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FIXED POINT AND NONLINEAR ERGODIC THEOREMS FOR SOME NEW TYPE SEMIGROUPS OF MAPPINGS

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Abstract. In the present paper, some new type of semigroups of mappings are introduced. Then, by using the theory of invariant means, fixed point theorem, and existence of nonspreading retraction for these semigroup are deduced and the illustrative examples are given. Also, weak convergence theorem of Mann's type, generalized Baillon's nonlinear ergodic theorem and strong convergence theorem of Halpern's type for such semigroups in the Hilbert space are considered. At the end, some applications are investigated.
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1. Introduction

Let $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ be the set of positive integer numbers, the set of rational numbers, and the set of real numbers, respectively. Also let H be a real Hilbert space with inner product $\langle .,. \rangle$ and $\|.\|$ be a norm from inner product $\langle .,. \rangle$ and C be a nonempty subset of H. The closed convex hull of C is denoted by $\overline{co}C$. Furthermore, the weak convergence is denoted by $\rightharpoonup$ and strong convergence is denoted by $\to$. Let T be a mapping of C into itself, and the set of fixed point of T, i.e., $\{x \in C : Tx = x\}$ is denoted by $F(T)$. A mapping $T\colon C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$. A self mapping T on C with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$, for all $x \in F(T)$, and $y \in C$. Now, the definitions of several classes of nonlinear mappings are recalled:

Definition 1.1. A mapping $T\colon C \to C$ is said to be:

(1) $r-$firmly nonexpansive, if there exist $r \in [0,1)$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|(1-r)(x-y) + r(Tx - Ty)\|.$$

(2) Nonspreading, if for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2.$$

(3) Hybrid, if for all $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2 + \|x - y\|^2.$$

(4) TJ-1, if for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - y\|^2.$$

(5) TJ-2, if for all $x, y \in C$,

$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|x - Ty\|^2.$$

Recently, Aoyama and Kohsaka [1] have introduced a new class of nonexpansive mappings, namely $\alpha-$nonexpansive mappings as follows:

Definition 1.2. A mapping $T\colon C \to C$ is said to be an $\alpha-$nonexpansive if for all $x, y \in C$ and $\alpha \in (-\infty, 1)$

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

They have obtained the fixed point theorem for the introduced mapings in the Definition 1. It is obvious that each $\alpha-$nonexpansive mapping, which has a fixed point is quasi-nonexpansive.

Remark 1.3. In [2], Ariza-Ruiz et al. showed that the concept of $\alpha-$nonexpansive mapping is trivial for $\alpha < 0$.

Remark 1.4. [2] Every nonexpansive mapping is 0-nonexpansive, and every nonspreading mapping is $\frac{1}{2}$-nonexpansive. Every hybrid mapping is $\frac{1}{3}$-nonexpansive, and every TJ-1 mapping is $\frac{1}{4}$-nonexpansive. Every TJ-2 mapping is nonspreading and hence is $\frac{1}{2}$-nonexpansive. Finally, every r-firmly nonexpansive mapping is α-nonexpansive with $\alpha = \frac{r}{1+r}$.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a.s$ and $s \mapsto s.a$ from S to S are continuous. A family $\mathcal{S} = \{T_t : t \in S\}$ of mappings of C into itself is called a continuous representation of S as mappings on C if the following propertis hold:

(1) $T_{ts}(x) = T_t T_s(x)$, $\forall s, t \in S$ and $x \in C$;
(2) for each $x \in C$, the mapping $t \to T_t x$ is continuous.

Let $\mathcal{S}$ be as above, $\mathcal{S}$ is said to be a nonexpansive semigroup on C if

$$\|T_t(x) - T_t(y)\| \leq \|x - y\|,$$

for all $x, y \in C$ and $t \in S$.

In this paper, we first introduce some new type semigroups of these mappings. Then motivated by [7] and [15], we prove a fixed point theorem for semigroups. After that, we show the existence of nonspreading retraction for the fixed point set of these semigroups. Morever, motivated and inspired by [10], [4], [14], [17], and [15], we prove weak convergence theorem of Mann's type and generalized nonlinear ergodic theorem for the introduced semigroups in Hilbert spaces. Finally, we deduce strong convergence theorem of Halpern's type for these semigroups. The presented results in this paper generalize and improve several results of the topics in the literature.

2. Preliminaries

Recall that the (nearest point) projection P from H into C assigning to $x \in H$ is the unique point $Px \in C$ which satisfy the following property

$$\| x - Px \| = \inf_{y \in C} \| x - y \| .$$

If $x \in H$ and $z \in C$, then $z = Px$ is equvalent to

$$\langle x - z, y - z \rangle \geq 0, \quad y \in C. \tag{2.1}$$

For more details we refer readers to [6] and [15].

Lemma 2.1. [10] *If $x, y, z, w \in H$ and $\alpha \in \mathbb{R}$, then*

(i) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$;
(ii) $2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$.

Lemma 2.2. [16] *Let D be a nonempty closed convex subset of H. Let P be a metric projection of H onto D and $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$, for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

Let l^∞ be the Banach space of bounded real number sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞), the value of μ at $f = (x_1, x_2, x_3, ...) \in l^\infty$ is denote by $\mu(f)$. Sometimes, we denote it by $\mu_n(x_n)$. $\mu \in (l^\infty)^*$ is called Banach limit on l^∞ if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, ...)$ and $\mu_n(x_{n+1}) = \mu_n(x_n)$. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, ...) \in l^\infty$, we have

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in l^\infty$ and $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then we can deduce $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other properties, see [15].

Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of C(S)). The value of μ at $f \in C(S)$ is denote by $\mu(f)$. Sometimes, we denote it by $\mu_t(f(t))$. For each $s \in S$ and $f \in C(S)$, we define two functions $l_s f$ and $r_s f$ as follows:

$$(l_s f)(t) = f(st) \quad \textit{and} \quad (r_s f)(t) = f(ts), \quad t \in S.$$

An element μ of $C(S)^*$ is called a mean on C(S) if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on C(S) if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on C(S) is called left invariant if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$, and $s \in S$. Similarly, a mean μ on C(S) is called right invariant if $\mu(r_s f) = \mu(f)$, for all $f \in C(S)$, and $s \in S$. A left and right invariant mean on C(S) is called an invariant mean on C(S). A net $\{\mu_\alpha\}$ of means on C(S) is said to be asymptotically invariant if for each $f \in C(S)$ and $s \in S$,

$$\lim_\alpha(\mu_\alpha(f) - \mu_\alpha(l_s f)) = 0 \quad \textit{and} \quad \lim_\alpha(\mu_\alpha(f) - \mu_\alpha(r_s f)) = 0.$$

A net $\{\mu_\alpha\}$ of means on C(S) is said to be strongly asymptotically invariant if for each $s \in S$

$$\|l_s^*\mu_\alpha - \mu_\alpha\| \to 0 \quad and \quad \|r_s^*\mu_\alpha - \mu_\alpha\| \to 0,$$

where l_s^* and r_s^* are the adjoint operators of l_s and r_s, respectively (For more details see [15]). We know that for a commutative semitopological semigroup, there exists an invariant mean on C(S) (see [15]). If $S = \mathrm{N}$, an invariant mean on C(S)=B(S) is a Banach limit on l^∞. Let C be a closed convex subset of H and let $U : S \to C$ be a continuous function such that $\sup_{s\in S}\|U(s)\| < +\infty$. For any $y \in H$, a real valued function h defined by

$$h(t) = \langle U(t), y\rangle,$$

for all $t \in S$, is in $C(S)$. Let μ be a mean on $C(S)$, and

$$g(y) = \mu(h) = \mu_t\langle U(t), y\rangle, \quad \forall y \in H.$$

Then, g is a linear functional on H and

$$|g(y)| = |\mu(h)| \le \|\mu\|\|h\| = \sup_{t\in S}|h(t)| = \sup_{t\in S}\langle U(t), y\rangle \le \sup_{t\in S}\|U(t)\|\|y\|.$$

Hence, from Riesz theorem, there is a unique element $x_0 \in H$ such that for all $y \in H$, we have $g(y) = \langle x_0, y\rangle$ or $\mu_t\langle U(t), y\rangle = \langle x_0, y\rangle$.

Theorem 2.3. [15] *If μ is a mean on $C(S)$ and x_0 is an element of H such that for all $y \in H$,*

$$\mu_t\langle U(t), y\rangle = \langle x_0, y\rangle.$$

Then, $x_0 \in \overline{co}\{U(t) : t \in S\} \subset C$.

In particular, if $\mathcal{S} = \{T_t : t \in S\}$ is a continuous representation of S as mappings on C such that $\{T_t : t \in S\}$ is bounded for some $x \in C$ and $U(t) = T_t x$ for all $t \in S$, then there exist a unique element $x_0 \in \overline{co}\{T_t x : t \in S\} \subset H$ such that

$$\mu_t\langle T_t x, y\rangle = \langle x_0, y\rangle, \quad \forall y \in H.$$

We denote such x_0 by $T_\mu x$.

The following theorem will be used in the Section 3.

Theorem 2.4.[17]. *Let S, $C(S)$ and H be as above. Let $U : S \to H$ be a function such that $\{U(s) : s \in S\} \subset C$ is bounded and let μ be a mean on $C(S)$. If $g : C \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s\|U(s) - z\|^2, \quad \forall z \in C.$$

Then there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Theorem 2.5.[15] *Let C be a closed convex subset of H and let S be a semitopological semigroup such that $C(S)$ has a left invariant mean. If $\mathcal{S} = \{T_t : t \in S\}$ is nonexpansive semigroup on C. Then, the following statements are equivalent:*

(i) *$\{T_t x : t \in S\}$ is bounded for some $x \in C$,*
(ii) *$\{T_t x : t \in S\}$ is bounded for every $x \in C$,*
(iii) *$F(\mathcal{S}) \neq \emptyset$.*

Theorem 2.6.[15] *Let C be a closed convex subset of H and let S be a semitopological semigroup such that $C(S)$ has an invariant mean μ. If $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, T_μ satisfies the following properties:*

(i) $T_\mu T_t = T_t T_\mu = T_\mu$, *for all $t \in S$,*

(ii) T_μ *is a nonexpansive retraction of C onto $F(\mathcal{S})$, i.e., for all $x, y \in C$,*

$$\|T_\mu(x) - T_\mu(y)\| \leq \|x - y\|,$$

(iii) $T_\mu(x) \in \overline{co}\{T_t x : t \in S\}$, *for all $x \in C$.*

The following theorem generalize nonlinear ergodic Theorem of Baillon's type for nonexpansive semigroups in Hilbert spaces.

Theorem 2.7.[15]*Let C be a closed convex subset of H and let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C and $F(\mathcal{S}) \neq \emptyset$. If $\{\mu_\alpha\}_{\alpha \in I}$ is a net of asymptotically invariant means on $C(S)$, then $\{T_{\mu_\alpha} x\}$ converges weakly to a point $x_0 \in F(\mathcal{S})$, for all $x \in C$. If, we put $Qx = x_0$ for all $x \in C$, then Q is a nonexpansive retraction of C onto $F(\mathcal{S})$ such that $QT_t = T_t Q = Q$ for all $t \in S$ and $\bigcap_t \overline{co}\{T_{st} x : s \in S\} \bigcap F(\mathcal{S}) = \{Qx\}$, for all $x \in C$.*

3. Existence of nonspreading retraction

Definition 3.1. Let C be a nonempty subset of H and S be a semitopological semigroup. Let $\mathcal{F} = \{f_t\}_{t \in S}$ be a net of mappings of C into itself and $\alpha = \{\alpha_t\}_{t \in S}$ be a net of real numbers in $[0, 1)$. A continuous representation $\mathcal{S} = \{T_t : t \in S\}$ of S as mappings on C is called an $(\alpha, \mathcal{F})$-semigroup on C if

$$\|T_t(x) - f_t(y)\|^2 \leq \alpha_t \|T_t(x) - y\|^2 + \alpha_t \|x - f_t(y)\|^2 + (1 - 2\alpha_t)\|x - y\|^2,$$

for all $x, y \in C$ and $t \in S$.

Remark 3.2. Notice that by the first condition of continuous representation, $(\alpha, \mathcal{F})$-semigroup is closed in the compounds of the mapping.

The set of common fixed points of $\mathcal{S}$ and the set of common fixed points of $\mathcal{F}$ are denoted by $F(\mathcal{S})$ and $F(\mathcal{F})$, respectively, i.e.,

$$F(\mathcal{S}) = \bigcap_{t \in S} F(T_t), \quad F(\mathcal{F}) = \bigcap_{t \in S} F(f_t).$$

Lemma 3.3. *Let C be a closed convex subset of H and let S be a semitopological semigroup. If $\mathcal{S} = \{T_t : t \in S\}$ is an $(\alpha, \mathcal{F})$-semigroup on C, then $F(\mathcal{F}) = F(\mathcal{S})$, and $F(\mathcal{S})$ is closed and convex.*

Proof. If $F(\mathcal{F}) = \emptyset$ and $F(\mathcal{S}) = \emptyset$, then $F(\mathcal{F}) = F(\mathcal{S})$. So, we assume that $F(\mathcal{F}) \neq \emptyset$ or $F(\mathcal{S}) \neq \emptyset$. Now, let $x \in F(\mathcal{F})$ and $t \in S$, then $f_t(x) = x$. Since $\mathcal{S}$ is an $(\alpha, \mathcal{F})-$semigroup on C, we can get

$$\begin{aligned} \|T_t(x) - x\|^2 &= \|T_t(x) - f_t(x)\|^2 \\ &\leq \alpha_t \|T_t(x) - x\|^2 + \alpha_t \|x - f_t(x)\|^2 + (1 - 2\alpha_t)\|x - x\|^2 \\ &= \alpha_t \|T_t(x) - x\|^2. \end{aligned} \tag{3.1}$$

Since $\alpha_t \in [0,1)$, from (3.1) we get $T_t(x) = x$, and then $x \in F(\mathcal{S})$. Therefore, $F(\mathcal{F}) \subset F(\mathcal{S})$. Again let $y \in F(\mathcal{S})$, and $t \in S$, then $T_t(y) = y$. Also, since $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C, we can obtain

$$\begin{aligned}\|y - f_t(y)\|^2 &= \|T_t(y) - f_t(y)\|^2 \\ &\leq \alpha_t\|T_t(y) - y\|^2 + \alpha_t\|y - f_t(y)\|^2 + (1-2\alpha_t)\|y - y\|^2 \\ &= \alpha_t\|y - f_t(y)\|^2. \end{aligned} \tag{3.2}$$

On the other hand, since $\alpha_t \in [0,1)$, from (3.2), we get $f_t(y) = y$, and $y \in F(\mathcal{F})$. Therefore $F(\mathcal{S}) \subset F(\mathcal{F})$. Now, we show $F(\mathcal{S})$ is closed. Let $\{x_n\} \subset F(\mathcal{S})$ and $x_n \to x^*$, then $x^* \in C$. For $t \in S$, we have

$$\begin{aligned}\|x_n - f_t(x^*)\|^2 &= \|T_t(x_n) - f_t(x^*)\|^2 \\ &\leq \alpha_t\|T_t(x_n) - x^*\|^2 + \alpha_t\|x_n - f_t(x^*)\|^2 + (1-2\alpha_t)\|x_n - x^*\|^2 \\ &= (1-\alpha_t)\|x_n - x^*\|^2 + \alpha_t\|x_n - f_t(x^*)\|^2. \end{aligned}$$

It follows that

$$\|x_n - f_t(x^*)\|^2 \leq \|x_n - x^*\|^2 \to 0.$$

Thus, $x^* \in F(\mathcal{F}) = F(\mathcal{S})$, and it illustrates $F(\mathcal{S})$ is closed. Finally, let $0 \leq \alpha \leq 1$, $u, v \in F(\mathcal{S})$ and $z = \alpha u + (1-\alpha)v$. Since $F(\mathcal{S}) = F(\mathcal{F})$, for all $t \in S$, from Lemma 2, we have

$$\begin{aligned}\|z - T_t z\|^2 &= \|\alpha u + (1-\alpha)v - T_t(z)\|^2 \\ &= \alpha\|u - T_t(z)\|^2 + (1-\alpha)\|v - T_t(z)\|^2 - \alpha(1-\alpha)\|u - v\|^2 \\ &= \alpha\|f_t(u) - T_t(z)\|^2 + (1-\alpha)\|f_t(v) - T_t z\|^2 - \alpha(1-\alpha)\|u - v\|^2 \\ &\leq \alpha[\alpha_t\|f_t(u) - z\|^2 + \alpha_t\|u - T_t(z)\|^2 + (1-2\alpha_t)\|u - z\|^2] \\ &\quad + (1-\alpha)[\alpha_t\|f_t(v) - z\|^2 + \alpha_t\|v - T_t(z)\|^2 + (1-2\alpha_t)\|v - z\|^2] \\ &\quad - \alpha(1-\alpha)\|u - v\|^2 \\ &\leq \alpha[\alpha_t\|u - z\|^2 + \alpha_t\|u - z\|^2 + (1-2\alpha_t)\|u - z\|^2] \\ &\quad + (1-\alpha)[\alpha_t\|v - z\|^2 + \alpha_t\|v - z\|^2 + (1-2\alpha_t)\|v - z\|^2] \\ &\quad - \alpha(1-\alpha)\|u - v\|^2 \\ &= \alpha\|u - z\|^2 + (1-\alpha)\|v - z\|^2 - \alpha(1-\alpha)\|u - v\|^2 \\ &= \alpha(1-\alpha)^2\|u - v\|^2 + (1-\alpha)\alpha^2\|u - v\|^2 - \alpha(1-\alpha)\|u - v\|^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|u - v\|^2 \\ &= 0, \end{aligned}$$

This implies $z \in F(\mathcal{S})$. So, we conclude that $F(\mathcal{S})$ is convex.

Theorem 3.4. *Let C be a closed convex subset of H and S be a semitopological semigroup such that $C(S)$ has a left invariant mean. If $\mathcal{S} = \{T_t : t \in S\}$ is an $(\alpha, \mathcal{F})$-semigroup on C, then, the following statements are equivalent:*

(i) *$\{T_t x : t \in S\}$ is bounded for some $x \in C$,*
(ii) *$\{T_t x : t \in S\}$ is bounded for every $x \in C$,*

(iii) $F(\mathcal{S}) \neq \emptyset$.

Proof. $(i) \Rightarrow (iii)$. Let μ be a left invariant mean on $C(S)$, $x \in C$ and $\{T_t x : t \in S\}$ be a bounded. By Theorem 2, there exists a unique member $x_0 \in C$ such that

$$\mu_t \|T_t(x) - x_0\|^2 = \min_{z \in C} \mu_t \|T_t(x) - z\|^2. \tag{3.3}$$

Since $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C, for all $s \in S$, we have

$$\begin{aligned} \|T_{st}(x) - f_s(x_0)\|^2 &= \|T_s T_t(x) - f_s(x_0)\|^2 \\ &\leq \alpha_s \|T_s T_t(x) - x_0\|^2 + \alpha_s \|T_t(x) - f_s(x_0)\|^2 \\ &\quad + (1 - 2\alpha_s)\|T_t(x) - x_0\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \mu_t \|T_{st}(x) - f_s(x_0)\|^2 \leq \mu_t(\alpha_s \|T_s T_t(x) - x_0\|^2) + \mu_t(\alpha_s \|T_t(x) - f_s(x_0)\|^2) \\ + \mu_t((1 - 2\alpha_s)\|T_t(x) - x_0\|^2). \end{aligned}$$

Since μ is left invariant, we have

$$\begin{aligned} \mu_t \|T_t(x) - f_s(x_0)\|^2 \leq \alpha_s \mu_t \|T_t(x) - x_0\|^2 + \alpha_s \mu_t \|T_t(x) - f_s(x_0)\|^2) \\ + (1 - 2\alpha_s)\mu_t \|T_t(x) - x_0\|^2. \end{aligned}$$

It follows that

$$\mu_t \|T_t(x) - f_s(x_0)\|^2 \leq \mu_t \|T_t(x) - x_0\|^2. \tag{3.4}$$

Since $f_s(x_0) \in C$, from (3.3) we can get

$$\mu_t \|T_t(x) - x_0\|^2 \leq \mu_t \|T_t(x) - f_s(x_0)\|^2. \tag{3.5}$$

Now, from (3.4) and (3.5) we obtain

$$\mu_t \|T_t(x) - f_s(x_0)\|^2 = \mu_t \|T_t(x) - x_0\|^2.$$

The uniqueness of x_0 in (3.3) implies that $f_s(x_0) = x_0$. Hence, $x_0 \in F(f_s)$, and we conclude that $x_0 \in F(\mathcal{F})$. So, by Lemma 3, we have $x_0 \in F(\mathcal{S})$.

$(iii) \Rightarrow (ii)$. Let $x^* \in F(\mathcal{S})$. By Lemma 3, $x^* \in F(\mathcal{F})$. So, we have

$$\begin{aligned} \|T_s(x) - x^*\|^2 &= \|T_s(x) - f_s(x^*)\|^2 \\ &\leq \alpha_s \|T_s(x) - x^*\|^2 + \alpha_s \|x - f_s(x^*)\|^2 + (1 - 2\alpha_s)\|x - x^*\|^2 \\ &= \alpha_s \|T_s(x) - x^*\|^2 + \alpha_s \|x - x^*\|^2 + (1 - 2\alpha_s)\|x - x^*\|^2, \end{aligned}$$

for all $x \in C$ and $s \in S$. It follows that $\|T_s(x) - x^*\|^2 \leq \|x - x^*\|^2$. Hence $\{T_s x : s \in S\}$ is bounded for every $x \in C$. $(ii) \Rightarrow (i)$. It is obvious.

Example 3.5. Let $S = (0, \frac{1}{2}]$, $C = [-1, 1]$, $H = \mathbb{R}$, and for each $t \in S$, we set $\alpha_t = \frac{1}{2}t$. Let T_t and f_t are defined as follows:

$$T_t : C \longrightarrow C, \qquad f_t : C \longrightarrow C,$$

$$T_t x = \begin{cases} tx, & 0 < x < 1, \\ 0, & x \in \{-1, 0, 1\}, \\ -tx, & -1 < x < 0, \end{cases} \qquad f_t x = \begin{cases} tx, & 0 < x < 1, \\ 0, & x \in [-1, 0] \bigcup \{1\}. \end{cases}$$

Let $\alpha = \{\alpha_t\}_{t \in S}$ and $\mathcal{S} = \{T_t : t \in S\}$. Obviously, S is a semitopological semigroup and $\mathcal{S} = \{T_t : t \in S\}$ is a continuous representation of S as mappings on C. Since T_t is not continuous for all $t \in S$, hence $\mathcal{S}$ is not a nonexpansive semigroup. Now, we prove that $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C. For $t \in S$ and $x, y \in C$, we consider the following cases:

- If $0 < x < 1$ and $0 < y < 1$, then

$$\begin{aligned} |T_t x - f_t y|^2 &= t^2 |x - y|^2 \\ &\le (1 - t)^2 |x - y|^2 \\ &\le (1 - t)|x - y|^2 \\ &\le \frac{1}{2} t |tx - y|^2 + \frac{1}{2} t |x - ty|^2 + (1 - t)|x - y|^2 \\ &= \alpha_t |T_t x - y|^2 + \alpha_t |x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2. \end{aligned}$$

- If $0 < x < 1$ and $y \in [-1, 0] \bigcup \{1\}$, then

$$\begin{aligned} |T_t x - f_t y|^2 = t^2 |x|^2 &\le \frac{1}{2} t |x|^2 \\ &\le \frac{1}{2} t |tx - y|^2 + \frac{1}{2} t |x - 0|^2 + (1 - t)|x - y|^2 \\ &= \alpha_t |T_t x - y|^2 + \alpha_t |x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2. \end{aligned}$$

- If $-1 < x < 0$ and $0 < y < 1$, then

$$\begin{aligned} |T_t x - f_t y|^2 = t^2 (x^2 + y^2 + 2xy) &\le t^2 (x^2 + y^2) \\ &\le \frac{1}{2} t (x^2 + y^2) \\ &\le \frac{1}{2} t (x^2 + y^2) + \frac{1}{2} t^3 (x^2 + y^2) + (1 - t)|x - y|^2 \\ &= \frac{1}{2} t |-tx - y|^2 + \frac{1}{2} t |x - ty|^2 + (1 - t)|x - y|^2 \\ &= \alpha_t |T_t x - y|^2 + \alpha_t |x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2. \end{aligned}$$

- If $-1 < x < 0$ and $y \in [-1, 0] \bigcup \{1\}$, then

$$\begin{aligned} |T_t x - f_t y|^2 = t^2 |-x|^2 &\le \frac{1}{2} t |-tx - y|^2 + \frac{1}{2} t |x - 0|^2 + (1 - t)|x - y|^2 \\ &= \alpha_t |T_t x - y|^2 + \alpha_t |x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2 \end{aligned}$$

- If $x \in \{-1, 0, 1\}$ and $0 < y < 1$, then

$$|T_t x - f_t y|^2 = t^2|y|^2 \leq \frac{1}{2}t|0 - y|^2 + \frac{1}{2}t|x - ty|^2 + (1-t)|x-y|^2$$
$$= \alpha_t|T_t x - y|^2 + \alpha_t|x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2$$

- If $x \in \{-1, 0, 1\}$ and $y \in [-1, 0] \bigcup \{1\}$, then

$$|T_t x - f_t y|^2 = 0 \leq \alpha_t|T_t x - y|^2 + \alpha_t|x - f_t y|^2 + (1 - 2\alpha_t)|x - y|^2$$

Therefore, $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C. Obviously $\{T_t x : t \in S\}$ is bounded for every $x \in C$ and $F(\mathcal{F}) = F(\mathcal{S}) = \{0\}$.

Remark 3.6. In the proof $(i) \Rightarrow (iii)$ in Theorem 3, we put $x_0 = T_\mu(x)$.

Theorem 3.7. *Let C be a closed convex subset of a Hilbert space H and let S be a semitopological semigroup such that $C(S)$ has an invariant mean μ.*

If $\mathcal{S} = \{T_t : t \in S\}$ is an $(\alpha, \mathcal{F})$-semigroup on C and $F(\mathcal{S}) \neq \emptyset$, then, T_μ satisfies the following properties:

(i) $T_\mu T_t = T_t T_\mu = T_\mu$, *for all* $t \in S$,

(ii) T_μ *is a nonspreading retraction of C onto $F(\mathcal{S})$, i.e., for all $x, y \in C$,*

$$2\|T_\mu(x) - T_\mu(y)\|^2 \leq \|T_\mu(x) - y\|^2 + \|T_\mu(y) - x\|^2, \quad \textit{and} \quad T_\mu^2 = T_\mu,$$

(iii) $\bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\} \bigcap F(\mathcal{S}) = \{T_\mu(x)\}$, *for all* $x \in C$.

Proof. (i). By the proof of Theorem 3, it is obvious that T_μ is a mapping of C onto $F(\mathcal{S})$, so

$$T_t T_\mu = T_\mu, \quad \forall t \in S.$$

Since μ is a right invariant mean, for all $s \in S$ and $x \in C$, we have

$$\langle T_\mu T_s(x), y\rangle = \mu_t\langle T_t T_s(x), y\rangle = \mu_t\langle T_{ts}(x), y\rangle = \mu_t\langle T_t(x), y\rangle = \langle T_\mu(x), y\rangle,$$

for all $y \in H$. Hence, $T_\mu T_s x = T_\mu x$, for all $s \in S$ and $x \in C$.

(ii). Let $x, y \in C$. From Lemma 2, we can get

$$\begin{aligned}
\|T_\mu(x) - T_\mu(y)\|^2 &= \langle T_\mu(x) - T_\mu(y), T_\mu(x) - T_\mu(y)\rangle \\
&= \mu_t\langle T_t(x) - T_t(y), T_\mu(x) - T_\mu(y)\rangle \\
&= \frac{1}{2}\mu_t(\|T_t(x) - T_\mu(y)\|^2 + \|T_t(y) - T_\mu(x)\|^2 \\
&\quad - \|T_t(x) - T_\mu(x)\|^2 - \|T_t(y) - T_\mu(y)\|^2) \\
&\leq \frac{1}{2}\mu_t(\|T_t(x) - T_\mu(y)\|^2 + \|T_t(y) - T_\mu(x)\|^2). \qquad (3.6)
\end{aligned}$$

Since $T_\mu(x) \in F(\mathcal{S})$ and $T_\mu(y) \in F(\mathcal{S})$. By Lemma 3, we have $T_\mu(x) \in F(\mathcal{F})$ and $T_\mu(y) \in F(\mathcal{F})$. Therefore, for $t \in S$, we have

$$f_t(T_\mu(x)) = T_\mu(x), \ \textit{and} \ f_t(T_\mu(y)) = T_\mu(y).$$

Since $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C, so we have

$$\begin{aligned}\|T_t(x) - T_\mu(y)\|^2 &= \|T_t(x) - f_t(T_\mu(y))\|^2\\ &\le \alpha_t\|T_t(x) - T_\mu(y)\|^2 + \alpha_t\|x - f_t(T_\mu(y))\|^2\\ &\quad + (1-2\alpha_t)\|x - T_\mu(y)\|^2\\ &= \alpha_t\|T_t(x) - T_\mu(y)\|^2 + \alpha_t\|x - T_\mu(y)\|^2\\ &\quad + (1-2\alpha_t)\|x - T_\mu(y)\|^2.\end{aligned}$$

It follows that

$$\mu_t(\|T_t(x) - T_\mu(y)\|^2 \le \|x - T_\mu(y)\|^2. \tag{3.7}$$

By using similar method as used in the proof of relation (3.7), we can prove that

$$\mu_t(\|T_t(y) - T_\mu(x)\|^2 \le \|y - T_\mu(x)\|^2. \tag{3.8}$$

Now, from (3.6), (3.7) and (3.8), we obtain that

$$2\|T_\mu(x) - T_\mu(y)\|^2 \le \|T_\mu(x) - y\|^2 + \|T_\mu(y) - x\|^2.$$

Next, we will show $T_\mu^2 = T_\mu$. For this purpose , for $x \in C$ and $y \in H$, from (i) we have

$$\langle T_\mu^2 x, y\rangle = \mu_t\langle T_t T_\mu x, y\rangle = \mu_t\langle T_\mu x, y\rangle = \langle T_\mu x, y\rangle.$$

Hence, $T_\mu^2 = T_\mu$.

(iii). By Theorem 2, we have

$$T_\mu(x) \in \overline{co}\{T_t(x) : t \in S\}, \quad \forall x \in C.$$

So by using (i), we get

$$T_\mu(x) = T_s T_\mu(x) = T_\mu T_s(x) \in \overline{co}\{T_{ts}(x) : t \in S\}, \quad \forall s \in S.$$

By Theorem 3, $T_\mu(x) \in F(\mathcal{S})$. Now, it is sufficient to show that $T_\mu(x)$ is the only unique member in

$$\bigcap_{s\in S} \overline{co}\{T_{ts}(x) : t \in S\} \bigcap F(\mathcal{S}).$$

Assume that

$$z_1 \in \bigcap_{s\in S} \overline{co}\{T_{ts}(x) : t \in S\} \bigcap F(\mathcal{S}).$$

We define a function $g : F(\mathcal{S}) \to \mathbb{R}$ as follows:

$$g(z) = \mu_s\|T_s(x) - z\|^2, \quad \forall z \in F(\mathcal{S}).$$

Since $F(\mathcal{S})$ is closed and convex, by Theorem 2, there exists a unique $z_0 \in F(\mathcal{S})$ such that

$$g(z_0) = \min\{g(z) : z \in F(\mathcal{S})\},$$

and

$$\mu_s\langle T_s(x), y\rangle = \langle z_0, y\rangle, \tag{3.9}$$

for all $s \in S$ and $y \in H$. From Lemma 2, for all $s \in S$, we have

$$2\langle z_1 - z_0, T_s(x) - z_1\rangle = \|T_s(x) - z_0\|^2 - \|z_1 - z_0\|^2 - \|T_s(x) - z_1\|^2.$$

It follows that

$$\begin{aligned}\|z_1 - z_0\|^2 &= \mu_s\|T_s(x) - z_0\|^2 - \mu_s\|T_s(x) - z_1\|^2 - 2\mu_s\langle z_1 - z_0, T_s(x) - z_1\rangle\\ &\le \mu_s\|T_s(x) - z_1\|^2 - \mu_s\|T_s(x) - z_1\|^2 - 2\mu_s\langle z_1 - z_0, T_s(x) - z_1\rangle\\ &= -2\mu_s\langle z_1 - z_0, T_s(x) - z_1\rangle\\ &= -2\mu_s\langle z_1, T_s(x)\rangle + 2\langle z_1, z_1\rangle + 2\mu_s\langle z_0, T_s(x)\rangle - 2\langle z_0, z_1\rangle.\end{aligned}$$

So, from (3.9) we get

$$\begin{aligned}\|z_1 - z_0\|^2 &\le -2\mu_s\langle z_1, T_s(x)\rangle + 2\langle z_1, z_1\rangle + 2\mu_s\langle z_0, T_s(x)\rangle - 2\langle z_0, z_1\rangle\\ &= -2\langle z_1, z_0\rangle + 2\langle z_1, z_1\rangle + 2\langle z_0, z_0\rangle - 2\langle z_0, z_1\rangle\\ &= -2\langle z_1 - z_0, z_0 - z_1\rangle.\end{aligned}$$

Hence, $z_1 = z_0$. Therefore

$$\bigcap_s \overline{co}\{T_{ts}x : t \in S\} \bigcap F(\mathcal{S}) = \{T_\mu(x)\}.$$

Definition 3.8. Let C be a nonempty subset of H and let S be a semitopological semigroup. Let $\alpha = \{\alpha_s\}_{s\in S}$ be a net of real numbers in $[0, 1)$ and $\mathcal{S} = \{T_t : t \in S\}$ be a continuous representation of S as mappings on C. Then, $\mathcal{S}$ is called an α-nonexpansive semigroup on C if

$$\|T_s(x) - T_s(y)\|^2 \le \alpha_s\|T_s(x) - y\|^2 + \alpha_s\|x - T_s(y)\|^2 + (1 - 2\alpha_s)\|x - y\|^2,$$

for all $x, y \in C$ and $s \in S$.

Theorem 3.9. *Let C be a closed convex subset of H and let S be a semitopological semigroup such that $C(S)$ has a left invariant mean. If $\mathcal{S} = \{T_t : t \in S\}$ is an α-nonexpansive semigroup on C. Then, the following statement are equivalent:*

(i) *$\{T_t x : t \in S\}$ is bounded for some $x \in C$,*
(ii) *$\{T_t x : t \in S\}$ is bounded for every $x \in C$,*
(iii) *$F(\mathcal{S}) \neq \emptyset$.*

Proof. By taking $\mathcal{S} = \mathcal{F}$ in Theorem 3 the proof is completed.

Theorem 3.10. *Let C be a closed convex subset of H and S be a semitopological semigroup such that $C(S)$ has an invariant mean μ. If $\mathcal{S} = \{T_t : t \in S\}$ is an α-nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$, then, T_μ satisfies the following properties:*

(i) *$T_\mu T_t = T_t T_\mu = T_\mu$, for all $t \in S$,*
(ii) *T_μ is a nonspreading retraction of C onto $F(\mathcal{S})$, i.e., for all $x, y \in C$,*

$$2\|T_\mu(x) - T_\mu(y)\|^2 \le \|T_\mu(x) - y\|^2 + \|T_\mu(y) - x\|^2, \quad \text{and} \quad T_\mu^2 = T_\mu,$$

(iii) *$\bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\} \bigcap F(\mathcal{S}) = \{T_\mu(x)\}$, for all $x \in C$.*

Proof. By taking $\mathcal{S} = \mathcal{F}$ in Theorem 3 the proof is completed.

Definition 3.11. Let C be a nonempty subset of H. Let S be a semitopological semigroup and $\mathcal{S}$ be a continuous representation of S as mappings on C. Then, $\mathcal{S}$ is called:

(1) r-firmly nonexpansive semigroup on C if there exist a net $r = \{r_s\}_{s\in S}$ of real numbers in $[0, 1)$ such that for all $x, y \in C$ and $s \in S$,

$$\|T_s(x) - T_s(y)\| \le \|(1 - r_s)(x - y) + r_s(T_s(x) - T_s(y))\|.$$

(2) Nonspreading semigroup on C if for all $x, y \in C$ and $s \in S$,

$$2\|T_s(x) - T_s(y)\|^2 \le \|T_s(x) - y\|^2 + \|x - T_s(y)\|^2.$$

(3) Hybrid semigroup on C if for all $x, y \in C$ and $s \in S$,

$$3\|T_s(x) - T_s(y)\|^2 \le \|T_s(x) - y\|^2 + \|x - T_s(y)\|^2 + \|x - y\|^2.$$

(4) TJ-1-semigroup on C if for all $x, y \in C$ and $s \in S$,

$$2\|T_s(x) - T_s(y)\|^2 \le \|T_s(x) - y\|^2 + \|x - y\|^2.$$

(5) TJ-2-semigroup on C if for all $x, y \in C$ and $s \in S$,

$$3\|T_s(x) - T_s(y)\|^2 \le 2\|T_s(x) - y\|^2 + \|x - T_s(y)\|^2.$$

Remark 3.12. By Remark 1, obviously, Theorem 3 and Theorem 3 are also true for nonexpansive semigroup, r-firmly nonexpansive semigroup, nonspreading semigroup, hybrid semigroup, TJ-1-semigroup and TJ-2-semigroup.

4. Convergence and nonlinear ergodic theorems

Theorem 4.1. *Let C be a closed convex subset of H and S be a semitopological semigroup with identity such that $C(S)$ has an invariant mean μ. Let $\mathcal{S} = \{T_t : t \in S\}$ be an $(\alpha, \mathcal{F})$-semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. If $\{\mu_n\}$ is a strongly asymptotically invariant sequence of means on $C(S)$ and for given $x_1 \in C$, $\{x_n\}$ is a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n}(x_n), \quad \forall n \in \mathbb{N};$$

where $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, the sequence $\{x_n\}$ converges weakly to $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} Px_n$, where P is the metric projection of H onto $F(\mathcal{S})$.

Proof. By Lemma 3, $F(\mathcal{S}) = F(\mathcal{F})$. Let $v \in F(\mathcal{S})$. By Theorem 3, $T_{\mu_n}(v) = v$. Since T_{μ_n} is a nonspreading retraction, then T_{μ_n} is quasi-nonexpansive. Then for all $n \in \mathbb{N}$, we have

$$\|T_{\mu_n}(x_n) - v\| \le \|x_n - v\|. \tag{4.1}$$

It follows that

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)T_{\mu_n}(x_n) - v\|^2\\ &\le \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|T_{\mu_n}(x_n) - v\|^2\\ &\le \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2\\ &= \|x_n - v\|^2.\end{aligned}$$

Therefore $\{\|x_n - v\|^2\}$ is a decreasing sequence of nonnegative real numbers and therefore convergent. Hence, $\{x_n\}$ is bounded. So by Lemma 2, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}(x_n) - v\|^2 \\ &= \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|T_{\mu_n}(x_n) - v\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|T_{\mu_n}(x_n) - x_n\|^2 \\ &\leq \alpha_n\|x_n - v\|^2 + (1-\alpha_n)\|x_n - v\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}(x_n) - x_n\|^2 \\ &= \|x_n - v\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}(x_n) - x_n\|^2.\end{aligned}$$

It follows that

$$\alpha_n(1-\alpha_n)\|T_{\mu_n}(x_n) - x_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$$

Since $\lim_{n\to\infty}\|x_n - v\|^2$ exists and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty}\|T_{\mu_n}(x_n) - x_n\| = 0. \tag{4.2}$$

Since $\{x_n\}$ is bounded, so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$. Now, by relation (4.2) we have

$$T_{\mu_{n_i}}x_{n_i} \rightharpoonup z. \tag{4.3}$$

From Lemma 2, for all $y \in C$ and $s, t \in S$, we have

$$2\langle T_{st}x_n - f_s(y), y - f_s(y)\rangle - \|f_s(y) - y\|^2 = \|T_{st}x_n - f_s(y)\|^2 - \|T_{st}x_n - y\|^2.$$

By applying μ_n to both sides of the recent equality, we have

$$\begin{aligned}&2(\mu_n)_t\langle T_{st}(x_n) - f_s(y), y - fy\rangle - \|f_s(y) - y\|^2 \\ &\qquad = (\mu_n)_t(\|T_{st}(x_n) - f_s(y)\|^2 - \|T_{st}(x_n) - y\|^2) \\ &\qquad = (\mu_n)_t(\|T_{st}(x_n) - f_s(y)\|^2) - (\mu_n)_t(\|T_{st}(x_n) - y\|^2)\end{aligned}$$

Since μ_n is a left invariant mean, so we have

$$\begin{aligned}&2(\mu_n)_t\langle T_t(x_n) - f_s(y), y - f_s(y)\rangle - \|f_s(y) - y\|^2 \\ &\qquad = (\mu_n)_t(\|T_t(x_n) - f_s(y)\|^2) - (\mu_n)_t(\|T_t(x_n) - y\|^2).\end{aligned} \tag{4.4}$$

On the other hand since $\mathcal{S}$ is an $(\alpha, \mathcal{F})$-semigroup on C, we have

$$\begin{aligned}\|T_{st}x_n - f_s(y)\|^2 \leq{}& \alpha_s\|T_{st}x_n - y\|^2 + \alpha_s\|T_t x_n - f_s(y)\|^2 \\ &+ (1-2\alpha_s)\|T_t x_n - y\|^2.\end{aligned}$$

Now, by applying μ_n to both sides of the recent inequality, we have

$$\begin{aligned}(\mu_n)_t\|T_{st}x_n - f_s(y)\|^2 \leq{}& \alpha_s(\mu_n)_t\|T_{st}x_n - y\|^2 + \alpha_s(\mu_n)_t\|T_t x_n - f_s(y)\|^2 \\ &+ (1-2\alpha_s)(\mu_n)_t\|T_t x_n - y\|^2.\end{aligned}$$

Since μ_n is a left invariant mean, we have

$$\begin{aligned}(\mu_n)_t\|T_t x_n - f_s(y)\|^2 \leq{}& \alpha_s(\mu_n)_t\|T_t x_n - y\|^2 + \alpha_s(\mu_n)_t\|T_t x_n - f_s(y)\|^2 \\ &+ (1-2\alpha_s)(\mu_n)_t\|T_t x_n - y\|^2.\end{aligned}$$

It follows that

$$(\mu_n)_t\|T_t x_n - f_s(y)\|^2 \leq (\mu_n)_t\|T_t x_n - y\|^2. \tag{4.5}$$

From (4.4) and (4.5), we can get

$$2\langle T_{\mu_n}(x_n) - f_s(y), y - f_s(y)\rangle - \|f_s(y) - y\|^2 \leq 0.$$

Using the last inequality and (4.3), we can get

$$2\langle z - f_s(y), y - f_s(y)\rangle - \|f_s(y) - y\|^2 \leq 0.$$

Putting $y = z$, we have $z \in F(f_s)$. Therefore $z \in F(\mathcal{F})$, so $z \in F(\mathcal{S})$. The rest of the proof is similar to the proof of Theorem 3.1 from [4], so we omit it.

Example 4.2. Let $S = ((0, \frac{1}{2}] \cup \{1\}) \cap \mathbb{Q}$. Since S is countable, it can be assumed $S = \{t_0, t_1 \cdots ,\}$. Let $t_0 = 1$ and C, $\mathcal{S}$, $\mathcal{F}$ and α be as in Example 3 and $T_{t_0}x = f_{t_0}x = 1$ for each $x \in C$. Obviously, $\mathcal{S} = \{T_t : t \in S\}$ is an $(\alpha, \mathcal{F})$-semigroup on C and $F(\mathcal{S}) \neq \emptyset$. Suppose for given $x_1 \in C$, $\{x_n\}$ be a sequence generated by:

$$x_{n+1} = \delta_n x_n + (1 - \delta_n)\frac{1}{n}\sum_{k=0}^{n-1} T_{t_k} x_n, \quad \forall n \in \mathbb{N},$$

where $\{\delta_n\}$ is a sequence of real numbers in $[0, 1]$ and $\liminf_{n\to\infty} \delta_n(1 - \delta_n) > 0$. Then, $\{x_n\}_{n=1}^{\infty}$ converges weakly to $z \in F(\mathcal{S})$. For $g = (x_{t_0}, x_{t_1}, x_{t_2}, ...) \in C(S)$, we define

$$\mu_n(g) = \frac{1}{n}\sum_{k=0}^{n-1} x_{t_k};$$

for all $n \in \mathbb{N}$. We first show that $\{\mu_n\}_{n=1}^{\infty}$ is an asymptotically invariant sequence of means on $C(S)$. It is obvious that for all $n \in \mathbb{N}$, μ_n is linear. Also, we have

$$|\mu_n(g)| \leq \frac{1}{n}\sum_{k=0}^{n-1} |x_{t_k}| \leq \frac{1}{n}\sum_{k=0}^{n-1} \|g\| = \|g\|,$$

for all $g \in C(S)$. Hence, $\|\mu_n\| \leq 1$. Also, we have

$$\mu_n(1) = \frac{1}{n}\sum_{k=0}^{n-1} 1 = 1,$$

hence $\|\mu_n\| = \mu_n(1) = 1$, i.e., μ_n is a mean. For $g = (x_{t_0}, x_{t_1}, x_{t_2}, ...) \in C(S)$ and $m \in S$, we have

$$\begin{aligned}
|\mu_n(g) - \mu_n(r_m g)| &= \left|\frac{1}{n}\sum_{k=0}^{n-1} x_{t_k} - \frac{1}{n}\sum_{k=0}^{n-1} x_{t_{k+m}}\right| \\
&= \left|\frac{1}{n}\left(\sum_{k=0}^{m-1} x_{t_k} - \sum_{k=n}^{n+m-1} x_{t_k}\right)\right| \\
&\leq \frac{1}{n} 2m\|g\| \to 0, \qquad (n \to \infty).
\end{aligned}$$

Then, $\{\mu_n\}$ is asymptotically invariant. Furthermore, we have

$$\langle T_{\mu_n}x, y\rangle = (\mu_n)_k\langle T_{t_k}x, y\rangle = \frac{1}{n}\sum_{k=0}^{n-1}\langle T_{t_k}x, y\rangle = \left\langle \frac{1}{n}\sum_{k=0}^{n-1} T_{t_k}x, y \right\rangle,$$

for all $x \in C$ and $y \in H$. Hence $T_{\mu_n}x = \frac{1}{n}\sum_{k=0}^{n-1} T_{t_k}x$. Finally, by using Theorem 4, x_n converges weakly to $z \in F(\mathcal{S})$ as $n \to \infty$.

Before proving Baillon's nonlinear ergodic theorem, we need the following Lemma.

Lemma 4.3. *Let C be a closed convex subset of H and let S be a semitopological semigroup such that $C(S)$ has an invariant mean. If $\mathcal{S} = \{T_t : t \in S\}$ is an $(\alpha, \mathcal{F})$-semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, $T_\lambda = T_\mu$ for both invariant means μ and λ on $C(S)$.*

Proof. By part (iii) of Theorem 3, the proof is completed.

Now, by using Theorem 3 and Lemma 4, we prove generalized Baillon's nonlinear ergodic theorem for the proposed semigroups.

Theorem 4.4. *Let C be a closed convex subset of H and let S be a semitopological semigroup with identity. Let $\mathcal{S} = \{T_t : t \in S\}$ be an $(\alpha, \mathcal{F})$-semigroup on C and suppose $F(\mathcal{S}) \neq \emptyset$. If $\{\mu_\alpha\}_{\alpha\in I}$ is a net of asymptotically invariant means on $C(S)$, then, $\{T_{\mu_\alpha}x\}_{\alpha\in I}$ converges weakly to a point $x_0 \in F(\mathcal{S})$, for all $x \in C$. In this case, putting $Qx = x_0$ for all $x \in C$, then, Q is a nonspreading retraction of C onto $F(\mathcal{S})$ such that, $QT_t = T_tQ = Q$ for all $t \in S$ and $\bigcap_t \overline{co}\{T_{st}x : s \in S\} \bigcap F(\mathcal{S}) = \{Qx\}$, for all $x \in C$.*

Proof. Since $\{\mu_\alpha\}_{\alpha\in I}$ is a net of means on C(S), it has a cluster point μ in the *weak** topology on $C(S)^*$. By Banach- Alaoglu Theorem, $\{\mu \in C(S)^* : \mu(1) = \|\mu\| = 1\}$ is compact in the weak* topology, it follows that μ is a mean on C(S). Since $\{\mu_\alpha\}_{\alpha\in I}$ is a net of asymptotically invariant means on C(S), for any $\epsilon > 0$ there exists $\alpha_0 \in I$ such that for all $\alpha \in I$ and $\alpha \succeq \alpha_0$, we have

$$|\mu_\alpha(g) - \mu_\alpha(l_s g)| \leq \frac{\epsilon}{3}, \quad \forall g \in C(S),\ s \in S.$$

Since μ is a cluster point of $\{\mu_\alpha\}_{\alpha\in I}$, we can choose $\beta \succeq \alpha_0$ such that

$$|\mu_\beta(g) - \mu(g)| \leq \frac{\epsilon}{3} \quad \textit{and} \quad |\mu_\beta(l_s g) - \mu(l_s g)| \leq \frac{\epsilon}{3}.$$

It follows that

$$\begin{aligned}|\mu(g) - \mu(l_s g)| &\leq |\mu(g) - \mu_\beta(g)| + |\mu_\beta(g) - \mu_\beta(l_s g)| + |\mu_\beta(l_s g) - \mu(l_s g)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we have

$$\mu(g) = \mu(l_s g), \quad \forall g \in C(S), s \in S.$$

Similarly, we can show that

$$\mu(g) = \mu(r_s g), \quad \forall g \in C(S), s \in S.$$

Hence μ is an invariant mean on $C(S)$. Now, Theorem 3 implies that T_μ is a nonspreading retraction of C onto $F(\mathcal{S})$ and

$$\bigcap_t \overline{co}\{T_{st}x : s \in S\} \bigcap F(\mathcal{S}) = \{T_\mu(x)\}, \quad \forall x \in C. \tag{4.6}$$

Let $x \in C$. Since $F(\mathcal{S}) \neq \emptyset$, Theorem 3 implies that $\{T_t(x) : t \in S\}$ is bounded in C. On the other hand S has an identity element, so, from Theorem 3 we get

$$\{T_{\mu_\alpha}(x)\}_{\alpha \in I} \subset \overline{co}\{T_t(x) : t \in S\}.$$

Therefore $\{T_{\mu_\alpha}(x)\}_{\alpha\in I}$ is a bounded net in C and hence, there exists a sub net $\{T_{\mu_{\alpha_\beta}}(x)\}_{\beta\in I}$ of $\{T_{\mu_\alpha}(x)\}_{\alpha\in I}$ converging weakly to some $x_0 \in C$. If λ is a cluster point of $\{\mu_{\alpha_\beta}\}$ in the *weak** topology, then λ is a cluster point of $\{\mu_\alpha\}$, too. So, λ is an invariant mean on C(S). From $T_{\mu_{\alpha_\beta}}x \rightharpoonup x_0$, we also have $\lambda_t\langle T_t x, y\rangle = \langle x_0, y\rangle$ for all $y \in H$, i.e., $T_\lambda x = x_0$. Since $T_\lambda = T_\mu$, from Lemma 4, by putting $Q = T_\mu$, we have $x_0 = Qx$ and hence $T_{\mu_\alpha}x \rightharpoonup Qx$, and the proof is completed.

Theorem 4.5. *Let C be a closed convex subset of H and let S be a semitopological semigroup with identity such that $C(S)$ has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be an $(\alpha, \mathcal{F})$-semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on $C(S)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $u \in C$ and $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T_{\mu_n}x_n, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} Px_n$, where P is the metric projection of H onto $F(\mathcal{S})$.

Proof. Let $q \in F(\mathcal{S})$. As the proof of Theorem 4, we have

$$\|T_{\mu_n}x_n - q\| \leq \|x_n - q\|. \tag{4.7}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - q\| &= \|\alpha_n u + (1 - \alpha_n)T_{\mu_n}x_n - q\| \\ &\leq \alpha_n\|u - q\| + (1 - \alpha_n)\|T_{\mu_n}x_n - q\| \\ &\leq \alpha_n\|u - q\| + (1 - \alpha_n)\|x_n - q\|.\end{aligned}$$

By mathematical induction, we have

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x_1 - q\|\}$$

for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. From (4.7), $\{T_{\mu_n}x_n\}$ is also bounded. Let $\{T_{\mu_{n_i}}x_{n_i}\}$ be a subsequence of $\{T_{\mu_n}x_n\}$ such that $T_{\mu_{n_i}}x_{n_i} \rightharpoonup v$ for some $v \in C$. As the proof of Theorem 4, we have $v \in F(\mathcal{S})$. The rest of the proof is similar to the proof of Theorem 4.1 from [4], so, we omit it.

Remark 4.6. By taking $\mathcal{S} = \mathcal{F}$, Theorem 4, Theorem 4 and Theorem 4 are true for α-nonexpansive semigroup. Also by Remark 1, obviously Theorem 4, Theorem 4 and Theorem 4 are true for nonexpansive semigroup, r-firmly nonexpansive semigroup, nonspreading semigroup, hybrid semigroup, TJ-1-semigroup and TJ-2-semigroup.

5. Applications

In this section, by using Theorem 4 and Theorem 4 , we prove some famous theorems in nonlinear ergodic theory.

Theorem 5.1. *Let C be a nonempty closed convex subset of H and T be an α-nonexpansive mapping on C such that $F(T) \neq \emptyset$. Then, for all $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x;$$

converges weakly to some $x_0 \in F(T)$ as $n \to \infty$.

Proof. Let $S = \{0, 1, 2, ...\}$. For $g = (z_0, z_1, z_2, ...) \in C(S)$, we define

$$\mu_n(g) = \frac{1}{n}\sum_{k=0}^{n-1} z_k;$$

for all $n \in \mathbb{N}$. Then, by [15], $\{\mu_n\}_{n=1}^{\infty}$ is an asymptotically invariant sequence of means on $C(S)$. Also,

$$T_{\mu_n} x = S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

Then, by using Theorem 4, $S_n x$ converges weakly to some $x_0 \in F(T)$ as $n \to \infty$. This complete the proof.

Theorem 5.2. (Baillon's nonlinear ergodic theorem [15]) *Let C be a nonempty closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Then, for all $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x;$$

converges weakly to some $x_0 \in F(T)$ as $n \to \infty$.

Proof. Since every 0-nonexpansive mapping is a nonexpansive mapping. So, by using Theorem 5, $S_n x$ converges weakly to some $x_0 \in F(T)$ as $n \to \infty$.

Remark 5.3. It is obvious that Theorem 5 is also true for nonspreading mappings, hybrid mappings, TJ-1 mappings, TJ-2 mappings and r-firmly nonexpansive mappings.

Let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \leq t < \infty\}$. Then a family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called an one-parameter α-nonexpansive semigroup on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \quad \forall t, s \in S \quad and \quad x \in C;$
(2) $S(0)x = x \quad \forall x \in C;$
(3) for all $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(4) for all $t \in \mathbb{R}^+$, $S(t)$ is α-nonexpansive mapping.

Similarly, we can define one-parameter nonexpansive semigroup (see [4]), one-parameter nonspreading semigroup, one-parameter hybrid semigroup, one-parameter TJ-1 semigroup, one-parameter TJ-2 semigroup and one-parameter r-firmly nonexpansive semigroup.

Theorem 5.4. *Let C be a closed convex subset of a Hilbert space H and let*

$$\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$$

be an one-parameter α-nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, for all $x \in C$,

$$S_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)xdt;$$

converges weakly to some $x_0 \in F(\mathcal{S})$ as $\lambda \to \infty$.

Proof. Let $S = \mathbb{R}^+$. For $f \in C(\mathbb{R}^+)$, we define

$$\mu_\lambda(f) = \frac{1}{\lambda}\int_0^\lambda f(t)dt;$$

for all $0 < \lambda < +\infty$. Then, $\{\mu_\lambda\}_{\lambda>0}$ is an asymptotically invariant net of means on $C(\mathbb{R}^+)$ see, [15, Theorem 3.5.2]. Also, $T_{\mu_\lambda}x = S_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)xdt$. Now, by Theorem 4, $\frac{1}{\lambda}\int_0^\lambda S(t)xdt$ converges weakly to some $x_0 \in F(\mathcal{S})$ as $\lambda \to \infty$.

Theorem 5.5. [15, Theorem 3.5.2] *Let C be a closed convex subset of H and*

$$\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$$

be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, for all $x \in C$,

$$S_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)xdt;$$

converges weakly to some $x_0 \in F(\mathcal{S})$ as $\lambda \to \infty$.

Proof. Since one-parameter 0-nonexpansive semigroup is an one-parameter nonexpansive semigroup. Hence by Theorem 5, the proof is completed.

Remark 5.6. It is obvious that Theorem 5 is also true for one-parameter nonspreading semigroups, one-parameter hybrid semigroups, one-parameter TJ-1 semigroups, one-parameter TJ-2 semigroups and one-parameter r-firmly nonexpansive semigroups.

The following theorem is concluded from Theorem 5 and Theorem 4.

Theorem 5.7. [12] *Let C be a closed convex subset of H and $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ be an one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $u \in C$ and for given $x_1 \in C$, $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} S(t)x_n dt,$$

where $0 < \lambda_n < +\infty$, $\lambda_n \to \infty$, $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{+\infty}\alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})}(u)$.

References

[1] K. Aoyama, F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear. Anal, **74**(2011), 4387-4391.

[2] D. Ariza, C. Hernandez, E. Llorens, E. Moreno, *On α-nonexpansive mappings in Banach spaces*, Carpathian J. Math, **32**(2016), 13-28.

[3] S. Atsushiba, W. Takahashi, *Approximating common fixed points of nonexpansive semigroups by the Mann iteration process*, Ann. Univ. Mariae Curie-Skodowska Sect. A, **51**(1997), 1-16.

[4] N. Hussain, W. Takahashi, *Weak and strong convergence theorems for semigroups of mappings without continuity in Hilbert spaces*, J. Nonlinear Convex Anal., **14**(2013), 769-783.
[5] T. Igarashi, W. Takahashi, K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama, (2008), 75-85.
[6] T. Jitpeera, P. Kumam, *An extragradient type method for a system of equilibrium problems, variational inequality problems and fixed point of finitely many nonexpansive mappings*, J. Nonlinear Anal. Optim., **1**(2010), 71-91.
[7] O. Kada, A. T. Lau, W. Takahashi, *Asymptotically invariant net and fixed point set for semigroup of nonexpansive mapping*, Nonlinear Anal., **29**(1997), 539-550.
[8] P. Kocourek, W. Takahashi, J.C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math., **14**(2010), 2497-2511.
[9] F. Kohsaka, W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operaors in Banach spaces*, Arch. Math., **91**(2008), 166-177.
[10] A.T. Lau, W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings*, Pacific J. Math., **126**(1987), 277-294.
[11] B.D. Rouhani, *Ergodic theorems for hybrid sequences in a Hilbert space with applications*, J. Math. Anal. Appl., **409**(2014), 205-211.
[12] T. Shimizu, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl., **211**(1997), 71-83.
[13] N. Shioji, W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal., **34**(1998), 87-99.
[14] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc, **97**(1986), 55-58.
[15] W. Takahashi, *Nonlinear Functional Analysis*, Yokohoma Publishers, Yokohoma, 2000.
[16] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118**(2003), 417-428.
[17] W. Takahashi, N.C. Wong, J.C. Yao, *Attractive point and mean convergence theorems for semigroups of mappings without continuity in Hilbert spaces*, Taiwanese J. Math., **15**(2014), 1087-1103.

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