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ITERATIVE CONSTRUCTION OF THE FIXED POINT OF SUZUKI'S GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. We approximate the fixed point of the Suzuki's generalized nonexpansive mappings via the Picard-Ishikawa hyprid iterative process, recently introduced by Okeke [18]. We prove some weak and strong convergence theorems of this type of mappings in the setting of uniformly convex Banach spaces. We apply our results in finding the solution of a mixed type Volterra-Fredholm functional nonlinear integral equation in Banach spaces. Finally, we give several numerical examples to validate our analytical results. Our results extend and improve several known results in literature, including the results of Okeke [18], Ullah and Arshad [30] and Craciun and Serban [7] among others.
Key Words and Phrases: Picard-Ishikawa hybrid iterative process, Suzuki's generalized nonexpansive mapping, weak convergence, Strong convergence, Volterra-Fredholm integral equation, data dependence, Banach space.
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1. Introduction

Whenever the existence of fixed point of a mapping T is proved, finding the value of the fixed point p of T is not always easy. One effective method developed by scientists in solving this problem is the fixed point iteration method. A critical factor in determining the choice of a fixed point iterative scheme to be used to approximate the fixed point of a given nonlinear mapping is the rate of convergence of the iteration under consideration in comparison with others. Several mathematicians have developed or used some iterative schemes for approximating the fixed point of some nonlinear mappings (see, e.g. [1], [2], [4], [3], [16], [13], [17], [23], [14], [6], [22], [20], [19], [15], [25], [18]).

Recently, Ullah and Arshad [30] introduced the M-iterative scheme which is the first three step iterative scheme with a single set of parameter. They proved the weak and the strong convergence theorems of the above mentioned scheme for Suzuki's generalized nonexpansive mapping. More recently, Okeke [18] in 2019 introduced the Picard-Ishikawa hybrid iterative scheme and proved that this new iterative process is faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor, Picard-Mann and Picard-Krasnoselskii [20] iterative processes in the sense of Berinde [5].

In 2011, Craciun and Serban [7] proved that the Picard iterative scheme converges strongly to the solution of mixed type Volterra-Fredholm functional nonlinear integral equation.

Motivated by Okeke [18], Ullah and Arshad [30] and others, we prove some weak and strong convergence theorems using the Picard-Ishikawa hybrid iterative scheme for Suzuki's generalized nonexpansive mappings in the setting of uniformly convex Banach space. We apply our results in finding the solution of a mixed type Volterra-Fredholm functional nonlinear integral equation in Banach spaces. We also produce some numerical examples to show the efficiency of this iterative scheme.

2. Preliminaries

First we recall some definitions, propositions and lemmas to be used in the next sections.

Definition 2.1. A Banach space X is called uniformly convex [11] if for each $\epsilon \in (0,2]$ there is a $\delta > 0$ such that for $x, y \, \epsilon X$ then

$$\|x\| \leq 1, \; \|y\| \leq 1, \; \|x-y\| > \epsilon \Rightarrow \|\frac{x+y}{2}\| \leq \delta. \tag{2.1}$$

Definition 2.2. A Banach space X is said to satisfy the Opial property [21] if for each sequence x_n in X, converging weakly to $x \in X$, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{2.2}$$

for all $y \in X$ such that $y \neq x$.

A point p is called fixed point of a mapping T if $T(p) = p$, and $F(T)$ represents the set of all fixed points of mapping T. Let C be a nonempty subset of a Banach space X.

Definition 2.3. A mapping $T : C \to C$ is a contraction if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \tag{2.3}$$

for all $x, y \in C$ and $L \in (0,1)$.

Definition 2.4. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.4}$$

for all $x, y \in C$ and quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have

$$\|Tx - p\| \leq \|x - p\|. \tag{2.5}$$

In 1974, Senter and Dotson [9] introduced the notion of a mappings satisfying condition (I) as follows:

Definition 2.5. A mapping $T : C \to C$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where

$$d(x, F(T)) = \inf_{p \in F(T)} \|x - p\|.$$

Suzuki [29] introduced the concept of Suzuki's generalized nonexpansive mapping which is called condition (C) in 2008.

Definition 2.6. A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2}\|x - Tx\| \leq \|x - y| \Rightarrow \|Tx - Ty\| \leq \|x - y\|. \tag{2.6}$$

Suzuki [29] showed that the mapping satisfying condition (C) is weaker than nonexpansive and stronger than quasi nonexpansive. The mapping that satisfy condition (C) is called Suzuki's generalized nonexpansive mapping.

Suzuki [29] obtained fixed point theorems and convergence theorems for Suzuki's generalized nonexpansive mapping. Recently, fixed point theorems for Suzuki's generalized nonexpansive mapping have been studied by number of authors (see e.g [30], [25]).

The following are some of the iterative schemes that will be needed in this study:

Let C be a nonempty convex subset of a normed space E, and let $T : C \to C$ be a mapping. Let $\mathbb{N}$ denote the set of all positive integers, I the identity mapping of C and $F(T)$ the set of all fixed points of T. The Picard or successive iterative process, introduced by Picard [24] is defined by the sequence $\{u_n\}_{n=0}^{\infty}$ as follows.

$$\begin{cases} u_1 = u \in C, \\ u_{n+1} = Tu_n, \ n \in \mathbb{N}. \end{cases} \tag{2.7}$$

The Mann iterative process, introduced by Mann [16] is defined by the sequence $\{v_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} v_1 = v \in C, \\ v_{n+1} = (1 - \alpha_n)v_n + \alpha_n Tv_n, \ n \in \mathbb{N}, \end{cases} \tag{2.8}$$

where $\{\alpha_n\}_{n=0}^{\infty} \in (0, 1)$.

The Ishikawa iterative process [13] is given by the sequence $\{z_n\}_{n=0}^{\infty}$ defined as follows

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = (1 - \alpha_n)z_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)z_n + \beta_n Tz_n, \ n \in \mathbb{N}, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are in $(0, 1)$.

The Picard-Ishikawa hybrid scheme introduced by Okeke [18] is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} x_1 = x \in C \\ x_{n+1} = Ty_n \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tz_n \\ z_n = (1 - \beta_n)x_n + \beta_n Tx_n, \forall n \in N \end{cases} \tag{2.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

The Picard-Mann [Normal-S] iterative process [15] is given by the sequence $\{z_n\}_{n=0}^{\infty}$ defined as follows

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \ n \in \mathbb{N}, \end{cases} \tag{2.11}$$

where $\{\beta_n\}_{n=0}^{\infty}$ is a real sequence in $(0,1)$.

Let C be a nonempty closed convex subset of a Banach space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is known that in uniformly convex Banach space, $A(C, \{x_n\})$ consists of exactly one point.

The following lemma will be needed in this study.

Lemma 2.1. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be any mapping.*
Then (i) [[29], *Proposition 1*] *if T is nonexpansive then T is Suzuki generalized nonexpansive mapping.*
(ii) [[29], *Proposition 2*] *If T is Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*
(iii) [[29], *Lemma 7*] *If T is Suzuki generalized nonexpansive mapping, then*

$$\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$$

for all $x, y \in C$.

Lemma 2.2 ([29], Proposition 3)**.** *Let T be mapping on a subset C of a Banach space X with the Opial property. Assume that T is Suzuki generalized mapping. If x_n converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$.*

Lemma 2.3 ([29], Theorem 5)**.** *Let C be a weakly compact convex subset of a uniformly convex Banach space X. Let T be a mapping on C. Assume that T is Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.4 ([28], lemma 1.3)**.** *Suppose that X is uniformly convex Banach and $\{t_n\}$ be any real sequence such that $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \ \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\limsup_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

hold for some $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.5. [12] *Let* $\{s_n\}_{n=0}^{\infty}$ *be a nonnegative sequence for which we assume there exists* $n_0 \in N$, *such that for all* $n \geq n_0$ *that satisfies the inequality*

$$s_{n+1} \leq (1-\mu_n)s_n + \mu_n\gamma_n. \tag{2.12}$$

If $\{\mu_n\}_{n=0}^{\infty} \subset (0,1)$ *and* $\sum_{n=0}^{\infty} \mu_n = \infty$, *and* $\gamma_n \geq 0$ *for all* $n \in N$. *Then the following inequality holds:*

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n. \tag{2.13}$$

3. Convergence analysis

We begin this section by proving the following convergence results.

Lemma 3.1. *Let* C *be a nonempty closed subset of a Banach space* X *and let* $T : C \to C$ *be a Suzuki generalized nonexpansive mapping with* $F(T) \neq \emptyset$. *For arbitrary chosen* $x_1 \in C$, *let the sequence* $\{x_n\}$ *be generated by (2.10), then* $\lim \|x_n - p\|$ *exist for any* $p \in F(T)$.

Proof. Let $p \in F(T)$ and $z \in C$. Since T is Suzuki generalized nonexpansive mapping, so $\frac{1}{2}\|p - Tp\| = 0 \leq \|p - z\|$ implies that $\|Tp - Tz\| \leq \|p - z\|$.
So by Proposition 2.1(ii), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.1}$$

Using (3.1) together with proposition 2.1(ii), we get

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n Tz_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tz_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.2}$$

Similarly, by (3.2) together with proposition 2.1(ii), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.3}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non increasing for all $p \in F(T)$.
Hence $\lim_{n\to\infty} \|x_n - p\|$ exists, as required. □

Next, we prove the following Theorems.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* X, *and let* $T : C \to C$ *be Suzuki generalized nonexpansive mapping. For arbitrary chosen* $x_1 \in C$, *let the sequence* $\{x_n\}$ *be generated by (2.10) for all* $n \geq 1$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequence of real numbers in* $[a,b]$ *for some a,b with* $0 < a \leq b < 1$. *Then* $F(T) \neq \emptyset$ *if and only if* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Without lost of generality, let

$$\lim_{n\to\infty} \|x_n - p\| = d. \tag{3.4}$$

From (3.1) and (3.4), we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = d. \tag{3.5}$$

By proposition 2.1(ii), we have the following

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = d. \tag{3.6}$$

On the other hand by using Proposition 2.1(ii), we get

$$\begin{array}{rcl} \|x_{n+1} - p\| &=& \|Ty_n - p\| - \|z_n - p\| + \|z_n - p\| \\ &\leq& \|y_n - p\| - \|z_n - p\| + \|z_n - p\| \\ &\leq& (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tz_n - p\| - \|z_n - p\| + \|z_n - p\| \\ &\leq& (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\| - \|z_n - p\| + \|z_n - p\| \\ &\leq& \|x_n - p\| - \alpha_n\|x_n - p\| + \alpha_n[(1-\beta_n)\|x_n - p\| \\ &+& \beta_n\|x_n - p\|] - [(1-\beta_n)\|x_n - p\| \\ &+& \beta_n\|x_n - p\|] + \|z_n - p\| \\ &=& \|z_n - p\|. \end{array} \tag{3.7}$$

Therefore

$$d = \liminf_{n\to\infty} \|z_n - p\|. \tag{3.8}$$

From (3.5) and (3.8), we get

$$\begin{array}{rcl} d &=& \lim_{n\to\infty} \|z_n - p\| \\ &=& \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &=& \lim_{n\to\infty} \|\beta_n(Tx_n - p) + (1-\beta_n)(x_n - p)\|. \end{array} \tag{3.9}$$

Using (3.4), (3.6) and (3.9) together with Lemma 2.4, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.10}$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.
Let $p \in A(C, \{x_n\})$. By Proposition 2.1(ii), we have the following

$$\begin{array}{rcl} r(Tp, \{x_n\}) &=& \limsup_{n\to\infty} \|x_n - Tp\| \\ &\leq& \limsup_{n\to\infty}(3\|Tx_n - x_n\| + \|x_n - p\|) \\ &\leq& \limsup_{n\to\infty} \|x_n - p\| \\ &=& r(p, \{x_n\}). \end{array} \tag{3.11}$$

□

Now, we prove the weak convergence theorem

Theorem 3.2. *Let C be a nonempty closed convex Banach space X with Opial property, and let $T : C \to C$ be Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (2.10) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$ and by Lemma 3.1, then $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Since X is uniformly convex hence reflexive, so there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $q_1 \in X$. Since C is closed and convex, $q_1 \in C$. By Lemma 2.2, $q_1 \in F(T)$. Now, we show that $\{x_n\}$ converges weakly to q_1. In fact, if this is not true, so there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_2 \in C$ and $q_2 \neq q_1$. By Lemma 2.2, $q_2 \in F(T)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. By Theorem 3.1 and Opial's property, we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - q_1\| &= \liminf_{j\to\infty} \|x_{n_j} - q_1\| \\ &< \lim_{j\to\infty} \|x_{n_j} - q_2\| \\ &= \lim_{n\to\infty} \|x_n - q_2\| \\ &= \liminf_{k\to\infty} \|x_{n_k} - q_2\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - q_1\| \\ &= \lim_{n\to\infty} \|x_n - q_1\|, \end{aligned} \tag{3.12}$$

which is a contradiction. Therefore $q_1 = q_2$. This implies that $\{x_n\}$ converges weakly to a common fixed point of T. □

Next, we prove the following strong convergence theorems.

Theorem 3.3. *Let C be a nonempty closed convex Banach space X with opial property,and let $T : C \to C$ be Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (2.10) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.3, we have that $F(T) \neq \emptyset$ and by Theorem 3.1, we obtained that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Since C is compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p, for some $p \in C$. By proposition 2.1(iii), we have

$$\|x_{n_k} - Tp\| \leq 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|,$$

for all $n \geq 1$. Letting $k \to \infty$, we get Tp = p, it implies that $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T)$, so $\{x_n\}$ converges strongly to p. □

Now, using condition (I) then we obtain the following results.

Theorem 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (2.10) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty}\|x_n - p\|$ for some $r \geq 0$.
If $r = 0$ then the result is obvious. Suppose $r > 0$, from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\|. \tag{3.13}$$

Since $F(T) \neq \emptyset$, so by Theorem 3.1, we have

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

So (3.13) implies that

$$f(d(x_n, F(T))) = 0. \tag{3.14}$$

Since f is nondecreasing function, so from (3.13) we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$\|x_{n_k} - y_k\| < \frac{1}{2^k}$$

for all $k \in N$.
Therefore using (3.3), we have the following

$$\|x_{n_{k+1}} - y_k\| \leq \|x_{n_k} - y_k\| < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} \|y_{k+1} - y_k\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0. \end{aligned}$$

as $k \to \infty$.

This shows that $\{y_k\}$ is Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty}\|x_n - p\|$ exists, we have that $x_n \to p \in F(T)$. Hence proved. □

4. Application to a nonlinear integral equation

In this section, We will prove that the Picard-Ishikawa hybrid (2.10) converges strongly to the solution of the following mixed type Volterra-Fredholm functional nonlinear integral equation.

$$x(t) = F(t, x(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t, s, x(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t, s, x(s))ds) \tag{4.1}$$

where $[a_1, b_1] \times ... \times [a_m, b_m]$ is an interval in $\mathbb{R}^m$, K, H : $[a_1, b_1] \times ... \times [a_m, b_m] \times [a_1, b_1] \times ... \times [a_m, b_m] \times \mathbb{R} \to \mathbb{R}$ is continuous functions, and $F : [a_1, b_1] \times ... \times [a_m, b_m] \times \mathbb{R}^3 \to \mathbb{R}$.

Theorem 4.1. [7] *Suppose that the following conditions are satisfied:*
(A_1) $K, H \in C([a_1,b_1] \times ... \times [a_m,b_m] \times [a_1,b_1] \times \cdots \times [a_m,b_m] \times \mathbb{R})$;
(A_2) $F \in C([a_1,b_1] \times ... \times [a_m,b_m] \times \mathbb{R}^3)$;
(A_3) *there exist nonnegative constants* α, β *and* γ *such that*

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \leq \alpha|u_1-u_2| + \beta|v_1-v_2| + \gamma|w_1-w_2|,$$

for all $t \in [a_1,b_1] \times ... \times [a_m,b_m], u_i, v_i, w_i \in \mathbb{R},\ i=1,2$*;*
(A_4) *there exist nonnegative constants* L_k *and* L_H *such that*

$$|K(t,s,u) - K(t,s,v)| \leq L_K|u-v|,$$
$$|H(t,s,u) - H(t,s,v)| \leq L_H|u-v|,$$

for all $t, s \in [a_1,b_1] \times ... \times [a_m,b_m],\ u, v \in \mathbb{R}$*;*
(A_5) $\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)...(b_m-a_m) < 1$.
Then (4.1) *has a unique solution* $x^* \in C([a_1,b_1] \times ... \times [a_m,b_m])$.

Theorem 4.2. *Let all the conditions* $(A_1) - (A_5)$ *in Theorem 4.1 be met and let* α_n *and* β_n *be real sequences contain in [0,1] and satisfy the following conditions*

$$\sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=0}^{\infty} \beta_n = \infty.$$

Then (4.1) *has a unique solution, say* x^**, in* $C([a_1,b_1] \times ... \times [a_m,b_m])$ *and the Picard-Ishikawa hybrid (2.10) converges strongly to* x^**.*

Proof. We consider the Banach space $B = C([a_1,b_1] \times ... \times [a_m,b_m], \|.\|_c)$, where $\|.\|_c$ is a Chevbyshev's norm. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by Picard-Ishikawa hybrid method for $T : B \to B$ defined by

$$T(x)(t) = F(t, x(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,x(s))ds) \tag{4.2}$$

We will show that x_n converges to x^* as n tends to ∞.
From (2.12), (4.1) and assumption $(A_1) - (A_4)$, we have

$$\begin{aligned}
\|z_n - x^*\| &\leq (1-\beta_n)|x_n(t) - x^*(t)| + \beta_n|T(x_n)(t) - T(x^*)(t)| \\
&= (1-\beta_n)|x_n(t) - x^*(t)| \\
&+\beta_n \left| F\left(t, x_n(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,x_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,x_n(s))ds\right) \right. \\
&\left. -F\left(t, x^*(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,x^*(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,x^*(s))ds\right)\right| \\
&\leq (1-\beta_n)|x_n(t) - x^*(t)| + \beta_n\alpha|x_n(t) - x^*(t)| \\
&+\beta_n\beta \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} L_K|x_n(t) - x^*(t)|ds + \beta_n\gamma \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} L_H|x_n(t) - x^*(t)|ds \\
&\leq \{1 - \beta_n(1 - (\alpha + (\beta L_K + \gamma L_H)(\Pi_{i=1}^m (b_i - a_i))))\} \times \|x_n - x^*\|.
\end{aligned} \tag{4.3}$$

Next, we compute the following:

$$\|y_n - x^*\| \leq (1-\alpha_n)|x_n(t) - x^*(t)| + \alpha_n|T(z_n)(t) - T(x^*)(t)|$$

$$= (1-\alpha_n)|x_n(t)-x^*(t)|$$
$$+\alpha_n \left| F\left(t, z_n(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,z_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,z_n(s))ds\right)\right.$$
$$\left. -F\left(t, x^*(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,x^*(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,x^*(s))ds\right)\right|$$
$$\leq (1-\alpha_n)|x_n(t)-x^*(t)| + \alpha_n\alpha|z_n(t)-x^*(t)|$$
$$+\alpha_n\beta \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} L_K|z_n(t)-x^*(t)|ds + \alpha_n\gamma \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} L_H|z_n(t)-x^*(t)|ds$$
$$\leq (1-\alpha_n)|x_n(t)-x^*(t)| + \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}$$
$$\times\|z_n-x^*\|. \tag{4.4}$$

Putting (4.3) into (4.4) and using (A_5), we have

$$\|y_n-x^*\| \leq (1-\alpha_n)|x_n(t)-x^*(t)| + \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}$$
$$\times\{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\|$$
$$\leq (1-\alpha_n)\|x_n-x^*\| + \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}$$
$$\times\{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\|.$$
$$\leq \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}$$
$$\times\{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\|$$
$$\leq \{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\|. \tag{4.5}$$

Hence, we have

$$\|x_{n+1}-x^*\| \leq |T(y_n)(t)-T(x^*)(t)|$$
$$= \left| F\left(t, y_n(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,y_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,y_n(s))ds\right)\right.$$
$$\left. -F\left(t, x^*(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,x^*(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,x^*(s))ds\right)\right|$$
$$\leq \alpha|y_n(t)-x^*(t)|$$
$$+\beta \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} L_K|y_n(t)-x^*(t)|ds + \gamma \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} L_H|y_n(t)-x^*(t)|ds$$
$$\leq \{(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i)))\} \times \|y_n-x^*\|. \tag{4.6}$$

Using (4.5) in (4.6), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| \quad \leq \quad & \{(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i)))\} \times \\
& \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \\
& \{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\| \\
\leq \quad & \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \\
& \{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\} \times \|x_n-x^*\|.
\end{aligned} \tag{4.7}$$

Using assumption (A_5), we obtain the following

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \{1-\alpha_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}\times \\ &\quad \{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}\times\|x_n-x^*\| \\ &\leq \{1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}\times\|x_n-x^*\|. \end{aligned} \tag{4.8}$$

Continuing the process by induction, we get

$$\|x_{n+1}-x^*\|\leq\|x_0-x^*\|\times\Pi_{k=0}^n\{1-\beta_k(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\}. \tag{4.9}$$

Therefore α_k and β_k belong to [0,1] for all $k\in\mathbf{N}$ and assumption (A_5) yields

$$[1-\beta_k(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))]<1. \tag{4.10}$$

Next, using the fact that $1-x\leq e^x$ for all $x\in[0,1]$, (4.9) becomes

$$\|x_{n+1}-x^*\| \leq \|x_0-x^*\|\times e^{-(1-(\alpha+(\beta L_K+\gamma L_H)(\Pi_{i=1}^m(b_i-a_i))))\Sigma_{k=0}^n\beta_k} \tag{4.11}$$

Hence, we have

$$\lim_{n\to\infty}\|x_n-x^*\|=0. \tag{4.12}$$

□

5. A data dependence result

Next, we prove the following data dependence results for the Picard-Ishikawa hybrid method (2.10).

Let B be as in Proof of Theorem 4.2 and $A,\tilde{A}:B\to B$ be two operators defined by

$$T(x)(t)=F(t,x(t),\int_{a_1}^{t_1}...\int_{a_m}^{t_m}K(t,s,x(s))ds,\int_{a_1}^{b_1}...\int_{a_m}^{b_m}H(t,s,x(s))ds), \tag{5.1}$$

$$\tilde{T}(x)(t)=F(t,x(t),\int_{a_1}^{t_1}...\int_{a_m}^{t_m}\tilde{K}(t,s,x(s))ds,\int_{a_1}^{b_1}...\int_{a_m}^{b_m}\tilde{H}(t,s,x(s))ds). \tag{5.2}$$

Where $K,\tilde{K},H,\tilde{H}\in C([a_1,b_1]\times...\times[a_m,b_m]\times[a_1,b_1]\times...\times[a_m,b_m]\times\mathbb{R})$.

Theorem 5.1. *Let F, K and H be defined as in Theorem 4.1 and let $\{x_n\}_{n=1}^\infty$ be a sequence defined by the Picard-Ishikawa hybrid method (2.10) associated with T. Let $\{\tilde{x}_n\}_{n=1}^\infty$ be an iterative sequence generated as follows*

$$\begin{cases}\tilde{x}_0\in B,\\ \tilde{x}_{n+1}=\tilde{T}\tilde{y}_n,\\ \tilde{y}_n=(1-\alpha_n)\tilde{x}_n+\alpha_n\tilde{T}\tilde{z}_n\\ \tilde{z}_n=(1-\beta_n)\tilde{x}_n+\beta_n\tilde{T}\tilde{x}_n,\ n\in\mathbb{N},\end{cases} \tag{5.3}$$

where α_n,β_n are real sequences in $(0,1)$ satisfying the following conditions:
(i) $\frac{1}{2}\leq\beta_n$ for all $n\in\mathbb{N}$, and
(ii) $\sum_{n=1}^\infty\beta_n=\infty$.
Suppose that (iii) there exist nonnegative constants ε_1 and ε_2 such that

$$|K(t,s,u)-\tilde{k}(t,s,u)|\leq\varepsilon_1 \text{ and } |H(t,s,u)-\tilde{H}(t,s,u)|\leq\varepsilon_2,$$

for all $u \in \mathbb{R}$ *and for all* $t, s \in [a_1, b_1] \times ... \times [a_m, b_m]$.
If x^* *and* $\tilde{x}^*$ *are solutions of corresponding equations (5.1) and (5.2), respectively we have*

$$\|x^* - \tilde{x}^*\| \leq \frac{5(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^m(b_i-a_i))}{[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^m(b_i-a_i)))]}. \tag{5.4}$$

Proof. Using (2.12), (5.1), (5.2), (5.3), assumption $(A_1)-(A_5)$ and (iii), we have the following

$$\|x_{n+1}-\tilde{x}_{n+1}\| = \|Ty_n - T\tilde{y}_n\|$$

$$= \Bigg|F(t, y_n(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} K(t,s,y_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} H(t,s,y_n(s))ds)$$

$$-F(t, \tilde{y}_n(t), \int_{a_1}^{t_1} ... \int_{a_m}^{t_m} \tilde{K}(t,s,\tilde{y}_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_m}^{b_m} \tilde{H}(t,s,\tilde{y}_n(s))ds)\Bigg|$$

$$\leq \alpha|y_n(t)-\tilde{y}_n(t)| + \beta\int_{a_1}^{t_1} ... \int_{a_m}^{t_m} |K(t,s,y_n(s)) - \tilde{K}(t,s,\tilde{y}_n(s))|ds$$

$$+\gamma\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} |H(t,s,y_n(s)) - \tilde{H}(t,s,\tilde{y}_n(s))|ds$$

$$\leq \alpha|y_n(t)-\tilde{y}_n(t)|$$

$$+\beta\int_{a_1}^{t_1} ... \int_{a_m}^{t_m} (|K(t,s,y_n(s)) - \tilde{K}(t,s,\tilde{y}_n(s))| + |K(t,s,y_n(s)) - \tilde{K}(t,s,\tilde{y}_n(s))|)ds$$

$$+\gamma\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} (|H(t,s,y_n(s)) - \tilde{H}(t,s,\tilde{y}_n(s))| + |H(t,s,y_n(s)) - \tilde{H}(t,s,\tilde{y}_n(s))|)ds$$

$$\leq \alpha|y_n(t)-\tilde{y}_n(t)| + \beta\int_{a_1}^{t_1} ... \int_{a_m}^{t_m} (L_K|y_n(s)-\tilde{y}_n(s)|+\varepsilon_1)ds$$

$$+\gamma\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} (L_H|y_n(s)-\tilde{y}_n(s)|+\varepsilon_2)ds$$

$$\leq \alpha\|y_n-\tilde{y}_n\| + \beta(L_K\|y_n-\tilde{y}_n\|+\varepsilon_1)\prod_{i=1}^m(b_i-a_i)$$

$$+\gamma(L_H\|y_n-\tilde{y}_n\|+\varepsilon_2)\prod_{i=1}^m(b_i-a_i)$$

$$\leq [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^m(b_i-a_i)]\|y_n-\tilde{y}_n\| + (\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^m(b_i-a_i). \tag{5.5}$$

$$\|y_n-\tilde{y}_n\| = \|(1-\alpha_n)x_n - \alpha_n Tz_n - (1-\alpha_n)\tilde{x}_n - \tilde{T}\tilde{z}_n\|$$

$$\leq (1-\alpha_n)\|x_n-\tilde{x}_n\| + \alpha_n\|Tz_n - \tilde{T}\tilde{z}_n\|$$

$$= (1-\alpha_n)|x_n(t)-\tilde{x}_n(t)| + \alpha_n|T(z_n)(t) - \tilde{T}\tilde{z}_n(t)|$$

$$\leq (1-\alpha_n)|x_n(t)-\tilde{x}_n(t)| + \alpha_n[\alpha|z_n(t)-\tilde{z}_n(t)|$$

$$+\beta\int_{a_1}^{t_1} ... \int_{a_m}^{t_m} (L_K|z_n(s)-\tilde{z}_n(s)|+\varepsilon_1)ds + \gamma\int_{a_1}^{b_1} ... \int_{a_m}^{b_m} (L_H|z_n(s)-\tilde{z}_n(s)|+\varepsilon_2)ds]$$

$$\leq (1-\alpha_n)|x_n(t)-\tilde{x}_n(t)| + \alpha_n[(\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i))\|z_n - \tilde{z}_n\|$$

$$+(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i)]. \tag{5.6}$$

$$\|z_n - \tilde{z}_n\| = \|(1-\beta_n)x_n + \beta_n T x_n - (1-\beta_n)\tilde{x}_n - \beta_n \tilde{T}\tilde{x}_n\|$$
$$\leq (1-\beta_n)\|x_n - \tilde{x}_n\| + \alpha_n\|Tx_n - \tilde{T}\tilde{x}_n\|$$
$$= (1-\beta_n)|x_n(t) - \tilde{x}_n(t)|$$
$$+\beta_n[\alpha|x_n(t) - \tilde{x}_n(t)| + \beta\int_{a_1}^{t_1}...\int_{a_m}^{t_m}(L_K|z_n(s) - \tilde{z}_n(s)| + \varepsilon_1)ds$$
$$+\gamma\int_{a_1}^{b_1}...\int_{a_m}^{b_m}(L_H|z_n(s) - \tilde{z}_n(s)| + \varepsilon_2)ds]$$
$$\leq (1-\beta_n)|x_n(t)-\tilde{x}_n(t)| + \beta_n[(\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i))\|x_n - \tilde{x}_n\|$$
$$+(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i)]$$
$$\leq [1 - \beta_n(1 - (\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i))]\|x_n - \tilde{x}_n\|$$
$$+\beta_n(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i)]. \tag{5.7}$$

Using (5.7) in (5.6) and assumption (A_5), we have

$$\|y_n - \tilde{y}_n\| \leq [1 - \alpha_n(1 - (\alpha + (\beta L_k + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)))]$$
$$\times[1 - \beta_n(1 - (\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)))]\|x_n - \tilde{x}_n\|$$
$$+(\alpha_n + \beta_n)(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i)$$
$$\leq [1 - \beta_n(1 - (\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)))]\|x_n - \tilde{x}_n\|$$
$$+(\alpha_n + \beta_n)(\beta\varepsilon_1 + \gamma\varepsilon_2)\prod_{i=1}^{m}(b_i - a_i). \tag{5.8}$$

Now, combining (5.5) with (5.8) and (A_5), it becomes

$$\|x_{n+1} - \tilde{x}_{n+1}\| \leq [\alpha + (\beta L_K + \gamma L_H)\prod_{i=1}^{m}(b_i - a_i)]$$

$$\times[1-\alpha_n(1-(\alpha+(\beta L_k+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]$$

$$\times[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]\|x_n-\tilde{x}_n\|$$

$$+(\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)+(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)$$

$$\leq[1-\alpha_n(1-(\alpha+(\beta L_k+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]$$

$$\times[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]\|x_n-\tilde{x}_n\|$$

$$+(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i)$$

$$\leq[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]\|x_n-\tilde{x}_n\|$$

$$+(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i). \tag{5.9}$$

Therefore, using assumptions (A_5) and $\frac{1}{2}\leq\beta_n$, we have the following

$$\|x_{n+1}-\tilde{x}_{n+1}\|\leq[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]\|x_n-\tilde{x}_n\|$$

$$+\beta_n[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]$$

$$\times\frac{5(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i))}{[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]}. \tag{5.10}$$

Now, let $\lambda_n:=\|x_n-\tilde{x}_n\|$,

$$\mu_n:=\beta_n[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]\in(0,1),$$

$$\gamma_n:=\frac{5(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i))}{[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]}\geq 0.$$

Using Lemma 2.5, it follows that

$$0\leq\limsup_{n\to\infty}\|x_n-\tilde{x}_n\|\leq\limsup_{n\to\infty}\frac{5(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i))}{[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]}. \tag{5.11}$$

From the results of Theorem 4.2, we know that $\lim_{n\to\infty} x_n = x^*$. Using this fact together with the assumption that $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}^*$, we have

$$\|x^* - \tilde{x}^*\| \leq \frac{5(1+\alpha_n+\beta_n)(\beta\varepsilon_1+\gamma\varepsilon_2)\prod_{i=1}^{m}(b_i-a_i))}{[1-\beta_n(1-(\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)))]}. \tag{5.12}$$

The proof of Theorem 5.1 is completed. □

6. Numerical examples

In this section, we give some numerical examples to validate our analytical results.

Example 6.1. Let $T : [0, 5] \longrightarrow [0, 5]$ be a self mapping defined by

$$T(x) = \begin{cases} 5 - x, \ if \ x \in \left[0, \dfrac{12}{5}\right) \\ \dfrac{x+35}{8}, \ if \ x \in \left[\dfrac{12}{5}, 5\right]. \end{cases} \tag{6.1}$$

We show that T is Suzuki's generalized nonexpansive mapping but not nonexpansive mapping.

Verification: Take $x = \dfrac{23}{10}$ and $y = \dfrac{12}{5}$. We have the following

$$\|T(x) - T(y)\| = \left|5 - \frac{23}{10} - \frac{187}{40}\right| = \left|\frac{2000-92-187}{40}\right| = \frac{79}{40}$$

and

$$\|x - y\| = \left|\frac{23}{10} - \frac{12}{5}\right| = \frac{1}{10}.$$

Therefore,

$$\|T(x) - T(y)\| = \frac{79}{40} > \frac{1}{10} = \|x - y\|.$$

Hence T is not nonexpansive mapping.
Next, we show that T is Suzuki generalized nonexpansive mapping. We proceed as follows:

Case I: If $x, y \in \left[0, \dfrac{12}{5}\right)$ then

$$\|x - y\| = |5 - x - (5 - y)| = |x - y|.$$

and if $x, y \in \left[\dfrac{12}{5}, 5\right]$, we have

$$\|x - y\| = \left|\frac{x+35}{8} - \left(\frac{y+35}{8}\right)\right| = \frac{1}{8}|x - y| \leq |x - y| = \|x - y\|.$$

Hence T is nonexpansive mapping and if T is nonexpansive mapping then it is Suzuki generalized nonexpansive mapping.

Case II: Let $x \in \left[0, \dfrac{12}{5}\right)$ then

$$\frac{1}{2}\|x - Tx\| = \frac{5-2x}{2} \in \left(\frac{1}{10}, \frac{5}{2}\right]$$

this implies that

$$\frac{5-2x}{2} \leq y - x$$

and

$$y \geq \frac{5}{2} \Rightarrow y \in \left[\frac{5}{2}, 5\right].$$

Hence

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

Therefore

$$\|Tx - Ty\| = \left|5 - x - \left(\frac{y+35}{8}\right)\right| = \left|\frac{5-8x-y}{8}\right| < \frac{1}{8}$$

and

$$\|x - y\| = |x - y| > \left|\frac{23}{10} - \frac{5}{2}\right| = \left|\frac{23-25}{10}\right| = \frac{1}{5} > \frac{1}{8} = \|Tx - Ty\|$$

Hence

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

Case III: Let $x \in \left[\frac{12}{5}, 5\right]$ then

$$\frac{1}{2}\|x - Tx\| = \frac{1}{2}\left|\frac{x+35}{8} - x\right| = \frac{35-7x}{16} \in \left[0, \frac{91}{80}\right].$$

Two possibilities arises:

(a) If $x < y$, we have

$$\frac{35-7x}{16} + x \leq y \Rightarrow y \geq \frac{35+9x}{16}$$

It implies that $y \in \left[\frac{283}{80}, 5\right] \subset \left[\frac{12}{5}, 5\right]$ and we have the following

$$\|Tx - Ty\| = \left|\frac{x+35}{8} - \frac{y+35}{8}\right| = \frac{1}{8}\|x - y\| \leq \|x - y\|$$

Hence

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

(b) Now, we assume $x > y$ then

$$\frac{35-7x}{16} \leq x - y$$

$$y \leq x - \left(\frac{35-7x}{16}\right) \Rightarrow y \leq \frac{23x-35}{16}.$$

This implies that $y \leq \frac{101}{80}$ and $y \leq 5$ then $y \in [1, 5]$.

If $y \leq \frac{23x-35}{16} \Rightarrow \frac{16y+35}{23} \leq x$.

Since $x \in \left[\frac{51}{23}, 5\right]$ and $y \in \left[\frac{12}{5}, 5\right]$.

Already treated in case (I) above and we consider $x \in \left[\frac{51}{23}, 5\right]$ and $y \in \left[0, \frac{12}{5}\right)$, then

$$\|Tx - Ty\| = \left|\frac{x+35}{8} - (5-y)\right| = \left|\frac{x+y-5}{8}\right| < \frac{1}{8}$$

and

$$\|x - y\| = |x - y| > \left|\frac{51}{23} - \frac{12}{5}\right| = \frac{21}{115} > \frac{1}{8}.$$

Then T is a suzuki generalized nonexpansive mapping.

Numerically we compare our iteration process with four existing iteration schemes. In Table 6.1 and Table 6.2 below, we compare the speed of convergence of various iterative schemes, viz: Mann, Picard-Ishikawa, Ishikawa, Picard and Picard-Mann (Normal-S). Choose $\alpha_n = \frac{n}{2n+3}$ and $\beta_n = \frac{1}{2^n}$ with initial value $x_0 = 4.5$ and the operator T as defined in Example 6.1, where $F(T) = \{5\}$. Using MATLAB, we obtain the following numerical results:

Step	Mann	Picard-Ishikawa	Picard	Ishikawa
0	4.500000000000000	4.500000000000000	4.500000000000000	4.500000000000000
1	4.690217391304348	4.961280076400094	4.937500000000000	4.690240611200748
2	4.808069470699433	4.997001535032835	4.992187500000000	4.808098242101428
3	4.881086519889866	4.999767799331108	4.999023437500000	4.881113257527673
4	4.926325343844809	4.999982018415682	4.999877929687500	4.926347430630877
5	4.954353745642980	4.999998607508858	4.999984741210938	4.954370850257452
6	4.971719168496194	4.999999892165699	4.999998092651367	4.971731884928638
7	4.982478180481338	4.999999991649328	4.999999761581421	4.982487371905977
8	4.989144090080829	4.999999999353325	4.999999970197678	4.989150598050654
9	4.993274055810949	4.999999999949922	4.999999996274710	4.993278591766666
10	4.995832838926349	4.999999999996122	4.999999999534339	4.9958359613875457
⋮	⋮	⋮	⋮	⋮

Table 6.1 Comparison of the speed of convergence among various iterative processes.

Step	Picard-Mann (Normal-S)
0	4.500000000000000
1	4.937553405761719
2	4.961313162272310
3	4.976032777581790
4	4.985151855665039
5	4.990801295771999
6	4.994301230001179
7	4.996469504976515
8	4.997812792038733
9	4.998644983597482
10	4.999160541894686
⋮	⋮

Table 6.2 Comparison of the speed of convergence among various iterative processes.

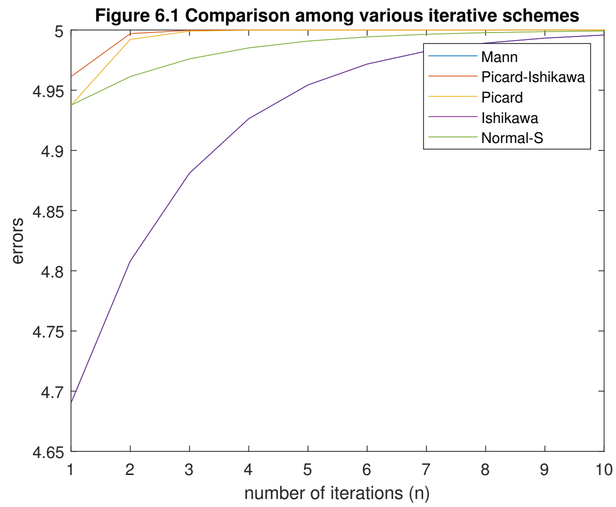

Figure 6.1

Remark 6.1. From Table 6.1, Table 6.2 and Figure 6.1 above, we see that the Picard-Ishikawa iterative scheme converges faster to the fixed point of T than all of Mann, Ishikawa, Picard and Picard-Mann (Normal-S) iterative schemes.

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