0^\aleph(\aleph-\upsilon)^{\varsigma-\daleth-1}v_2(\upsilon)d\upsilon\\&\quad-\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^\kappa\int_0^\upsilon(\upsilon-m)^{\varsigma-1}v_2(m)dmd\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-1}v_2(\upsilon)d\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma-1)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-2}v_2(\upsilon)d\upsilon\bigg|\\&\leq\frac{1}{\Gamma(\varsigma-\vartheta)}\int_0^\aleph(\aleph-\upsilon)^{\varsigma-\daleth-1}|v_1(\upsilon)-v_2(\upsilon)|d\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^\kappa\int_0^\upsilon(\upsilon-m)^{\varsigma-1}|v_1(m)-v_1(m)|dmd\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-1}|v_1(\upsilon)-v_2(\upsilon)d\upsilon\\&\quad+\frac{2\Gamma(2)t^{1-\vartheta}}{(4-\kappa^2)\Gamma(\varsigma-1)\Gamma(2-\vartheta)}\int_0^1(1-\upsilon)^{\varsigma-2}|v_1(\upsilon)-v_2(\upsilon)|d\upsilon\\&\leq\frac{\|m\|_\infty}{2\sqrt{2}}\left(\frac{1}{\Gamma(\varsigma-\vartheta+1)}+\frac{2}{3\Gamma(\varsigma+2)\Gamma(2-\vartheta)}+\frac{2}{3\Gamma(\varsigma+1)\Gamma(2-\vartheta)}\right.\\&\quad\left.+\frac{2}{3\Gamma(\varsigma)\Gamma(2-\vartheta)}\right)\left(\frac{1}{\sqrt{{\Lambda_1}^2+{\Lambda_2}^2}\|m\|_\infty}\right)\frac{\phi(\|\Im-\flat\|)}{\sqrt{4\|\Im-\flat\|+1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}|^cD^\vartheta\varrho_1(\aleph)-{}^cD^\vartheta\varrho_2(\aleph)|^2\leq\frac{1}{8}&\left(\frac{1}{\Gamma(\varsigma-\vartheta+1)}+\frac{2}{3\Gamma(\varsigma+2)\Gamma(2-\vartheta)}\right.\\&\left.+\frac{2}{3\Gamma(\varsigma+1)\Gamma(2-\vartheta)}\right)^2\frac{1}{{\Lambda_1}^2+{\Lambda_2}^2}\frac{(\phi(\|\Im-\flat\|^2))^2}{4\|\Im-\flat\|+1},\end{aligned}$$

for all $\aleph \in J$. Hence,

$$\begin{aligned}\|\varrho_1 - \varrho_2\|^2 &= \sup_{\aleph\in J} |\varrho_1(\aleph) - \varrho_2(\aleph)|^2 + \sup_{\aleph\in J} |{}^cD^{\vartheta}\varrho_1(\aleph) -{}^c D^{\vartheta}\varrho_2(\aleph)|^2 \\ &\leq \frac{1}{8} \times \left(\frac{2\varsigma^2 + 7\varsigma + 7}{3\Gamma(\varsigma + 2)}\right)^2 \times \frac{1}{{\Lambda_1}^2 + {\Lambda_2}^2} \\ &\quad \times \frac{(\phi(\|\Im - \flat\|^2))^2}{4(\sup_{\aleph\in J} |\Im(\aleph) - \flat(\aleph)|^2 + \sup_{\aleph\in J} |{}^cD^{\vartheta}\Im(\aleph) -{}^c D^{\vartheta}\flat(\aleph)|^2) + 1} \\ &\quad + \frac{1}{8}(\frac{1}{\Gamma(\varsigma - \vartheta + 1)} + \frac{2}{3\Gamma(\varsigma + 2)\Gamma(2 - \vartheta)} + \frac{2}{3\Gamma(\varsigma + 1)\Gamma(2 - \vartheta)})^2 \\ &\quad \times \frac{1}{{\Lambda_1}^2 + {\Lambda_2}^2} \times \frac{(\phi(\|\Im - \flat\|^2))^2}{4\|\Im - \flat\| + 1} \\ &= \frac{1}{8}\frac{\phi(\|\Im - \flat\|^2))^2}{4\|\Im - \flat\|^2 + 1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha(\Im, \flat)\phi(8H_d(\Omega_\hbar(\Im), \Omega_\hbar(\flat))) &\leq 8\alpha(\Im, \flat)\phi(H_d(\Omega_\hbar(\Im), \Omega_\hbar(\flat))) \\ &\leq \frac{\phi(d(\Im, \flat))^2}{4d(\Im, \flat) + 1} \leq \frac{\phi(d(\Im, \flat))^2}{4\phi(d(\Im, \flat)) + 1} \\ &= \daleth(\phi(d(\Im, \flat)))\phi(d(\Im, \flat)),\ \daleth \in \mathcal{B}.\end{aligned}$$

Consequently, $\Omega_\hbar$ is an α-ϕ-Geraghty contractive multifunction. Assume $\Im \in \mathcal{Y}$ and $\flat \in \Omega_\hbar(\Im)$ be such that $\alpha(\Im, \flat) \geq 1$. Then,

$$\xi((\Im(\aleph),{}^c D^{\vartheta}\Im(\aleph)), (\flat(\aleph),{}^c D^{\vartheta}\flat(\aleph))) \geq 0.$$

Therefore, there exists $z \in \Omega_\hbar(\flat)$ such that $\xi((\flat(\aleph),{}^c D^{\vartheta}\flat(\aleph)), (z(\aleph),{}^c D^{\vartheta}z(\aleph))) \geq 0$. Hence, $\alpha(\flat, z) \geq 1$ and $\Omega_\hbar$ is α-admissible. Choose $\Im_0 \in \mathcal{Y}$ and $\flat \in \Omega_\hbar(\Im_0)$ such that

$$\xi((\Im_0(\aleph),{}^c D^{\vartheta}\Im_0(\aleph)), (\flat(\aleph),{}^c D^{\vartheta}\flat(\aleph))) \geq 0.$$

Thus, $\alpha(\Im_0, \flat) \geq 1$. Now, by Lemma 1.6, there exists $\Im^* \in \mathcal{Y}$ such that $\Im^* \in \Omega_\hbar(\Im^*)$. It is easy to see that $\Im^*$ is a solution of the problem (2.1). □

In the sequel, we consider the fractional boundary value inclusion

$$\begin{aligned}&{}^cD^{\varsigma}\Im(\aleph) \in \hbar(\aleph, \Im(\aleph)), \\ &\Im(0) = j\int_0^{\iota} \Im(\upsilon)d\upsilon,\ \Im(1) = i\int_0^{\kappa} \Im(\upsilon)d\upsilon,\end{aligned} \tag{2.2}$$

where $\aleph \in J$, $1 < \varsigma \leq 2$, $0 < \iota, \kappa < 1$, $j, i \in \mathbb{R}$, ${}^cD^{\varsigma}$ is the standard Caputo differentiation and $\hbar : J \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

In 2011, Ahmad and Ntouyas discussed this inclusion problem by utilizing Lemma 1.7 ([10]). In this manuscript, we are going to show that one can solve this inclusion problem by making use of Lemma 1.6.

Let $\upsilon \in C(J,\mathbb{R})$. As a result, the unique solution of the fractional differential equation ${}^cD^{\varsigma}\Im(\aleph) = \upsilon(\aleph)$ with the boundary conditions

$$\Im(0) = j\int_0^{\iota} \Im(\upsilon)d\upsilon \text{ and } \Im(1) = i\int_0^{\kappa} \Im(\upsilon)d\upsilon$$

is given by

$$\begin{aligned}\Im(\aleph) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon \\ &+ \frac{a}{\gamma\Gamma(\varsigma)}\left(\frac{2-i\kappa^2}{2}+(b\kappa-1)\aleph\right)\int_0^{\iota}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm\right)d\upsilon \\ &+ \frac{b}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{\kappa}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm\right)d\upsilon \\ &- \frac{1}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{1}(1-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon,\end{aligned}$$

where $0 \le \aleph \le 1$, $1 < \varsigma \le 2$, $0 < \iota, \kappa < 1$ and

$$\gamma = \frac{1}{2}[(a\iota-1)(b\kappa^2-2) - a\iota(b\kappa-1)] \neq 0$$

(see [10]). Note that $w \in C(J,\mathbb{R})$ is a solution of the problem (2.2) whenever it satisfies the boundary conditions and there exists a function $v \in L^1J$ such that $v(\aleph) \in \hbar(\aleph, \Im(\aleph))$ for almost all $\aleph \in J$ (see [10]) and

$$\begin{aligned}\Im(\aleph) &= \frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon \\ &+ \frac{a}{\gamma\Gamma(\varsigma)}(\frac{2-i\kappa^2}{2}+(b\kappa-1)\aleph)\int_0^{\iota}(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm)d\upsilon \\ &+ \frac{b}{\gamma\Gamma(\varsigma)}(\frac{a\iota^2}{2}+(1-\iota a)\aleph)\int_0^{\kappa}(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm)d\upsilon \\ &- \frac{1}{\gamma\Gamma(\varsigma)}(\frac{a\iota^2}{2}+(1-\iota a)\aleph)\int_0^{1}(1-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon.\end{aligned}$$

Theorem 2.3. *Suppose that $\hbar : J\times\mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that $\hbar$ is integrable and bounded and $\hbar(.,\Im) : J \to P_{cp}(\mathbb{R})$ is measurable for all $\Im \in \mathbb{R}$. Assume that there exist a function $\xi : \mathbb{R}^2 \to \mathbb{R}$, $\phi \in \Phi$ and $m \in C(J,[0,\infty))$ such that*

$$H_d(\hbar(\aleph,\Im),\hbar(\aleph,\flat)) \le \frac{m(\aleph)}{2\sqrt{2}}\frac{\phi(|\Im-\flat|^2)}{\sqrt{4\|\Im-\flat\|^2+1}}\left(\frac{2|\gamma|\Gamma(\varsigma+2)}{(2|\gamma|(\varsigma+1)+(\Lambda_1+\Lambda_2))\|m\|_{\infty}}\right),$$

for all $\aleph \in J$ and $\Im, \flat \in \mathbb{R}$, where $\Lambda_1 = |j|(|2-i\kappa^2|+2|i\kappa-1|)\iota^{\varsigma+1}$ and

$$\Lambda_2 = (|j|\iota^2 + 2|1-\iota j|)(|i|\kappa^{\varsigma+1}+1).$$

Also, suppose the following three conditions ((i)-(iii)) hold,
(i) If $\{\Im_n\}$ is a sequence in $\mathcal{Y}$ such that $\Im_n \to \Im$ and $\xi(\Im_n(\aleph),\Im_{n+1}(\aleph)) \ge 0$ for all $\aleph \in J$, then there exists a subsequence $\{\Im_{n_k}\}$ of $\{\Im_n\}$ such that $\xi(\Im_{n_k}(\aleph),\Im(\aleph)) \ge 0$ for all $\aleph \in J$.

(ii) For each $\Im \in \mathcal{Y}$ *and* $\flat \in \Omega_\hbar(\Im)$ *with* $\xi(\Im(\aleph), \flat(\aleph)) \geq 0$*, there exists* $z \in \Omega_\hbar(\flat)$ *such that* $\xi((\flat(\aleph), z(\aleph)) \geq 0$*, where the operator* $\Omega_\hbar : \mathcal{Y} \to P(\mathcal{Y})$ *is defined by*

$$\Omega_\hbar(\Im) = \{h \in \mathcal{Y} :\ \exists\ v \in S_{\hbar,\Im} \text{ such that } h(\aleph) = \Im(\aleph)\ \forall\ \aleph \in J\}$$

where

$$\begin{aligned}
\Im(\aleph) = &\frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon\\
&+\frac{a}{\gamma\Gamma(\varsigma)}\left(\frac{2-i\kappa^2}{2}+(b\kappa-1)\aleph\right)\int_0^{\iota}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm\right)d\upsilon\\
&+\frac{b}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{\kappa}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v(m)dm\right)d\upsilon\\
&-\frac{1}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{1}(1-\upsilon)^{\varsigma-1}v(\upsilon)d\upsilon.
\end{aligned}$$

(iii) There exist $\Im_0 \in \mathcal{Y}$ *and* $h \in \Omega_\hbar(\Im_0)$ *with* $\xi(\Im_0(\aleph), h(\aleph)) \geq 0$ *for* $\aleph \in J$*.*
Then, the boundary value inclusion (2.2) has a solution.

Proof. We show that the operator $\Omega_\hbar$ has a fixed point. By using a similar proof of Theorem 2.2, one can show that the operator $\Omega_\hbar$ has closed and bounded values. Define the function $\alpha : \mathcal{Y}\times\mathcal{Y} \to [0,\infty)$ by $\alpha(\Im,\flat) = 1$ whenever $\xi(\Im(\aleph),\flat(\aleph)) \geq 0$ for $\aleph \in J$ and $\alpha(\Im,\flat) = 0$ otherwise. Let $\Im,\flat \in \mathcal{Y}$ and $\varrho_1 \in \Omega_\hbar(\flat)$. Choose $v_1 \in S_{\hbar,y}$ such that

$$\begin{aligned}
\varrho_1(\aleph) = &\frac{1}{\Gamma(\varsigma)}\int_0^{\aleph}(\aleph-\upsilon)^{\varsigma-1}v_1(\upsilon)d\upsilon\\
&+\frac{a}{\gamma\Gamma(\varsigma)}\left(\frac{2-i\kappa^2}{2}+(b\kappa-1)\aleph\right)\int_0^{\iota}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v_1(m)dm\right)d\upsilon\\
&+\frac{b}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{\kappa}\left(\int_0^{\upsilon}(\upsilon-m)^{\varsigma-1}v_1(m)dm\right)d\upsilon\\
&-\frac{1}{\gamma\Gamma(\varsigma)}\left(\frac{a\iota^2}{2}+(1-\iota a)\aleph\right)\int_0^{1}(1-\upsilon)^{\varsigma-1}v_1(\upsilon)d\upsilon,
\end{aligned}$$

for all $\aleph \in J$. Since

$$H_d(\hbar(\aleph,\Im(\aleph)),\hbar(\aleph,\flat(\aleph))) \leq \frac{m(\aleph)}{2\sqrt{2}}\frac{\phi(|\Im-\flat|^2)}{\sqrt{4\|\Im-\flat\|^2+1}}\left(\frac{2|\gamma|\Gamma(\varsigma+2)}{(2|\gamma|(\varsigma+1)+(\Lambda_1+\Lambda_2))\|m\|_\infty}\right)$$

for all $\Im,\flat \in \mathcal{Y}$ with $\xi(\Im(\aleph),\flat(\aleph)) \geq 0$ for $\aleph \in J$, there exists $g \in \hbar(\aleph,\Im(\aleph))$ such that

$$|v_1(\aleph)-g| \leq \frac{m(\aleph)}{2\sqrt{2}}\frac{\phi(|\Im-\flat|^2)}{\sqrt{4\|\Im-\flat\|^2+1}}\left(\frac{2|\gamma|\Gamma(\varsigma+2)}{(2|\gamma|(\varsigma+1)+(\Lambda_1+\Lambda_2))\|m\|_\infty}\right).$$

Define $U : J \to P(\mathbb{R})$ by

$$U(\aleph) = \Bigg\{ g \in \mathbb{R} : |v_1(\aleph) - g| \leq \frac{m(\aleph)}{2\sqrt{2}} \frac{\phi(|\Im - \flat|^2)}{\sqrt{4\|\Im - \flat\|^2 + 1}} \left(\frac{2|\gamma|\Gamma(\varsigma + 2)}{(2|\gamma|(\varsigma + 1) + (\Lambda_1 + \Lambda_2))\|m\|_\infty} \right) \Bigg\}.$$

Since v_1 and

$$\frac{m(\aleph)}{2\sqrt{2}} \frac{\phi(|\Im - \flat|^2)}{\sqrt{4\|\Im - \flat\|^2 + 1}} (\frac{2|\gamma|\Gamma(\varsigma + 2)}{(2|\gamma|(\varsigma + 1) + (\Lambda_1 + \Lambda_2))\|m\|_\infty})$$

are measurable, it is easy to see that the multifunction $U(.) \bigcap \hbar(., \Im(.))$ is measurable. Thus, we can choose v_2 such that $v_2(\aleph) \in \hbar(\aleph, \Im(\aleph))$ and

$$|v_1(\aleph) - v_2(\aleph)| \leq \frac{m(\aleph)}{2\sqrt{2}} \frac{\phi(|\Im - \flat|^2)}{\sqrt{4\|\Im - \flat\|^2 + 1}} \left(\frac{2|\gamma|\Gamma(\varsigma + 2)}{(2|\gamma|(\varsigma + 1) + (\Lambda_1 + \Lambda_2))\|m\|_\infty} \right),$$

for all $\aleph \in J$. Now, consider $\varrho_2 \in \Omega_\hbar(\Im)$ which is defined by

$$\begin{aligned}
\varrho_2(\aleph) = & \frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma - 1} v_2(\upsilon) d\upsilon \\
& + \frac{a}{\gamma\Gamma(\varsigma)} \left(\frac{2 - i\kappa^2}{2} + (b\kappa - 1)\aleph \right) \int_0^\iota \left(\int_0^\upsilon (\upsilon - m)^{\varsigma - 1} v_2(m) dm \right) d\upsilon \\
& + \frac{b}{\gamma\Gamma(\varsigma)} \left(\frac{a\iota^2}{2} + (1 - \iota a)\aleph \right) \int_0^\kappa \left(\int_0^\upsilon (\upsilon - m)^{\varsigma - 1} v_2(m) dm \right) d\upsilon \\
& - \frac{1}{\gamma\Gamma(\varsigma)} \left(\frac{a\iota^2}{2} + (1 - \iota a)\aleph \right) \int_0^1 (1 - \upsilon)^{\varsigma - 1} v_2(\upsilon) d\upsilon,
\end{aligned}$$

for all $\aleph \in J$. Thus,

$$\begin{aligned}
& |\varrho_1(\aleph) - \varrho_2(\aleph)| \\
& \leq \frac{1}{\Gamma(\varsigma)} \int_0^\aleph (\aleph - \upsilon)^{\varsigma - 1} |v_1(\upsilon) - v_2(\upsilon)| d\upsilon \\
& \quad + \frac{a}{\gamma\Gamma(\varsigma)} \left(\frac{2 - i\kappa^2}{2} + (b\kappa - 1)\aleph \right) \int_0^\iota \left(\int_0^\upsilon (\upsilon - m)^{\varsigma - 1} |v_1(m) - v_2(m)| dm \right) d\upsilon \\
& \quad + \frac{b}{\gamma\Gamma(\varsigma)} \left(\frac{a\iota^2}{2} + (1 - \iota a)\aleph \right) \int_0^\kappa \left(\int_0^\upsilon (\upsilon - m)^{\varsigma - 1} |v_1(m) - v_2(m)| dm \right) d\upsilon \\
& \quad + \frac{1}{\gamma\Gamma(\varsigma)} \left(\frac{a\iota^2}{2} + (1 - \iota a)\aleph \right) \int_0^1 (1 - \upsilon)^{\varsigma - 1} |v_1(\upsilon) - v_2(\upsilon)| d\upsilon,
\end{aligned}$$

for all $\aleph \in J$. Hence,

$$\begin{aligned}
\|\varrho_1 - \varrho_2\| = \sup_{\aleph \in J} |\varrho_1(\aleph) - \varrho_2(\aleph)| \leq & (\frac{2|\gamma|(\varsigma + 1) + (\Lambda_1 + \Lambda_2)}{2|\gamma|\Gamma(\varsigma + 2)}) \times \frac{\|m\|_\infty}{2\sqrt{2}} \\
& \times \left(\frac{2|\gamma|\Gamma(\varsigma + 2)}{(2|\gamma|(\varsigma + 1) + (\Lambda_1 + \Lambda_2))\|m\|_\infty} \right) \times \frac{\phi(\|\Im - \flat\|^2)}{\sqrt{4\|\Im - \flat\|^2 + 1}}.
\end{aligned}$$

Define $\daleth : [0,\infty) \to [0, \frac{1}{4})$ by $\daleth(\varsigma) = \frac{q}{4q+1}$ and let $\upsilon = 2$. Hence,

$$\alpha(\Im,\flat)\phi(8H_d(\Omega_\hbar(\Im),\Omega_\hbar(\flat))) \le 8\alpha(\Im,\flat)\phi(H_d(\Omega_\hbar(\Im),\Omega_\hbar(\flat))) \le \frac{\phi(d(\Im,\flat))^2}{4d(\Im,\flat)+1}$$
$$\le \frac{\phi(d(\Im,\flat))^2}{4\phi(d(\Im,\flat))+1} = \daleth(\phi(d(\Im,\flat)))\phi(d(\Im,\flat)),\ \ \daleth \in \mathcal{B}.$$

Therefore,

$$\alpha(\Im,\flat)\phi(8H_d(\Omega_\hbar(\Im),\Omega_\hbar(\flat))) \le \daleth(\phi(\|\Im-\flat\|))\phi(\|\Im-\flat\|),$$

for all $\Im,\flat \in \mathcal{Y}$, Thus, $\Omega_\hbar$ is an α-ϕ Geraghty contractive multifunction. Choose $\hbar \in \mathcal{Y}$ and $\flat \in \Omega_\hbar(\Im)$ such that $\alpha(\Im,\flat) \ge 1$. Then, $\xi(\Im(\aleph),\flat(\aleph)) \ge 0$ and therefore there exists $z \in \Omega_\hbar(\flat)$ such that $\xi(\flat(\aleph),z(\aleph)) \ge 0$. Hence, $\alpha(\flat,z) \ge 1$ and $\Omega_\hbar$ is α-admissible. Choose $\Im_0 \in \mathcal{Y}$ and $\flat \in \Omega_\hbar(\Im_0)$ such that $\xi(\Im_0(\aleph),\flat(\aleph)) \ge 0$. This implies that $\alpha(\Im_0,\flat) \ge 1$. Now, by Lemma 1.6, there exists $\Im^* \in \mathcal{Y}$ such that $\Im^* \in \Omega_\hbar(\Im^*)$. It is easy to see that $\Im^*$ is a solution of the problem (2.2). □

By the similar proof of Theorem 2.3, the following corollary can be proven.

Corollary 2.4. *Suppose that $\hbar : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that $\hbar$ is integrable and bounded and $\hbar(.,\Im) : J \to P_{cp}(\mathbb{R})$ is measurable for all $\Im \in \mathbb{R}$. Assume that there exist a function $\xi : \mathbb{R}^2 \to \mathbb{R}$, $\phi \in \Phi$ and $m \in C(J,[0,\infty))$ such that*

$$H_d(\hbar(\aleph,\Im),\hbar(\aleph,\flat)) \le \frac{m(\aleph)}{2\sqrt{2}}\frac{\sqrt{\phi(|\Im-\flat|^2)}}{2}\left(\frac{2|\gamma|\Gamma(\varsigma+2)}{(2|\gamma|(\varsigma+1)+(\Lambda_1+\Lambda_2))\|m\|_\infty}\right),$$

for all $\aleph \in J$ and $\Im,\flat \in \mathbb{R}$, where

$$\Lambda_1 = |j|(|2-i\kappa^2|+2|i\kappa-1|)\iota^{\varsigma+1},$$
$$\Lambda_2 = (|j|\iota^2+2|1-\iota j|)(|i|\kappa^{\varsigma+1}+1).$$

If in addition conditions $(i)-(iii)$ in Theorem 2.3 are added to our hypotheses, then the boundary value inclusion (2.2) has a solution.

Example 2.5. Consider the fractional boundary value problem

$$\begin{aligned} &{}^cD^{\frac{3}{2}}\Im(\aleph) \in \hbar(\aleph,\Im(\aleph)),\\ &\Im(0) = \int_0^{\frac{1}{3}}\Im(\upsilon)d\upsilon,\ \Im(1) = \int_0^{\frac{1}{2}}\Im(\upsilon)d\upsilon, \end{aligned} \tag{2.3}$$

where $\aleph \in J$, $\varsigma = \frac{3}{2}$, $\iota = \frac{1}{3}, \kappa = \frac{1}{2}$, $j,i = 1$, ${}^cD^{\frac{3}{2}}$ is the standard Caputo differentiation and define the compact valued multifunction map $\hbar : J \times J \to 2^{\mathbb{R}}$ with

$$\hbar(\aleph,\Im) = \left[0, \frac{\aleph\mid\Im\mid}{200(1+\mid\Im\mid)}\right].$$

Let $\phi(\aleph) = \frac{\aleph}{2}$, $\xi(\Im,\flat) = (\Im\flat)^2$, $m(\aleph) = \frac{\aleph}{200}$ and $\Im_n = \Im + \dfrac{1}{n+1}$. It is obvious that conditions in Corollary 2.4 hold. Hence, the problem (2.3) has at least one solution.

3. Conclusion

This paper intend to provide an affirmative answer to this inquiry by verifying the notion of existence of solutions for fractional differential inclusions by the help of the fixed point technique based on α-$\psi-$Geraghty contractive type mappings. An example is presented as particular case for our proposed theorem. It is proved that the obtained results are consistent with our theoretical findings.

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References

[1] H. Afshari, H.H. Alsulami, E. Karapinar, *On the extended multivalued Geraghty type contractions*, J. Nonlinear Sci. Appl., **9**(2016), 4695-4706.

[2] H. Afshari, H. Aydi, E. Karapinar, *Existence of fixed points of set-valued mappings in b-metric spaces*, East Asian Math. J., **32**(2016), no. 3, 319-332.

[3] H. Afshari, S. Kalantari, D. Baleanu, *Solution of fractional differential equations via $\alpha-\psi$-Geraghty type mappings*, Advances in Difference Equations, (2018), 2018:347.

[4] R.P. Agarwal, B. Ahmad, *Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions*, J. Appl. Math. Comput., **62**(2011), 1200-1214.

[5] R.P. Agarwal, D. Baleanu, V. Hedayati, S. Rezapour, *Two fractional derivative inclusion problems via integral boundary condition*, J. Appl. Math. Comput., **257**(2015), 205-212.

[6] R.P. Agarwal, M. Belmekki, M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Diff. Eq., (2009), Article ID 981728, 47 pages.

[7] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109**(2010), 973-1033.

[8] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2004.

[9] B. Ahmad, J.J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topol. Methods Nonlinear Anal., **35**(2010), 295-304.

[10] B. Ahmad, S.K. Ntouyas, *Boundary value problem for Fractional differential inclusions with four-point integral boundary conditions*, Surveys Math. Appl., **6**(2011), 175-193.

[11] R. Ameen, F. Jarad, T. Abdeljawad, *Ulam stability for delay fractional differential equations with a generalized Caputo derivative*, Filomat, **32**(15)(2018), 5265-5274.

[12] G.A. Anastassiou, *Principles of delta fractional calculus on time scales and inequalities*, Math. Comput. Model., **52**(2010), 556-566.

[13] M. Benchohra, N. Hamidi, *Fractional order differential inclusions on the Half-Lin*, Surveys Math. Appl., **5**(2010), 99-111.

[14] M. Benchohra, S.K. Ntouyas, *On second order differential inclusions with periodic boundary conditions*, Acta Math. Univ. Comenianae, **69**(2000), no. 2, 173-181.

[15] K. Deimling, *Multi-valued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[16] B.D. Dhage, *Fixed point theorems for discontinuous multivalued operators on ordered spaces with applications*, Computer Math. Appl., **51**(2006), 589-604.

[17] A.M.A. El-Sayed, A.G. Ibrahim, *Multivalued fractional differential equations*, Appl. Math. Comput., **68**(1995), 15-25.

[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier Science, 2006.

[19] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, 1991.

[20] X. Liu, Z., Liu, *Existence result for fractional differential inclusions with multivalued term depending on lower-order derivative*, Abstract and Applied Analysis, (2012), Article ID 423796, 24 pages.

[21] H.R. Marasi, H. Afshari, M. Daneshbastam, C.B. Zhai, *Fixed points of mixed monotone operators for existence and uniqueness of nonlinear fractional differential equations*, J. Contemporary Mathematical Analysis, **52**(2017), p. 8C13.

[22] A. Ouahab, *Some results for fractional boundary value problem of differential inclusions*, Nonlinear Analysis, **69**(2008), 3877-3896.

[23] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[24] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Analysis, **75**(2012), 2154-2165.

[25] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[26] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Letters, **22**(2009), 64-69.

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