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THE EXISTENCE AND COMPACTNESS OF THE SET OF SOLUTIONS FOR A 2-ORDER NONLINEAR INTEGRODIFFERENTIAL EQUATION IN N VARIABLES IN A BANACH SPACE

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Abstract. In this paper, by applying the fixed point theorem of Krasnosel'skii, we prove the existence and compactness of the set of solutions for a 2-order nonlinear integrodifferential equation in N variables in an arbitrary Banach space E. Here, an appropriate Banach space X_1 for the above equation is defined and a sufficient condition for relatively compact subsets in X_1 is proved. An example is given to verify the efficiency of the used method.
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1. Introduction

The integral and integrodifferential equations usually have attracted many interests of scientists, because these equations can be used to model many problems of science and theoretical physics such as engineering, mechanic, electrostatics, population dynamics, economics, and other fields of science. They occur in a natural way in the description of many physical phenomena, for example, see the books written by Corduneanu [5], Deimling [7].

In this paper, we consider the following nonlinear integrodifferential equation in N variables

$$\begin{aligned} u(x) \quad = \quad & g(x) + \int_\Omega K(x, y; u(y), D_1D_2u(y), D_2D_1u(y))dy \qquad (1.1) \\ & + \int_\Omega H(x, y; u(y), D_1D_2u(y), D_2D_1u(y))dy, \end{aligned}$$

where $(x_1, \cdots, x_N) \in \Omega = [0,1]^N$ and $g : \Omega \to E$, K, $H : \Omega \times \Omega \times E^3 \to E$ are given functions, $(E, \|\cdot\|_E)$ is an arbitrary Banach space. Denote by

$$D_2 D_1 u = \frac{\partial^2 u}{\partial x_1 \partial x_2},\ D_1 D_2 u = \frac{\partial^2 u}{\partial x_2 \partial x_1},$$

the 2-order partial derivatives with respect to the variables x_1, x_2 of a function $u : \Omega \to E$.

It is well known that many types of Eq. (1.1) are studied by many different methods, in which the fixed point theorems are often applied, see [1]-[19] and the references therein.

In [4], Bica et al. used Perov's fixed point theorem to obtain the existence, the uniqueness and the global approximation of the solution of the following neutral Fredholm integro-differential equation

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s))ds,\ t \in [a, b],$$

where E is an arbitrary Banach space, $f \in C([a, b] \times [a, b] \times E \times E; E)$, $g \in C^1([a, b]; E)$ and $f(\cdot, s, u, v) \in C^1([a, b]; E)$ for any $s \in [a, b]$, $u, v \in E$. In the case $E = \mathbb{R}^d$, motivated by the results in [4], based on the application of the Banach fixed point theorem coupled with a Bielecki-type norm and a certain integral inequality with explicit estimates, B.G. Pachpatte [16] proved the uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s))ds,\ t \in [a, b],$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. By the same methods, B. G. Pachpatte [17] studied the existence, the uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables as the following

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g\left(x, y, s, t, u(s, t), D_1 u(s, t), D_2 u(s, t)\right) dt ds.$$

In [2], M.A. Abdou et al. considered the existence of an integrable solution of a nonlinear integral equation of Hammerstein-Volterra type of the second kind by using the technique of measure of weak noncompactness and the Schauder fixed point theorem.

In [3], A. Aghajani et al. studied the Fredholm type integro-differential equation in two variables of the form

$$u(x, y) = f(x, y) + \int_a^b \int_c^d g\left(x, y, s, t, u(s, t), D_1 u(s, t), D_2 u(s, t)\right) dt ds,$$

where g, f are given real valued functions, u is the unknown function to be found, $D_i u(x_1, x_2) = \frac{\partial u}{\partial x_i}(x_1, x_2)$, $i = 1, 2$. By using the concept of generalized metric and Perov's fixed point theorem, the authors in [3] proved some results on the existence, the uniqueness, and the estimation of the solutions of the equation considered.

In [11]-[13], by using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, the solvability and the asymptotic stability of nonlinear functional integral equations in one variable, two variables, and N variables were investigated.

In [6], [8], [14], the fixed point theorems of Banach, Schauder and Krasnosel'skii type were also applied to obtain the existence result. On the other hand, the sets of solutions are compact (as in [6], [8]) or a continuum (i.e. nonempty, compact and connected, as in [14]). Such a structure of the solutions set for differential equations and integral equations have been studied by many authors, for examples, we refer to [7], [9], [10], [15] and references therein.

Because of mathematical context, motivated by the above mentioned works, we study the existence and compactness of the set of solutions for Eq. (1.1). This paper is organized as follows. Section 2 is devoted to preliminaries, where we present the definition of an appropriate Banach space (Lemma 2.1) and a sufficient condition for relatively compact subsets (Lemma 2.2). In Section 3, by applying the fixed point theorem of Krasnosel'skii, we prove the Theorem 3.1. It follows that the solution set is nonempty. Furthermore, the solution set is compact. In order to illustrate the results obtained here, in Section 4, we give an example.

2. Preliminaries

First, we construct an appropriate Banach space for (1.1) as follows.
Let $X = C(\Omega; E)$ be the space of all continuous functions from $\Omega = [0,1]^N$ into E equipped with the usual norm

$$\|u\|_X = \sup_{x\in\Omega} \|u(x)\|_E\,,\ u \in X. \tag{2.1}$$

Put

$$X_1 = \{u \in X : D_1u,\ D_2u,\ D_2D_1u,\ D_1D_2u \in X\}. \tag{2.2}$$

Remark 1. In order to solve Eq.(1.1), the space X_1 chosen as above is rather natural and, in general, it is very efficient by the following properties.

(i) $C^2(\Omega; E) \subsetneqq X_1 \subsetneqq C^1(\Omega; E)$ if $N = 2$;

(ii) $X_1 \cap C^1(\Omega; E) \neq \phi$, $X_1 \backslash C^1(\Omega; E) \neq \phi$, $C^1(\Omega; E)\backslash X_1 \neq \phi$, if $N \geq 3$.

Indeed, let $e_1 \in E$, $e_1 \neq 0$.

(i) Case $N = 2$.

Proof of $X_1 \subsetneqq C^1(\Omega; E)$. Consider $u(x) = u(x_1, x_2) = \Phi(x_1, x_2)e_1$, where

$$\Phi(x) = \frac{(x_1 - \frac{1}{2})^2(x_2 - \frac{1}{2})^2}{(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2},\ x = (x_1, x_2) \in \Omega.$$

We have $u \in C^1(\Omega; E)$, but $u \notin X_1$, it is proved below. Note that

$$D_1\Phi(x) = \frac{2(x_1 - \frac{1}{2})(x_2 - \frac{1}{2})^4}{\left[(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2\right]^2},\ x = (x_1, x_2) \in \Omega.$$

Hence $D_1u = (D_1\Phi)\, e_1 \in X$.

Similarly

$$D_2\Phi(x) = \frac{2(x_2-\frac{1}{2})(x_1-\frac{1}{2})^4}{\left[(x_1-\frac{1}{2})^2+(x_2-\frac{1}{2})^2\right]^2},\ x=(x_1,x_2)\in\Omega,$$

it follows that $D_2u = (D_2\Phi)\, e_1 \in X$. Therefore $u \in C^1(\Omega;E)$.

On the other hand,

$$D_2D_1\Phi(x_1,x_2) = \frac{8(x_1-\frac{1}{2})^3(x_2-\frac{1}{2})^3}{\left[(x_1-\frac{1}{2})^2+(x_2-\frac{1}{2})^2\right]^3},\ x=(x_1,x_2)\in\Omega,$$

it follows that there is no the limit $\nexists \lim\limits_{(x_1,x_2)\to(\frac{1}{2},\frac{1}{2})} D_2D_1\Phi(x_1,x_2)$, which implies that $(D_2D_1\Phi)\, e_1 \notin X$. Thus $u \notin X_1$.

Proof of $C^2(\Omega;E) \subsetneqq X_1$. Considering

$$w = w(x) = \left(\left|x_1-\frac{1}{2}\right|^{3/2} + \left|x_2-\frac{1}{2}\right|^{3/2}\right) e_1,$$

we have $w \in X_1$, but $w \notin C^2(\Omega;E)$, it is proved below. We have:

$$D_1w(x) = \frac{3}{2}\left|x_1-\frac{1}{2}\right|^{-1/2}\left(x_1-\frac{1}{2}\right)e_1,\ D_2w(x) = \frac{3}{2}\left|x_2-\frac{1}{2}\right|^{-1/2}\left(x_2-\frac{1}{2}\right)e_1.$$

Moreover $D_2D_1w(x) = D_1D_2w(x) = 0$, it follows that $w \in X_1$.

On the other hand, $D_1^2w(x) = \dfrac{3}{4}\left|x_1-\frac{1}{2}\right|^{-1/2} e_1$, which implies that $D_1^2w \notin X$. Thus $w \notin C^2(\Omega;E)$.

(ii) Case $N \geq 3$.

Proof of $X_1 \cap C^1(\Omega;E) \neq \phi$. It is obvious that $C^2(\Omega;E) \subset X_1 \cap C^1(\Omega;E)$.

Proof of $C^1(\Omega;E)\backslash X_1 \neq \phi$. Consider

$$u(x) = u(x_1,\cdots,x_N) = \left(\Phi(x_1,x_2) + \sum_{i=3}^{N} e^{x_i}\right) e_1,$$

where the function $\Phi(x_1,x_2)$ as in **(i)**, we have $u \in C^1(\Omega;E)$, $u \notin X_1$.
Thus $C^1(\Omega;E)\backslash X_1 \neq \phi$.

Proof of $X_1\backslash C^1(\Omega;E) \neq \phi$. Considering

$$u(x) = u(x_1,\cdots,x_N) = \left(\left|x_1-\frac{1}{2}\right|^{3/2} + \left|x_2-\frac{1}{2}\right|^{3/2} + \sum\nolimits_{i=3}^{N}\left|x_i-\frac{1}{2}\right|\right) e_1,$$

we have $u \in X_1$, $u \notin C^1(\Omega;E)$. Thus $X_1\backslash C^1(\Omega;E) \neq \phi$.

In particular, the space X_1 have the following useful property.

Lemma 2.1. *X_1 is a Banach space with the norm defined by*

$$\|u\|_{X_1} = \|u\|_X + \|D_1u\|_X + \|D_2u\|_X + \|D_2D_1u\|_X + \|D_1D_2u\|_X\,,\ u \in X_1. \quad (2.3)$$

Proof. Let $\{u_p\} \subset X_1$ be a Cauchy sequence in X_1, it means that

$$\begin{aligned}\|u_p-u_q\|_{X_1} &= \|u_p-u_q\|_X + \|D_1u_p-D_1u_q\|_X + \|D_2u_p-D_2u_q\|_X \\ &\quad + \|D_2D_1u_p-D_2D_1u_q\|_X + \|D_1D_2u_p-D_1D_2u_q\|_X \to 0 \text{ as } p,\ q\to\infty.\end{aligned}$$

Then $\{u_p\}$, $\{D_1u_p\}$, $\{D_2u_p\}$, $\{D_2D_1u_p\}$ and $\{D_1D_2u_p\}$ are also Cauchy sequences in X.
Since X is complete, there exist u, v_1, v_2, v_{21}, $v_{12} \in X$ such that
We shall show that $D_1u = v_1$, $D_2u = v_2$, $D_2D_1u = v_{21}$, $D_1D_2u = v_{12}$.
We have

$$u_p(x_1, x_2, x') - u_p(0, x_2, x') = \int_0^{x_1} D_1u_p(s, x_2, x')ds, \ \forall(x_1, x_2, x') \in \Omega, \tag{2.5}$$

where (and in what follows) $x' = (x_3, \cdots, x_N) \in [0,1]^{N-2}$.
By $\|u_p - u\|_X \to 0$, we get

$$u_p(x_1, x_2, x') - u_p(0, x_2, x') \to u(x_1, x_2, x') - u(0, x_2, x') \text{ in } E, \ \forall(x_1, x_2, x') \in \Omega. \tag{2.6}$$

On the other hand, it follows from $\|D_1u_p - v_1\|_X \to 0$ that

$$\int_0^{x_1} D_1u_p(s, x_2, x')ds \to \int_0^{x_1} v_1(s, x_2, x')ds, \ \forall(x_1, x_2, x') \in \Omega \tag{2.7}$$

since

$$\begin{aligned} &\left\| \int_0^{x_1} D_1u_p(s, x_2, x')ds - \int_0^{x_1} v_1(s, x_2, x')ds \right\|_E \\ &\le \int_0^{x_1} \|D_1u_p(s, x_2, x') - v_1(s, x_2, x')\|_E \, ds \\ &\le \|D_1u_p - v_1\|_X \to 0. \end{aligned}$$

Combining (2.5)-(2.7) leads to

$$u(x_1, x_2, x') - u(0, x_2, x') = \int_0^{x_1} v_1(s, x_2, x')ds, \ \forall(x_1, x_2, x') \in \Omega. \tag{2.8}$$

It implies that $D_1u = v_1 \in X$. Similarly $D_2u = v_2 \in X$.
By the same argument, it follows from

$$D_1u_p(x_1, x_2, x') - D_1u_p(x_1, 0, x') = \int_0^{x_2} D_2D_1u_p(x_1, t, x')dt, \ \forall(x_1, x_2, x') \in \Omega,$$

and $\|D_2D_1u_p - v_{21}\|_X \to 0$, that

$$D_1u(x_1, x_2, x') - D_1u(x_1, 0, x') = \int_0^{x_2} v_{21}(x_1, t, x')dt, \ \forall(x_1, x_2, x') \in \Omega.$$

It implies that $D_2D_1u = v_{21} \in X$. Similarly $D_1D_2u = v_{12} \in X$.
Therefore $u \in X_1$ and $u_p \to u$ in X_1. Lemma 2.1 is proved. □
Next, for our purpose related to solving Eq.(1.1), it is very useful to propose a sufficient condition for relatively compact subsets of X_1 as follows.

Lemma 2.2. *Let $\mathcal{F} \subset X_1$. Then $\mathcal{F}$ is relatively compact in X_1 if and only if the following conditions are satisfied*

$$\begin{array}{l} \text{(i) } \forall x \in \Omega, \mathcal{F}(x) = \{u(x) : u \in \mathcal{F}\},\ D_1\mathcal{F}(x) = \{D_1u(x) : u \in \mathcal{F}\}, \\ \qquad D_2\mathcal{F}(x) = \{D_2u(x) : u \in \mathcal{F}\},\ D_2D_1\mathcal{F}(x) = \{D_2D_1u(x) : u \in \mathcal{F}\}, \\ \ \textit{and } D_1D_2\mathcal{F}(x) = \{D_1D_2u(x) : u \in \mathcal{F}\}, \\ \ \textit{are relatively compact subsets of } E; \\ \text{(ii) } \forall \varepsilon > 0, \exists \delta > 0 : \forall x,\, \bar{x} \in \Omega,\, |x - \bar{x}| < \delta \Longrightarrow \sup_{u \in \mathcal{F}} [u(x) - u(\bar{x})]_* < \varepsilon, \end{array} \tag{2.9}$$

where

$$\begin{aligned} [u(x) - u(\bar{x})]_* \quad = \quad & \|u(x) - u(\bar{x})\|_E + \|D_1u(x) - D_1u(\bar{x})\|_E + \|D_2u(x) - D_2u(\bar{x})\|_E \\ & + \|D_2D_1u(x) - D_2D_1u(\bar{x})\|_E + \|D_1D_2u(x) - D_1D_2u(\bar{x})\|_E . \end{aligned}$$

Proof. **(a)** Let $\mathcal{F}$ be relatively compact in X_1.

First, we show that (2.9) (i) is true.

We begin by considering $\mathcal{F}(x) = \{u(x) : u \in \mathcal{F}\}$. To prove that $\mathcal{F}(x)$ is relatively compact in E, let $\{u_p(x)\}$ be a sequence in $\mathcal{F}(x)$, we show that $\{u_p(x)\}$ contains a convergent subsequence in E. Because $\overline{\mathcal{F}}$ is compact in X_1, we have $\{u_p\} \subset \mathcal{F}$ contains a convergent subsequence $\{u_{p_k}\}$ in X_1. Then, there exists $u \in X_1$ such that

$$\|u_{p_k} - u\|_{X_1} \to 0 \text{ as } k \to \infty.$$

By $\|u_{p_k}(x) - u(x)\|_E \leq \|u_{p_k} - u\|_X \leq \|u_{p_k} - u\|_{X_1} \to 0$. Hence $u_{p_k}(x) \to u(x)$ in E. Thus $\mathcal{F}(x)$ is relatively compact in E.

By the same argument, by

$$\|D_1u_{p_k}(x) - D_1u(x)\|_E \leq \|D_1u_{p_k} - D_1u\|_X \leq \|u_{p_k} - u\|_{X_1} \to 0,$$

we have $D_1\mathcal{F}(x)$ is also relatively compact in E.

Similarly, we have also $D_2\mathcal{F}(x)$ is also relatively compact in E.

On the other hand, by

$$\|D_2D_1u_{p_k}(x) - D_2D_1u(x)\|_E \leq \|D_2D_1u_{p_k} - D_2D_1u\|_X \leq \|u_{p_k} - u\|_{X_1} \to 0,$$

it gives $D_2D_1u_{p_k}(x) \to D_2D_1u(x)$ in E. Thus $D_2D_1\mathcal{F}(x)$ is relatively compact in E.

Similarly, we have also $D_1D_2\mathcal{F}(x)$ is also relatively compact in E.

It implies that (2.9) (i) is true.

Next, we show that (2.9) (ii) is also true.

For every $\varepsilon > 0$, considering a collection of open balls in X_1 centered at $u \in \mathcal{F}$ with radius $\frac{\varepsilon}{4}$, as the following

$$B(u, \frac{\varepsilon}{4}) = \{\bar{u} \in X_1 : \|u - \bar{u}\|_{X_1} < \frac{\varepsilon}{4}\},\ u \in \mathcal{F}.$$

It is clear that $\overline{\mathcal{F}} \subset \bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{4})$. Because $\overline{\mathcal{F}}$ is compact in X_1, the open cover $\bigcup_{u \in \mathcal{F}} B(u, \frac{\varepsilon}{4})$ of $\overline{\mathcal{F}}$ contains a finite subcover and there are $u_1, \cdots, u_q \in \mathcal{F}$ such that $\overline{\mathcal{F}} \subset \bigcup_{j=1}^{q} B(u_j, \frac{\varepsilon}{4})$.

By the functions u_j, D_1u_j, D_2u_j, $D_2D_1u_j$, $D_1D_2u_j$, $j=\overline{1,q}$ are uniformly continuous on Ω, there exists $\delta>0$ such that

$$\forall x,\bar{x}\in\Omega,\ \ |x-\bar{x}|<\delta \Longrightarrow [u_j(x)-u_j(\bar{x})]_*<\frac{\varepsilon}{2}\ \forall j=\overline{1,q}.$$

For all $u\in\mathcal{F}$, $u\in B(u_{j_0},\frac{\varepsilon}{4})$ for some $j_0=\overline{1,q}$. Thus, for all x, $\bar{x}\in\Omega$, if $|x-\bar{x}|<\delta$ then we obtain

$$\begin{aligned}[][u(x)-u(\bar{x})]_* &\leq [u(x)-u_{j_0}(x)]_*+[u_{j_0}(x)-u_{j_0}(\bar{x})]_*+[u_{j_0}(\bar{x})-u(\bar{x})]_*\\ &\leq 2\left\|u-u_{j_0}\right\|_{X_1}+[u_{j_0}(x)-u_{j_0}(\bar{x})]_*<\frac{2\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

It implies that (2.9) (ii) is true.

(b) Conversely, let (2.9) be correct.

To prove that $\mathcal{F}$ is relatively compact in X_1, let $\{u_p\}$ be a sequence in $\mathcal{F}$, we show that $\{u_p\}$ contains a convergent subsequence.

Put $\mathcal{F}_0=\{u_p : p\in\mathbb{N}\}$. By (2.9), $\mathcal{F}_0(x)=\{u_p(x) : p\in\mathbb{N}\}$ is a relatively compact subset of E, for all $x\in\Omega$ and $\mathcal{F}_0$ is equicontinuous in X. Applying the Ascoli-Arzela theorem to $\mathcal{F}_0$, it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u\in X$ such that

$$\left\|u_{p_k}-u\right\|_X\to 0 \text{ as } k\to\infty.$$

Similarly, $\mathcal{F}_1=\{D_1u_{p_k} : k\in\mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D_1u_{p_k}\}$, denoted by the same symbol, and $v_1\in X$ such that

$$\left\|D_1u_{p_k}-v_1\right\|_X\to 0 \text{ as } k\to\infty.$$

Since

$$u_{p_k}(x)-u_{p_k}(0,x_2,x')=\int_0^{x_1}D_1u_{p_k}(s,x_2,x')ds, \forall x=(x_1,x_2,x')\in\Omega,$$

furthermore, since $\left\|u_{p_k}-u\right\|_X\to 0$ and $\left\|D_1u_{p_k}-v_1\right\|_X\to 0$, we obtain

$$u(x)-u(0,x_2,x')=\int_0^{x_1}v_1(s,x_2,x')ds, \forall x=(x_1,x_2,x')\in\Omega.$$

It gives $D_1u=v_1\in X$.

Similarly, $\mathcal{F}_2=\{D_2u_{p_k} : k\in\mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D_2u_{p_k}\}$, denoted by the same symbol, such that

$$\left\|D_2u_{p_k}-D_2u\right\|_X\to 0 \text{ as } k\to\infty.$$

By the same argument, by $\mathcal{F}_{21}=\{D_2D_1u_{p_k} : k\in\mathbb{N}\}$ is also relatively compact in X, we obtain the existence of a subsequence of $\{D_2D_1u_{p_k}\}$, denoted by the same symbol, and $v_{21}\in X$ such that

$$\left\|D_2D_1u_{p_k}-v_{21}\right\|_X\to 0 \text{ as } k\to\infty.$$

Since

$$D_1u_{p_k}(x)-D_1u_{p_k}(x_1,0,x')=\int_0^{x_2}D_2D_1u_{p_k}(x_1,t,x')dt, \forall x=(x_1,x_2,x')\in\Omega,$$

furthermore, since $\|D_1 u_{p_k} - D_1 u\|_X \to 0$ and $\|D_2 D_1 u_{p_k} - v_{21}\|_X \to 0$, we obtain

$$D_1 u(x) - D_1 u(x_1, 0, x') = \int_0^{x_2} v_{21}(x_1, t, x')dt, \forall x = (x_1, x_2, x') \in \Omega.$$

It gives $D_2 D_1 u = v_{21} \in X$.

Similarly, by $\mathcal{F}_{12} = \{D_1 D_2 u_{p_k} : k \in \mathbb{N}\}$ is also relatively compact in X. We obtain the existence of a subsequence of $\{D_1 D_2 u_{p_k}\}$, denoted by the same symbol, such that

$$\|D_1 D_2 u_{p_k} - D_1 D_2 u\|_X \to 0 \text{ as } k \to \infty.$$

Therefore $u \in X_1$ and $u_{p_k} \to u$ in X_1. Lemma 2.2 is proved. □

For convenient, we now recall the fixed point theorem of Krasnosel'skii in the following.

Theorem 2.3. (see [5], [10]). *Let B be a nonempty bounded closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that $U : B \to X$ is a contraction and $C : B \to X$ is a completely continuous operator such that $U(x) + C(y) \in B$, $\forall x, y \in B$. Then $U + C$ has a fixed point in B.*

3. The existence and compactness of the set of solutions

We make the following assumptions.

(A_1) $g \in X_1$,

(A_2) $K \in C(\Omega \times \Omega \times E^3; E)$

such that $D_1 K,\ D_2 K,\ D_1 D_2 K,\ D_2 D_1 K \in C(\Omega \times \Omega \times E^3; E)$,

and there exist nonnegative functions $k_0,\ k_1,\ k_2,\ k_{21},\ k_{12} : \Omega \times \Omega \to \mathbb{R}$ with the following properties

(i) $\|K(x, y; u, v, w) - K(x, y; \bar{u}, \bar{v}, \bar{w})\|_E$
$\le k_0(x, y) \left(\|u - \bar{u}\|_E + \|v - \bar{v}\|_E + \|w - \bar{w}\|_E\right)$,

(ii) $\|D_i K(x, y; u, v, w) - D_i K(x, y; \bar{u}, \bar{v}, \bar{w})\|_E$
$\le k_i(x, y) \left(\|u - \bar{u}\|_E + \|v - \bar{v}\|_E + \|w - \bar{w}\|_E\right)$,

(iii) $\|D_1 D_2 K(x, y; u, v, w) - D_1 D_2 K(x, y; \bar{u}, \bar{v}, \bar{w})\|_E$
$\le k_{12}(x, y) \left(\|u - \bar{u}\|_E + \|v - \bar{v}\|_E + \|w - \bar{w}\|_E\right)$,

(iv) $\|D_2 D_1 K(x, y; u, v, w) - D_2 D_1 K(x, y; \bar{u}, \bar{v}, \bar{w})\|_E$
$\le k_{21}(x, y) \left(\|u - \bar{u}\|_E + \|v - \bar{v}\|_E + \|w - \bar{w}\|_E\right)$,

$\forall (x, y) \in \Omega \times \Omega$, $\forall (u, v, w)$, $(\bar{u}, \bar{v}, \bar{w}) \in E^3$;

(A_3) $H \in C(\Omega \times \Omega \times E^3; E)$ such that
$D_1 H, D_2 H, D_1 D_2 H, D_2 D_1 H \in C(\Omega \times \Omega \times E^3; E)$,

and there exist nonnegative functions $\bar{h}_0,\ \bar{h}_1,\ \bar{h}_2,\ \bar{h}_{21},\ \bar{h}_{12} : \Omega \times \Omega \to \mathbb{R}$ with the following properties

(i) $\|H(x, y; u, v, w)\|_E \le \bar{h}_0(x, y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right)$,

(ii) $\|D_i H(x, y; u, v, w)\|_E \le \bar{h}_i(x, y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right)$, $i = 1, 2$,

(iii) $\|D_1 D_2(x, y; u, v, w)\|_E \le \bar{h}_{12}(x, y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right)$,

(iv) $\|D_2 D_1 H(x, y; u, v, w)\|_E \le \bar{h}_{21}(x, y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right)$,

$\forall (x, y) \in \Omega \times \Omega$, $\forall (u, v, w) \in E^3$;

(A_4) $H, D_1 H, D_2 H, D_1 D_2 H, D_2 D_1 H : \Omega \times \Omega \times E^3 \to E$

are completely continuous such that for any bounded subset J of E^3, for all $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$\begin{aligned}
&\forall x,\, \bar{x} \in \Omega,\, |x - \bar{x}| < \delta \\
&\Longrightarrow \|H(x,y;u,v,w) - H(\bar{x},y;u,v,w)\|_E \\
&+ \|D_1 H(x,y;u,v,w) - D_1 H(\bar{x},y;u,v,w)\|_E \\
&+ \|D_2 H(x,y;u,v,w) - D_2 H(\bar{x},y;u,v,w)\|_E \\
&+ \|D_1 D_2 H(x,y;u,v,w) - D_1 D_2 H(\bar{x},y;u,v,w)\|_E \\
&+ \|D_2 D_1 H(x,y;u,v,w) - D_2 D_1 H(\bar{x},y;u,v,w)\|_E < \varepsilon,
\end{aligned}$$

$\forall y \in \Omega$, $\forall (u,v,w) \in J$;

(A_5) $\beta_1^* + \beta_2^* < 1$, where

$$\begin{aligned}
\beta_1^* &= \sum\nolimits_{i=0}^{2} \sup_{x\in\Omega} \int_\Omega k_i(x,y)dy + \sup_{x\in\Omega} \int_\Omega k_{21}(x,y)dy + \sup_{x\in\Omega} \int_\Omega k_{12}(x,y)dy, \\
\beta_2^* &= \sum\nolimits_{i=0}^{2} \sup_{x\in\Omega} \int_\Omega \bar{h}_i(x,y)dy + \sup_{x\in\Omega} \int_\Omega \bar{h}_{21}(x,y)dy + \sup_{x\in\Omega} \int_\Omega \bar{h}_{12}(x,y)dy.
\end{aligned}$$

Theorem 3.1. *Let the functions g, K, H in Eq.* (1.1) *satisfy the assumptions $(A_1) - (A_5)$. Then Eq.* (1.1) *has a solution in X_1. Furthermore, the set of solutions is compact.*

Proof of Theorem 3.1. We rewrite (1.1) as follows

$$u(x) = (Au)(x), \tag{3.1}$$

where

$$\begin{aligned}
&(Au)(x) = (Uu)(x) + (Cu)(x), \\
&(Uu)(x) = g(x) + \textstyle\int_\Omega K(x,y;u(y), D_1D_2u(y), D_2D_1u(y))dy, \\
&(Cu)(x) = \textstyle\int_\Omega H(x,y;u(y), D_1D_2u(y), D_2D_1u(y))dy, \\
&x \in \Omega, u \in X_1.
\end{aligned} \tag{3.2}$$

A simple verification shows that Uu, $Cu \in X_1$, $\forall u \in X_1$.

For $\rho > 0$, we consider a closed ball in X_1 as follows

$$B_\rho = \{u \in X_1 : \|u\|_{X_1} \le \rho\}. \tag{3.3}$$

We will show that there exists $\rho > 0$ such that

(i) $Uu + Cv \in B_\rho$, for every u, $v \in B_\rho$,

and the operators U, C satisfy the conditions $(ii) - (iv)$ below.

(ii) $U : B_\rho \to X_1$ is a contraction map,

(iii) $C : B_\rho \to X_1$ is continuous,

(iv) $\mathcal{F} = C(B_\rho)$ is relatively compact in X_1.

Proof of (i). Let $\rho > 0$. For every $u \in B_\rho$, for all $x \in \Omega$, we have

$$\begin{aligned}
&\|(Uu)(x)\|_E \le \|g(x)\|_E + \textstyle\int_\Omega \|K(x,y;u(y),D_1D_2u(y),D_2D_1u(y))\|_E\,dy \\
&\le \|g\|_X + \textstyle\int_\Omega \|K(x,y;0,0,0)\|_E\,dy \\
&+ \textstyle\int_\Omega \|K(x,y;u(y),D_1D_2u(y),D_2D_1u(y)) - K(x,y;0,0,0)\|_E\,dy \\
&\le \|g\|_X + \textstyle\int_\Omega \|K(x,y;0,0,0)\|_E\,dy \\
&+ \textstyle\int_\Omega k_0(x,y)\left(\|u(y)\|_E + \|D_1D_2u(y)\|_E + \|D_2D_1u(y)\|_E\right)dy \\
&\le \|g\|_X + \textstyle\int_\Omega \|K(x,y;0,0,0)\|_E\,dy + \|u\|_{X_1}\textstyle\int_\Omega k_0(x,y)dy \\
&\le \|g\|_X + \textstyle\int_\Omega \|K(x,y;0,0,0)\|_E\,dy + \rho\textstyle\int_\Omega k_0(x,y)dy,
\end{aligned} \tag{3.4}$$

so

$$\|Uu\|_X \le \|g\|_X + \sup_{x\in\Omega}\int_\Omega \|K(x,y;0,0,0)\|_E\,dy + \rho\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy. \tag{3.5}$$

On the other hand, for all $x \in \Omega$, we have

$$(D_1(Uu))(x) = D_1g(x) + \int_\Omega D_1K(x,y;u(y),D_1D_2u(y),D_2D_1u(y))dy,$$

Similarly, with $D_1(Uu)$, $D_2(Uu)$, $D_2D_1(Uu)$, $D_1D_2(Uu)$, we get

$$\|D_1(Uu)\|_X \le \|D_1g\|_X + \sup_{x\in\Omega}\int_\Omega \|D_1K(x,y;0,0,0)\|_E\,dy + \rho\sup_{x\in\Omega}\int_\Omega k_1(x,y)dy, \tag{3.6}$$

$$\begin{aligned}
&\|D_2(Uu)\|_X \le \|D_2g\|_X + \sup_{x\in\Omega}\textstyle\int_\Omega \|D_2K(x,y;0,0,0)\|_E\,dy + \rho\sup_{x\in\Omega}\textstyle\int_\Omega k_2(x,y)dy, \\
&\|D_2D_1(Uu)\|_X \le \|D_2D_1g\|_X + \sup_{x\in\Omega}\textstyle\int_\Omega \|D_2D_1K(x,y;0,0,0)\|_E\,dy \\
&\qquad + \rho\sup_{x\in\Omega}\textstyle\int_\Omega k_{21}(x,y)dy, \\
&\|D_1D_2(Uu)\|_X \le \|D_1D_2g\|_X + \sup_{x\in\Omega}\textstyle\int_\Omega \|D_1D_2K(x,y;0,0,0)\|_E\,dy \\
&\qquad + \rho\sup_{x\in\Omega}\textstyle\int_\Omega k_{12}(x,y)dy.
\end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7), it gives

$$\|Uu\|_{X_1} \le \|g\|_{X_1} + \alpha_1^* + \rho\beta_1^*, \tag{3.8}$$

where

$$\begin{aligned}
&\alpha_1^* = \sup_{x\in\Omega}\textstyle\int_\Omega \|K(x,y;0,0,0)\|_E\,dy + \sup_{x\in\Omega}\textstyle\int_\Omega \|D_1K(x,y;0,0,0)\|_E\,dy \\
&+ \sup_{x\in\Omega}\textstyle\int_\Omega \|D_2K(x,y;0,0,0)\|_E\,dy \\
&+ \sup_{x\in\Omega}\textstyle\int_\Omega \|D_2D_1K(x,y;0,0,0)\|_E\,dy + \sup_{x\in\Omega}\textstyle\int_\Omega \|D_1D_2K(x,y;0,0,0)\|_E\,dy, \\
&\beta_1^* = \textstyle\sum_{i=0}^2 \sup_{x\in\Omega}\textstyle\int_\Omega k_i(x,y)dy + \sup_{x\in\Omega}\textstyle\int_\Omega k_{21}(x,y)dy + \sup_{x\in\Omega}\textstyle\int_\Omega k_{12}(x,y)dy.
\end{aligned} \tag{3.9}$$

On the other hand, for every $v \in B_\rho$, for all $x \in \Omega$, we have

$$\begin{aligned}
&\|(Cv)(x)\|_E \le \textstyle\int_\Omega \|H(x,y;v(y),D_1D_2v(y),D_2D_1v(y))\|_E\,dy \\
&\le \textstyle\int_\Omega h_0(x,y)\left(1 + \|v(y)\|_E + \|D_1D_2v(y)\|_E + \|D_2D_1v(y)\|_E\right)dy \\
&\le \left(1 + \|v(y)\|_{X_1}\right)\textstyle\int_\Omega \bar{h}_0(x,y)dy \\
&\le (1+\rho)\sup_{x\in\Omega}\textstyle\int_\Omega \bar{h}_0(x,y)dy.
\end{aligned} \tag{3.10}$$

Thus

$$\|Cv\|_X \leq (1+\rho)\sup_{x\in\Omega}\int_\Omega \bar{h}_0(x,y)dy. \tag{3.11}$$

Similarly, we have

$$\begin{aligned}
&\|D_1(Cv)\|_X \leq (1+\rho)\sup_{x\in\Omega}\textstyle\int_\Omega \bar{h}_1(x,y)dy,\\
&\|D_2(Cv)\|_X \leq (1+\rho)\sup_{x\in\Omega}\textstyle\int_\Omega \bar{h}_2(x,y)dy,\\
&\|D_2D_1(Cv)\|_X \leq (1+\rho)\sup_{x\in\Omega}\textstyle\int_\Omega \bar{h}_{21}(x,y)dy,\\
&\|D_1D_2(Cv)\|_X \leq (1+\rho)\sup_{x\in\Omega}\textstyle\int_\Omega \bar{h}_{12}(x,y)dy.
\end{aligned} \tag{3.12}$$

It implies from (3.11) and (3.12) that

$$\|Cv\|_{X_1} \leq (1+\rho)\,\beta_2^*, \tag{3.13}$$

where

$$\beta_2^* = \sum\nolimits_{i=0}^{2}\sup_{x\in\Omega}\int_\Omega \bar{h}_i(x,y)dy + \sup_{x\in\Omega}\int_\Omega \bar{h}_{21}(x,y)dy + \sup_{x\in\Omega}\int_\Omega \bar{h}_{12}(x,y)dy. \tag{3.14}$$

From (3.8) and (3.13), we obtain

$$\|Uu+Cv\|_{X_1} \leq \|g\|_{X_1} + \alpha_1^* + \beta_2^* + (\beta_1^*+\beta_2^*)\,\rho. \tag{3.15}$$

Choose $\rho \geq \dfrac{\|g\|_{X_1}+\alpha_1^*+\beta_2^*}{1-\beta_1^*-\beta_2^*}$, then $Uu+Cv\in B_\rho$, for every u, $v\in B_\rho$.
Proof of (ii). In view of (A_2), $U: B_\rho \to X_1$ is a contraction map, if we show that

$$\|Uu-Uv\|_{X_1} \leq \beta_1^*\,\|u-v\|_{X_1}\,,\ \forall u,v\in B_\rho. \tag{3.16}$$

For every u, $v\in B_\rho$, for all $x\in\Omega$, using (A_2,i), (3.2) leads to

$$\begin{aligned}
&\|(Uu)(x)-(Uv)(x)\|_E\\
&\leq \int_\Omega \|K(x,y;u(y),D_1D_2u(y),D_2D_1u(y))-K(x,y;v(y),D_1D_2v(y),D_2D_1v(y))\|_E\,dy\\
&\leq \int_\Omega k_0(x,y)[\|u(y)-v(y)\|_E + \|D_1D_2u(y)-D_1D_2v(y)\|_E\\
&+\|D_2D_1u(y)-D_2D_1v(y)\|_E]dy\\
&\leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy.
\end{aligned}$$

Thus

$$\|Uu-Uv\|_X \leq \|u-v\|_{X_1}\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy. \tag{3.17}$$

Similarly, we also have

$$\begin{aligned}
&\|D_1(Uu) - D_1(Uv)\|_X \leq \|u - v\|_{X_1} \sup_{x\in\Omega} \textstyle\int_\Omega k_1(x,y)dy, \\
&\|D_2(Uu) - D_2(Uv)\|_X \leq \|u - v\|_{X_1} \sup_{x\in\Omega} \textstyle\int_\Omega k_2(x,y)dy, \\
&\|D_2D_1(Uu) - D_2D_1(Uv)\|_X \leq \|u - v\|_{X_1} \sup_{x\in\Omega} \textstyle\int_\Omega k_{21}(x,y)dy, \\
&\|D_1D_2(Uu) - D_1D_2(Uv)\|_X \leq \|u - v\|_{X_1} \sup_{x\in\Omega} \textstyle\int_\Omega k_{12}(x,y)dy.
\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), obviously, (3.16) holds.

Proof of (iii). To prove (iii), let $\{u_m\} \subset B_\rho$, $u_0 \in B_\rho$, $\|u_m - u_0\|_{X_1} \to 0$, as $m \to \infty$, we have to prove that

$$\begin{aligned}
&\|Cu_m - Cu_0\|_X \to 0, \\
&\|D_1(Cu_m) - D_1(Cu_0)\|_X \to 0, \\
&\|D_2(Cu_m) - D_2(Cu_0)\|_X \to 0, \\
&\|D_2D_1(Cu_m) - D_2D_1(Cu_0)\|_X \to 0, \\
&\|D_1D_2(Cu_m) - D_1D_2(Cu_0)\|_X \to 0.
\end{aligned} \tag{3.19}$$

Remark that

$$\begin{aligned}
&\|(Cu_m)(x) - (Cu_0)(x)\|_E \\
&\leq \textstyle\int_\Omega ||H(x,y;u_m(y), D_1D_2u_m(y), D_2D_1u_m(y)) \\
&\qquad\qquad -H(x,y;u_0(y), D_1D_2u_0(y), D_2D_1u_0(y))||_E dy.
\end{aligned} \tag{3.20}$$

Put

$$\begin{aligned}
S_1 &= \{u_m(y) : y \in \Omega,\ m = 0,1,2,\cdots\}, \\
S_2 &= \{D_1D_2u_m(y) : y \in \Omega,\ m = 0,1,2,\cdots\}, \\
S_3 &= \{D_2D_1u_m(y) : y \in \Omega,\ m = 0,1,2,\cdots\}.
\end{aligned} \tag{3.21}$$

We prove that S_1, S_2, S_3 are compact in E, because of $\|u_m - u_0\|_{X_1} \to 0$.

(j) S_1 is compact in E.

Indeed, let $\{u_{m_j}(y_j)\}_j$ be a sequence in S_1. We can assume that $\lim_{j\to\infty} y_j = y_0$ and $\lim_{j\to\infty} \left\|u_{m_j} - u_0\right\|_{X_1} = 0$. We have

$$\begin{aligned}
&\left\|u_{m_j}(y_j) - u_0(y_0)\right\|_E \leq \left\|u_{m_j}(y_j) - u_0(y_j)\right\|_E + \|u_0(y_j) - u_0(y_0)\|_E \\
&\leq \left\|u_{m_j} - u_0\right\|_{X_1} + \|u_0(y_j) - u_0(y_0)\|_E \to 0 \text{ as } j \to \infty,
\end{aligned} \tag{3.22}$$

which shows that $\lim_{j\to\infty} u_{m_j}(y_j) = u_0(y_0)$ in E. This means that S_1 is compact in E.

(jj) Similarly S_2, S_3 are also compact in E.

For given $\varepsilon > 0$, since H is uniformly continuous on $\Omega \times \Omega \times S_1 \times S_2 \times S_3$, there exists $\delta > 0$ such that

$$\begin{aligned}
\forall (u,v,w),\ (\bar{u},\bar{v},\bar{w}) \quad &\in \quad S_1 \times S_2 \times S_3,\ \ \|u - \bar{u}\|_E + \|v - \bar{v}\|_E + \|w - \bar{w}\|_E < \delta \\
&\Longrightarrow \quad \|H(x,y;u,v,w) - H(x,y;\bar{u},\bar{v},\bar{w})\|_E < \varepsilon,\ \forall (x,y) \in \Omega \times \Omega.
\end{aligned}$$

We have $\|u_m - u_0\|_X \to 0$, $\|D_1D_2u_m - D_1D_2u_0\|_X \to 0$ and

$$\|D_2D_1u_m - D_2D_1u_0\|_X \to 0,$$

so with $\delta > 0$ as above, there exists $m_0 \in \mathbb{N}$ such that, $\forall m \in \mathbb{N}$, if $m \geq m_0$ then it gives

$$\|u_m - u_0\|_X + \|D_1D_2u_m - D_1D_2u_0\|_X + \|D_2D_1u_m - D_2D_1u_0\|_X < \delta.$$

It implies that there exists $m_0 \in \mathbb{N}$ as above such that $\forall m \in \mathbb{N}$, if $m \geq m_0$ then the following inequality is fulfilled

$$\|H(x, y; u_m(y), D_1D_2u_m(y), D_2D_1u_m(y)) - H(x, y; u_0(y), D_1D_2u_0(y), D_2D_1u_0(y))\|_E < \varepsilon, \ \forall (x, y) \in \Omega \times \Omega.$$

Consequently

$$\|(Cu_m)(x) - (Cu_0)(x)\|_E < \varepsilon \ \forall x \in \Omega, \ \forall m \geq m_0.$$

It means that

$$\|Cu_m - Cu_0\|_X < \varepsilon \ \forall m \geq m_0, \tag{3.23}$$

i.e., $\|Cu_m - Cu_0\|_X \to 0$ as $m \to \infty$.

By the same argument, we obtain that

$$\begin{aligned} &\|D_1(Cu_m) - D_1(Cu_0)\|_X \to 0, \\ &\|D_2(Cu_m) - D_2(Cu_0)\|_X \to 0, \\ &\|D_1D_2(Cu_m) - D_1D_2(Cu_0)\|_X \to 0, \\ &\|D_2D_1(Cu_m) - D_2D_1(Cu_0)\|_X \to 0, \text{ as } m \to \infty. \end{aligned} \tag{3.24}$$

The continuity of C is proved.

Proof of (iv). To prove (iv), we use Lemma 2.2.

The condition (2.9) (i) holds, i.e., the sets

$$\begin{aligned} &C(B_\rho)(x) = \{Cu(x) : u \in B_\rho\}, \\ &D_1C(B_\rho)(x) = \{D_1(Cu)(x) : u \in B_\rho\}, \\ &D_2C(B_\rho)(x) = \{D_2(Cu)(x) : u \in B_\rho\}, \\ &D_1D_2C(B_\rho)(x) = \{D_1D_2(Cu)(x) : u \in B_\rho\}, \\ &\text{and } D_2D_1C(B_\rho)(x) = \{D_2D_1(Cu)(x) : u \in B_\rho\}, \end{aligned} \tag{3.25}$$

are relatively compact in E.

Indeed, put

$$\begin{aligned} &R_1 = \{u(y) : y \in \Omega, \ u \in B_\rho\}, \\ &R_2 = \{D_1D_2u(y) : y \in \Omega, \ u \in B_\rho\}, \\ &R_3 = \{D_2D_1u(y) : y \in \Omega, \ u \in B_\rho\}. \end{aligned} \tag{3.26}$$

Then R_1, R_2, R_3 are bounded in E.

Since $H : \Omega \times \Omega \times E^3 \to E$ is completely continuous, $H(\Omega \times \Omega \times R_1 \times R_2 \times R_3)$ is relatively compact in E. It implies that $\overline{H(\Omega \times \Omega \times R_1 \times R_2 \times R_3)}$ is compact in E. So is $\overline{conv}(H(\Omega \times \Omega \times R_1 \times R_2 \times R_3))$, where $\overline{conv}(H(\Omega \times \Omega \times R_1 \times R_2 \times R_3))$ is the convex closure of $H(\Omega \times \Omega \times R_1 \times R_2 \times R_3)$.

For every $x \in \Omega$, for all $u \in B_\rho$, it follows from

$$H(x, y; u(y), D_1D_2u(y), D_2D_1u(y)) \in H(\Omega \times \Omega \times R_1 \times R_2 \times R_3), \ \forall y \in \Omega, \tag{3.27}$$

that

$$\begin{aligned} &\overline{C(B_\rho)(x)} \subset |\Omega| \, \overline{conv}(H(\Omega \times \Omega \times R_1 \times R_2 \times R_3)) \\ &= \overline{conv}(H(\Omega \times \Omega \times R_1 \times R_2 \times R_3)). \end{aligned} \tag{3.28}$$

Hence, the set $C(B_\rho)(x)$ is relatively compact in E.

Similarly,

$$\begin{aligned}
&\overline{D_1C(B_\rho)(x)} \subset \overline{conv}\,(D_1H\,(\Omega\times\Omega\times R_1\times R_2\times R_3))\,,\\
&\overline{D_2C(B_\rho)(x)} \subset \overline{conv}\,(D_2H\,(\Omega\times\Omega\times R_1\times R_2\times R_3))\,,\\
&\overline{D_1D_2C(B_\rho)(x)} \subset \overline{conv}\,(D_1D_2H\,(\Omega\times\Omega\times R_1\times R_2\times R_3))\,,\\
&\overline{D_2D_1C(B_\rho)(x)} \subset \overline{conv}\,(D_2D_1H\,(\Omega\times\Omega\times R_1\times R_2\times R_3))\,.
\end{aligned} \tag{3.29}$$

Hence the sets $D_1C(B_\rho)(x)$, $D_2C(B_\rho)(x)$, $D_1D_2C(B_\rho)(x)$, $D_2D_1C(B_\rho)(x)$ are relatively compact in E.

The condition (2.9) (ii) also holds.

Indeed, let $\varepsilon>0$ be given. By (A_4), there exists $\delta_1>0$ such that $\forall x,\ \bar{x}\in\Omega$, if $|x-\bar{x}|<\delta_1$ then

$$\begin{aligned}
&[H(x,y;u,v,w)-H(\bar{x},y;u,v,w)]_*\\
&=\|H(x,y;u,v,w)-H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1H(x,y;u,v,w)-D_1H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2H(x,y;u,v,w)-D_2H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2D_1H(x,y;u,v,w)-D_2D_1H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1D_2H(x,y;u,v,w)-D_1D_2H(\bar{x},y;u,v,w)\|_E\\
&<\varepsilon,\ \forall y\in\Omega,\ \forall(u,v,w)\in R_1\times R_2\times R_3.
\end{aligned} \tag{3.30}$$

Hence

$$\begin{aligned}
&[(Cu)(x)-(Cu)(\bar{x})]_*\\
&\le\int_\Omega[H(x,y;u(y),D_1D_2u(y),D_2D_1u(y))-H(\bar{x},y;u(y),D_1D_2u(y),D_2D_1u(y))]_*dy\\
&<\varepsilon.
\end{aligned} \tag{3.31}$$

Using Lemma 2.2, it implies that $\mathcal{F}=C(B_\rho)$ is relatively compact in X_1.

Applying the Krasnosel'skii fixed point theorem (Theorem 2.3), the existence of a solution for (1.1) is proved.

Now, we show that the set of solutions for (1.1),

$$S=\{u\in B_\rho : u=Au\},$$

is compact in X_1.

It is clear that

$$S=\{u\in B_\rho : u=Uu+Cu\}=\{u\in B_\rho : u=(I-U)^{-1}Cu\}, \tag{3.32}$$

so $S=(I-U)^{-1}C(S)$.

Therefore, from the compactness of the operator $C:B_\rho\to B_\rho$ and the continuity of $(I-U)^{-1}:B_\rho\to B_\rho$, we only show that S is closed.

Let $\{u_m\}\subset S$, $u\in X_1$, $\|u_m-u\|_{X_1}\to 0$. The continuity of $(I-U)^{-1}C$ leads to

$$\begin{aligned}
&\left\|u-(I-U)^{-1}Cu\right\|_{X_1}\le\|u-u_m\|_{X_1}+\left\|u_m-(I-U)^{-1}Cu\right\|_{X_1}\\
&=\|u-u_m\|_{X_1}+\left\|(I-U)^{-1}Cu_m-(I-U)^{-1}Cu\right\|_{X_1}\to 0,
\end{aligned} \tag{3.33}$$

so $u=(I-U)^{-1}Cu\in S$. Theorem 3.1 is proved. □

4. An example

In this section, we illustrate the results obtained in Section 3 by the following example.

Let $E = C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $v : [0,1] \to \mathbb{R}$ equipped with the norm

$$\|v\|_E = \sup_{0\le t\le 1} |v(t)|, \; v \in E. \tag{4.1}$$

Let $X = C(\Omega; E)$ be the space of all continuous functions from $\Omega = [0,1]\times[0,1]$ into E equipped with the following norm

$$\|u\|_X = \sup_{x\in\Omega} \|u(x)\|_E, \; u \in X. \tag{4.2}$$

Put

$$X_1 = \{u \in X : D_1 u, \; D_2 u, \; D_2 D_1 u, \; D_1 D_2 u \in X\}. \tag{4.3}$$

Then, for all $u \in X_1$ and $x \in \Omega$, $u(x)$ is an element of E and we denote

$$u(x)(t) = u(x;t), \; 0 \le t \le 1. \tag{4.4}$$

Remark that $C^2(\Omega; E) \subsetneqq X_1 \subsetneqq C^1(\Omega; E)$. We consider (1.1) with the functions K, $H : \Omega\times\Omega\times E^3 \to E$, $g : \Omega \to E$, as the following

(i) Function $K : \Omega\times\Omega\times E^3 \to E$

$$\begin{aligned} &(x,y;u,v,w) \longmapsto K(x,y;u,v,w), \\ &K(x,y;u,v,w)(t) = k(x;t)\Big[(y_1y_2)^{\alpha_1}\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right) \\ &+(y_1y_2)^{\alpha_2}\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right) + (y_1y_2)^{\alpha_3}\cos\left(\tfrac{2\pi w(t)}{D_2D_1\theta_0(y;t)}\right)\Big], \end{aligned} \tag{4.5}$$

$0 \le t \le 1$, $(x,y;u,v,w) \in \Omega\times\Omega\times E^3$, with

$$\begin{cases} k, \, \theta_0 : \Omega \to E, \\ k(x;t) = e^{-t}\left(\left|x_1 - \frac{1}{3}\right|^{\tilde\gamma_1} + \left|x_2 - \frac{1}{3}\right|^{\tilde\gamma_2} + e^{x_1+x_2}\right), \, 0\le t\le 1, \, x\in\Omega, \\ \theta_0(x;t) = e^{-t}\left(\left|x_1 - \frac{1}{2}\right|^{\gamma_1} + \left|x_2 - \frac{1}{2}\right|^{\gamma_2} + e^{x_1+x_2}\right), \, 0\le t\le 1, \, x\in\Omega, \end{cases} \tag{4.6}$$

where $\alpha_1, \, \alpha_2, \, \alpha_3, \, \gamma_1, \, \gamma_2, \, \tilde\gamma_1, \, \tilde\gamma_2$ are positive constants, with $\gamma_1, \, \gamma_2, \, \tilde\gamma_1, \, \tilde\gamma_2 \in (1,2)$.

(ii) Function $H : \Omega\times\Omega\times E^3 \to E$

$$\begin{aligned} &(x,y;u,v,w) \longmapsto H(x,y;u,v,w), \\ &H(x,y;u,v,w)(t) \\ &= h(x;t)\left[(y_1y_2)^{\bar\alpha_1}\int_0^t \left|\tfrac{u(s)}{\theta_0(y;s)}\right|^{1/2} ds + (y_1y_2)^{\bar\alpha_2}\int_0^t \left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3} ds \right. \\ &\qquad \left. +(y_1y_2)^{\bar\alpha_3}\int_0^t \left(\tfrac{w(s)}{D_2D_1\theta_0(y;s)}\right)^{1/5} ds\right], 0\le t\le 1, (x,y;u,v,w)\in\Omega\times\Omega\times E^3, \end{aligned} \tag{4.7}$$

with

$$\begin{aligned} &h : \Omega \to E, \\ &h(x;t) = e^{-t}\left(\left|x_1 - \frac{1}{4}\right|^{\bar\gamma_1} + \left|x_2 - \frac{1}{4}\right|^{\bar\gamma_2} + e^{x_1+x_2}\right), \; 0\le t\le 1, \; x\in\Omega, \end{aligned} \tag{4.8}$$

where $\bar\alpha_2, \, \bar\alpha_3, \, \bar\gamma_1, \, \bar\gamma_2$ are positive constants, with $\bar\gamma_1, \, \bar\gamma_2 \in (1,2)$.

(ii) Function $g:\Omega\to E$,

$$g(x;t)=\theta_0(x;t)-\sum_{i=1}^{3}\left(\tfrac{k(x;t)}{(1+\alpha_i)^2}+\tfrac{h(x;t)}{(1+\bar{\alpha}_i)^2}\right),\ 0\le t\le 1,\ x\in\Omega. \tag{4.9}$$

The above positive constants $\alpha_1,\ \alpha_2,\ \alpha_3,\ \bar{\alpha}_1,\ \bar{\alpha}_2,\ \bar{\alpha}_3,\ \tilde{\gamma}_1,\ \tilde{\gamma}_2,\ \bar{\gamma}_1,\ \bar{\gamma}_2$ satisfying

$$\begin{aligned}&4\pi e\left(2+\tilde{\gamma}_1+\tilde{\gamma}_2+5e^2\right)\sum_{i=1}^{3}\tfrac{1}{(1+\alpha_i)^2}\\ &+e^{1/2}\left(2+\bar{\gamma}_1+\bar{\gamma}_2+5e^2\right)\sum_{i=1}^{3}\tfrac{1}{(1+\bar{\alpha}_i)^2}<1.\end{aligned} \tag{4.10}$$

We now prove that (A_1), (A_2) hold.

It is obvious that (A_1) holds by $\theta_0,\ k,\ h\in X_1$.

Assumption (A_2) holds, it is proved below.

First, we show that $K:\Omega\times\Omega\times E^3\to E$ is continuous. For all $(x,y;u,v,w)$, $(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})\in\Omega\times\Omega\times E^3$, $0\le t\le 1$,

$$\begin{aligned}&K(x,y;u,v,w)(t)-K(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})(t)\\ &=[k(x;t)-k(\bar{x};t)]\Big[(y_1y_2)^{\alpha_1}\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right)+(y_1y_2)^{\alpha_2}\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right)\\ &\qquad+(y_1y_2)^{\alpha_3}\cos\left(\tfrac{4\pi w(t)}{D_2D_1\theta_0(y;t)}\right)\Big]\\ &+k(\bar{x};t)(y_1y_2)^{\alpha_1}\left[\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right)-\sin\left(\tfrac{\pi\bar{u}(t)}{2\theta_0(\bar{y};t)}\right)\right]\\ &+k(\bar{x};t)\left[(y_1y_2)^{\alpha_1}-(\bar{y}_1\bar{y}_2)^{\alpha_1}\right]\sin\left(\tfrac{\pi\bar{u}(t)}{2\theta_0(\bar{y};t)}\right)\\ &+k(\bar{x};t)(y_1y_2)^{\alpha_2}\left[\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right)-\cos\left(\tfrac{2\pi\bar{v}(t)}{D_1D_2\theta_0(\bar{y};t)}\right)\right]\\ &+k(\bar{x};t)\left[(y_1y_2)^{\alpha_2}-(\bar{y}_1\bar{y}_2)^{\alpha_2}\right]\cos\left(\tfrac{2\pi\bar{v}(t)}{D_1D_2\theta_0(\bar{y};t)}\right)\\ &+k(\bar{x};t)(y_1y_2)^{\alpha_3}\left[\cos\left(\tfrac{4\pi w(t)}{D_2D_1\theta_0(y;t)}\right)-\cos\left(\tfrac{4\pi\bar{w}(t)}{D_2D_1\theta_0(\bar{y};t)}\right)\right]\\ &+k(\bar{x};t)\left[(y_1y_2)^{\alpha_3}-(\bar{y}_1\bar{y}_2)^{\alpha_3}\right]\cos\left(\tfrac{4\pi\bar{w}(t)}{D_2D_1\theta_0(\bar{y};t)}\right).\end{aligned} \tag{4.11}$$

We have

$$\begin{aligned}&\theta_0(x;t)=e^{-t}\left(\left|x_1-\tfrac{1}{2}\right|^{\gamma_1}+\left|x_2-\tfrac{1}{2}\right|^{\gamma_2}+e^{x_1+x_2}\right),\\ &D_1\theta_0(x;t)=e^{-t}\left(\gamma_1\left|x_1-\tfrac{1}{2}\right|^{\gamma_1-2}\left(x_1-\tfrac{1}{2}\right)+e^{x_1+x_2}\right),\\ &D_2D_1\theta_0(x;t)=D_1D_2\theta_0(x;t)=e^{-t}e^{x_1+x_2},\ 0\le t\le 1,\ x\in\Omega,\end{aligned} \tag{4.12}$$

so $\theta_0,\ D_1D_2\theta_0,\ D_2D_1\theta_0\in X$ and $\theta_0(x;t)\ge\dfrac{1}{e}$, $D_1D_2\theta_0(x;t)\ge\dfrac{1}{e}$, $D_2D_1\theta_0(x;t)\ge\dfrac{1}{e}$, it follows that

$$\begin{aligned}&\left|K(x,y;u,v,w)(t)-K(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})(t)\right|\\ &\le 3\left\|k(x)-k(\bar{x})\right\|_E+\left\|k(\bar{x})\right\|_E\left|\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right)-\sin\left(\tfrac{\pi\bar{u}(t)}{2\theta_0(\bar{y};t)}\right)\right|\\ &+\left\|k(\bar{x})\right\|_E\left|(y_1y_2)^{\alpha_1}-(\bar{y}_1\bar{y}_2)^{\alpha_1}\right|\\ &+\left\|k(\bar{x})\right\|_E\left|\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right)-\cos\left(\tfrac{2\pi\bar{v}(t)}{D_1D_2\theta_0(\bar{y};t)}\right)\right|\\ &+\left\|k(\bar{x})\right\|_E\left|(y_1y_2)^{\alpha_2}-(\bar{y}_1\bar{y}_2)^{\alpha_2}\right|\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&+\|k(\bar{x})\|_E \left|\cos\left(\frac{4\pi w(t)}{D_2D_1\theta_0(y;t)}\right)-\cos\left(\frac{4\pi \bar{w}(t)}{D_2D_1\theta_0(\bar{y};t)}\right)\right| \\
&+\|k(\bar{x})\|_E \left|(y_1y_2)^{\alpha_3}-(\bar{y}_1\bar{y}_2)^{\alpha_3}\right| \\
\leq\ & 3\|k(x)-k(\bar{x})\|_E+\|k(\bar{x})\|_E \sum_{i=1}^{3}\left|(y_1y_2)^{\alpha_i}-(\bar{y}_1\bar{y}_2)^{\alpha_i}\right| \\
&+\frac{\pi}{2}\|k(\bar{x})\|_E \left|\frac{u(t)}{\theta_0(y;t)}-\frac{\bar{u}(t)}{\theta_0(\bar{y};t)}\right| \\
&+2\pi\|k(\bar{x})\|_E \left|\frac{v(t)}{D_1D_2\theta_0(y;t)}-\frac{\bar{v}(t)}{D_1D_2\theta_0(\bar{y};t)}\right| \\
&+4\pi\|k(\bar{x})\|_E \left|\frac{w(t)}{D_2D_1\theta_0(y;t)}-\frac{\bar{w}(t)}{D_2D_1\theta_0(\bar{y};t)}\right|.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\left|\tfrac{u(t)}{\theta_0(y;t)}-\tfrac{\bar{u}(t)}{\theta_0(\bar{y};t)}\right| = \left|\tfrac{[u(t)-\bar{u}(t)]\theta_0(\bar{y};t)+\bar{u}(t)[\theta_0(\bar{y};t)-\theta_0(y;t)]}{\theta_0(y;t)\theta_0(\bar{y};t)}\right| \\
&\leq e^2\left[\|\theta_0(\bar{y})\|_E\|u-\bar{u}\|_E+\|\bar{u}\|_E\|\theta_0(\bar{y})-\theta_0(y)\|_E\right] \\
&\leq e^2\left[\|\theta_0\|_{X_1}\|u-\bar{u}\|_E+\|\bar{u}\|_E\|\theta_0(\bar{y})-\theta_0(y)\|_E\right].
\end{aligned}
\tag{4.14}
$$

Similarly

$$
\begin{aligned}
&\left|\tfrac{v(t)}{D_1D_2\theta_0(y;t)}-\tfrac{\bar{v}(t)}{D_1D_2\theta_0(\bar{y};t)}\right| \\
&\leq e^2\left[\|D_1D_2\theta_0(\bar{y})\|_E\|v-\bar{v}\|_E+\|\bar{v}\|_E\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E\right] \\
&\leq e^2\left[\|\theta_0\|_{X_1}\|v-\bar{v}\|_E+\|\bar{v}\|_E\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E\right], \\
&\left|\tfrac{w(t)}{D_2D_1\theta_0(y;t)}-\tfrac{\bar{w}(t)}{D_2D_1\theta_0(\bar{y};t)}\right| \\
&\leq e^2\left[\|D_2D_1\theta_0(\bar{y})\|_E\|w-\bar{w}\|_E+\|\bar{w}\|_E\|D_2D_1\theta_0(\bar{y})-D_2D_1\theta_0(y)\|_E\right] \\
&\leq e^2\left[\|\theta_0\|_{X_1}\|w-\bar{w}\|_E+\|\bar{w}\|_E\|D_2D_1\theta_0(\bar{y})-D_2D_1\theta_0(y)\|_E\right].
\end{aligned}
\tag{4.15}
$$

This gives

$$
\begin{aligned}
&\|K(x,y;u,v,w)-K(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})\|_E \\
&\leq 3\|k(x)-k(\bar{x})\|_E+\|k\|_{X_1}\textstyle\sum_{i=1}^{3}\left|(y_1y_2)^{\alpha_i}-(\bar{y}_1\bar{y}_2)^{\alpha_i}\right| \\
&+\frac{\pi e^2}{2}\|k\|_{X_1}\left[\|\theta_0\|_{X_1}\|u-\bar{u}\|_E+\|\bar{u}\|_E\|\theta_0(\bar{y})-\theta_0(y)\|_E\right] \\
&+2\pi e^2\|k\|_{X_1}\left[\|\theta_0\|_{X_1}\|v-\bar{v}\|_E+\|\bar{v}\|_E\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E\right] \\
&+4\pi e^2\|k\|_{X_1}\left[\|\theta_0\|_{X_1}\|w-\bar{w}\|_E+\|\bar{w}\|_E\|D_2D_1\theta_0(\bar{y})-D_2D_1\theta_0(y)\|_E\right] \\
&= 3\|k(x)-k(\bar{x})\|_E+\|k\|_{X_1}\textstyle\sum_{i=1}^{3}\left|(y_1y_2)^{\alpha_i}-(\bar{y}_1\bar{y}_2)^{\alpha_i}\right| \\
&+\pi e^2\|k\|_{X_1}\|\theta_0\|_{X_1}\left[\frac{1}{2}\|u-\bar{u}\|_E+2\|v-\bar{v}\|_E+4\|w-\bar{w}\|_E\right] \\
&+\frac{1}{2}\pi e^2\|k\|_{X_1}\|\bar{u}\|_E\|\theta_0(\bar{y})-\theta_0(y)\|_E \\
&+2\pi e^2\|k\|_{X_1}\|\bar{v}\|_E\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E \\
&+4\pi e^2\|k\|_{X_1}\|\bar{w}\|_E\|D_2D_1\theta_0(\bar{y})-D_2D_1\theta_0(y)\|_E.
\end{aligned}
\tag{4.16}
$$

and the continuity of K is proved.

Similarly, we also have D_1K, D_2K, D_1D_2K, $D_2D_1K : \Omega\times\Omega\times E^3 \to E$ are continuous.

Next, the assumption $(A_2,(i),(ii),(iii),(iv))$ is true by the following.

For all $(x,y;u,v,w)$, $(x,y;\bar{u},\bar{v},\bar{w}) \in \Omega\times\Omega\times E^3$, $0\le t\le 1$,

$$\begin{aligned}
&|K(x,y;u,v,w)(t)-K(x,y;\bar{u},\bar{v},\bar{w})(t)| \\
&\le k(x;t)(y_1y_2)^{\alpha_1}\left|\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right)-\sin\left(\tfrac{\pi\bar{u}(t)}{2\theta_0(y;t)}\right)\right| \\
&+k(x;t)(y_1y_2)^{\alpha_2}\left|\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right)-\cos\left(\tfrac{2\pi\bar{v}(t)}{D_1D_2\theta_0(y;t)}\right)\right| \\
&+k(x;t)(y_1y_2)^{\alpha_3}\left|\cos\left(\tfrac{4\pi w(t)}{D_2D_1\theta_0(y;t)}\right)-\cos\left(\tfrac{4\pi\bar{w}(t)}{D_2D_1\theta_0(y;t)}\right)\right| \\
&\le \frac{\pi}{2}k(x;t)(y_1y_2)^{\alpha_1}\left|\tfrac{u(t)}{\theta_0(y;t)}-\tfrac{\bar{u}(t)}{\theta_0(y;t)}\right| \\
&+2\pi k(x;t)(y_1y_2)^{\alpha_2}\left|\tfrac{v(t)}{D_1D_2\theta_0(y;t)}-\tfrac{\bar{v}(t)}{D_1D_2\theta_0(y;t)}\right| \\
&+4\pi k(x;t)(y_1y_2)^{\alpha_3}\left|\tfrac{w(t)}{D_2D_1\theta_0(y;t)}-\tfrac{\bar{w}(t)}{D_2D_1\theta_0(y;t)}\right| \\
&\le \frac{\pi e}{2}k(x;t)(y_1y_2)^{\alpha_1}|u(t)-\bar{u}(t)|+2\pi e k(x;t)(y_1y_2)^{\alpha_2}|v(t)-\bar{v}(t)| \\
&+4\pi e k(x;t)(y_1y_2)^{\alpha_3}|w(t)-\bar{w}(t)| \\
&\le \frac{\pi e}{2}\|k(x)\|_E(y_1y_2)^{\alpha_1}\|u-\bar{u}\|_E+2\pi e\|k(x)\|_E(y_1y_2)^{\alpha_2}\|v-\bar{v}\|_E \\
&+4\pi e\|k(x)\|_E(y_1y_2)^{\alpha_3}\|w-\bar{w}\|_E \\
&\le 4\pi e\|k(x)\|_E\textstyle\sum_{i=1}^3(y_1y_2)^{\alpha_i}\left[\|u-\bar{u}\|_E+\|v-\bar{v}\|_E+\|w-\bar{w}\|_E\right].
\end{aligned}\tag{4.17}$$

Hence

$$\|K(x,y;u,v,w)-K(x,y;\bar{u},\bar{v},\bar{w})\|_E\le k_0(x,y)\left[\|u-\bar{u}\|_E+\|v-\bar{v}\|_E+\|w-\bar{w}\|_E\right],\tag{4.18}$$

in which

$$k_0(x,y)=4\pi e\|k(x)\|_E\sum_{i=1}^3(y_1y_2)^{\alpha_i}.\tag{4.19}$$

Similarly, because of

$$\begin{aligned}
D_iK(x,y;u,v,w)(t) &= D_ik(x;t)K_1(x,y;u,v,w)(t),\ i=1,2;\\
D_iD_jK(x,y;u,v,w)(t) &= D_iD_jk(x;t)K_1(x,y;u,v,w)(t),\ (i,j)\in\{(1,2),\ (2,1)\},
\end{aligned}$$

where

$$\begin{aligned}
K_1(x,y;u,v,w)(t)=&\Big[(y_1y_2)^{\alpha_1}\sin\left(\tfrac{\pi u(t)}{2\theta_0(y;t)}\right)+(y_1y_2)^{\alpha_2}\cos\left(\tfrac{2\pi v(t)}{D_1D_2\theta_0(y;t)}\right)\\
&+(y_1y_2)^{\alpha_3}\cos\left(\tfrac{4\pi w(t)}{D_2D_1\theta_0(y;t)}\right)\Big],
\end{aligned}\tag{4.20}$$

we have

$$\begin{aligned}
&\|D_iK(x,y;u,v,w)-D_iK(x,y;\bar{u},\bar{v},\bar{w})\|_E\\
&\le k_i(x,y)\left[\|u-\bar{u}\|_E+\|v-\bar{v}\|_E+\|w-\bar{w}\|_E\right],\\
&\|D_iD_jK(x,y;u,v,w)-D_iD_jK(x,y;\bar{u},\bar{v},\bar{w})\|_E\\
&\le k_{ij}(x,y)\left[\|u-\bar{u}\|_E+\|v-\bar{v}\|_E+\|w-\bar{w}\|_E\right],\ (i,j)\in\{(1,2),\ (2,1)\},
\end{aligned}\tag{4.21}$$

with

$$\begin{aligned}
&k_i(x,y)=4\pi e\|D_ik(x)\|_E\textstyle\sum_{i=1}^3(y_1y_2)^{\alpha_i},\ i=1,2,\\
&k_{ij}(x,y)=4\pi e\|D_iD_jk(x)\|_E\textstyle\sum_{i=1}^3(y_1y_2)^{\alpha_i},\ (i,j)\in\{(1,2),\ (2,1)\}.
\end{aligned}\tag{4.22}$$

Thus, assumption (A_2) holds.
Assumption (A_3) also holds, the proof is as below.
Indeed, we first show that $H:\Omega\times\Omega\times E^3\to E$ is continuous.

For all $(x, y; u, v, w)$, $(\bar{x}, \bar{y}; \bar{u}, \bar{v}, \bar{w}) \in \Omega \times \Omega \times E^3$, $0 \le t \le 1$,

$$\begin{aligned}
&H(x,y;u,v,w)(t) - H(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})(t)\\
&= [h(x;t) - h(\bar{x};t)] \left[(y_1y_2)^{\bar{\alpha}_1} \int_0^t \left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} ds \right.\\
&\left. +(y_1y_2)^{\bar{\alpha}_2} \int_0^t \left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right)^{1/3} ds + (y_1y_2)^{\bar{\alpha}_3} \int_0^t \left(\tfrac{w(s)}{D_2D_1\theta_0(y;s)} \right)^{1/5} ds \right]\\
&+h(\bar{x};t) \left[(y_1y_2)^{\bar{\alpha}_1} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_1} \right] \int_0^t \left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} ds\\
&+h(\bar{x};t)(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_1} \int_0^t \left(\left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} - \left| \tfrac{\bar{u}(s)}{\theta_0(\bar{y};s)} \right|^{1/2} \right) ds\\
&+h(\bar{x};t) \left[(y_1y_2)^{\bar{\alpha}_2} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_2} \right] \int_0^t \left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right)^{1/3} ds\\
&+h(\bar{x};t) \left[(y_1y_2)^{\bar{\alpha}_2} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_2} \right] \int_0^t \left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right)^{1/3} ds\\
&+h(\bar{x};t)(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_2} \int_0^t \left[\left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right)^{1/3} - \left(\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)} \right)^{1/3} \right] ds\\
&+h(\bar{x};t) \left[(y_1y_2)^{\bar{\alpha}_3} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_3} \right] \int_0^t \left(\tfrac{w(s)}{D_2D_1\theta_0(y;s)} \right)^{1/5} ds\\
&+h(\bar{x};t)(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_3} \int_0^t \left[\left(\tfrac{w(s)}{D_2D_1\theta_0(y;s)} \right)^{1/5} - \left(\tfrac{\bar{w}(s)}{D_2D_1\theta_0(\bar{y};s)} \right)^{1/5} \right] ds.
\end{aligned} \tag{4.23}$$

By

$$\theta_0(x;s) \ge \frac{1}{e},\ D_1D_2\theta_0(x;s) \ge \frac{1}{e},\ D_2D_1\theta_0(x;s) \ge \frac{1}{e},$$

it follows that

$$\begin{aligned}
&\|H(x,y;u,v,w) - H(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})\|_E\\
&\le \|h(x) - h(\bar{x})\|_E \left[\int_0^1 \left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} ds + \int_0^1 \left| \tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right|^{1/3} ds \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \int_0^1 \left| \tfrac{w(s)}{D_2D_1\theta_0(y;s)} \right|^{1/5} ds \right]\\
&+ \|h(\bar{x})\|_E \left| (y_1y_2)^{\bar{\alpha}_1} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_1} \right| \int_0^1 \left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} ds\\
&+ \|h(\bar{x})\|_E \int_0^1 \left| \left| \tfrac{u(s)}{\theta_0(y;s)} \right|^{1/2} - \left| \tfrac{\bar{u}(s)}{\theta_0(\bar{y};s)} \right|^{1/2} \right| ds\\
&+ \|h(\bar{x})\|_E \left| (y_1y_2)^{\bar{\alpha}_2} - (\bar{y}_1\bar{y}_2)^{\bar{\alpha}_2} \right| \int_0^1 \left| \tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right|^{1/3} ds\\
&+ \|h(\bar{x})\|_E \int_0^1 \left| \left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)} \right)^{1/3} - \left(\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)} \right)^{1/3} \right| ds
\end{aligned} \tag{4.24}$$

$$
\begin{aligned}
& +\|h(\bar{x})\|_E \left|(y_1y_2)^{\bar{\alpha}_3}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_3}\right| \int_0^1 \left|\frac{w(s)}{D_2D_1\theta_0(y;s)}\right|^{1/5} ds \\
& +\|h(\bar{x})\|_E \int_0^1 \left|\left(\frac{w(s)}{D_2D_1\theta_0(y;s)}\right)^{1/5}-\left(\frac{\bar{w}(s)}{D_2D_1\theta_0(\bar{y};s)}\right)^{1/5}\right| ds \\
\leq\ & \|h(x)-h(\bar{x})\|_E \left[e^{1/2}\|u\|_E^{1/2}+e^{1/3}\|v\|_E^{1/3}+e^{1/5}\|w\|_E^{1/5}\right] \\
& +\|h\|_{X_1}\left|(y_1y_2)^{\bar{\alpha}_1}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_1}\right| e^{1/2}\|u\|_E^{1/2} \\
& +\|h\|_{X_1}\int_0^1 \left|\left|\frac{u(s)}{\theta_0(y;s)}\right|^{1/2}-\left|\frac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right|^{1/2}\right| ds \\
& +\|h\|_{X_1}\left|(y_1y_2)^{\bar{\alpha}_2}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_2}\right| e^{1/3}\|v\|_E^{1/3} \\
& +\|h\|_{X_1}\int_0^1 \left|\left(\frac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3}-\left(\frac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right)^{1/3}\right| ds \\
& +\|h\|_{X_1}\left|(y_1y_2)^{\bar{\alpha}_3}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_3}\right| e^{1/5}\|w\|_E^{1/5} \\
& +\|h\|_{X_1}\int_0^1 \left|\left(\frac{w(s)}{D_2D_1\theta_0(y;s)}\right)^{1/5}-\left(\frac{\bar{w}(s)}{D_2D_1\theta_0(\bar{y};s)}\right)^{1/5}\right| ds \\
\leq\ & e^{1/2}\left[\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}\right] \\
& \times\left[\|h(x)-h(\bar{x})\|_E+\|h\|_{X_1}\sum_{i=1}^3\left|(y_1y_2)^{\bar{\alpha}_i}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_i}\right|\right] \\
& +\|h\|_{X_1}\int_0^1 \left|\left|\frac{u(s)}{\theta_0(y;s)}\right|^{1/2}-\left|\frac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right|^{1/2}\right| ds \\
& +\|h\|_{X_1}\int_0^1 \left|\left(\frac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3}-\left(\frac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right)^{1/3}\right| ds \\
& +\|h\|_{X_1}\int_0^1 \left|\left(\frac{w(s)}{D_2D_1\theta_0(y;s)}\right)^{1/5}-\left(\frac{\bar{w}(s)}{D_2D_1\theta_0(\bar{y};s)}\right)^{1/5}\right| ds \\
=\ & R_1+R_2+R_3+R_4.
\end{aligned}
$$

We estimate the terms on the right - hand side of (4.24) as follows.
Estimating R_1. It is easy to see that

$$
\begin{aligned}
R_1 = {} & e^{1/2}(\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}) \\
& \times(\|h(x)-h(\bar{x})\|_E+\|h\|_{X_1}\textstyle\sum_{i=1}^3\left|(y_1y_2)^{\bar{\alpha}_i}-(\bar{y}_1\bar{y}_2)^{\bar{\alpha}_i}\right|) \\
& \to 0, \text{ as } |x-\bar{x}|+|\bar{y}-y|\to 0.
\end{aligned} \tag{4.25}
$$

Estimating R_2. We have

$$
\begin{aligned}
& \left|\tfrac{u(s)}{\theta_0(y;s)}-\tfrac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right| = \left|\tfrac{[\theta_0(\bar{y};s)-\theta_0(y;s)]u(s)+\theta_0(y;s)[u(s)-\bar{u}(s)]}{\theta_0(y;s)\theta_0(\bar{y};s)}\right| \\
& \leq e^2\left[\|\theta_0(\bar{y})-\theta_0(y)\|_E\|u\|_E+\|\theta_0\|_{X_1}\|u-\bar{u}\|_E\right].
\end{aligned} \tag{4.26}
$$

Applying the following inequalities

$$\begin{aligned} &||a|^q-|b|^q|\le |a-b|^q \ \ \forall a,b\in\mathbb{R},\ \forall q\in(0,1],\\ &(a+b)^q\le a^q+b^q \ \forall a,b\ge 0,\forall q\in(0,1], \end{aligned} \tag{4.27}$$

we obtain

$$\begin{aligned} &\left|\left|\frac{u(s)}{\theta_0(y;s)}\right|^{1/2}-\left|\frac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right|^{1/2}\right|\le\left|\tfrac{u(s)}{\theta_0(y;s)}-\tfrac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right|^{1/2}\\ &\le e\left[\|\theta_0(\bar{y})-\theta_0(y)\|_E\|u\|_E+\|\theta_0\|_{X_1}\|u-\bar{u}\|_E\right]^{1/2}\\ &\le e\left[\|\theta_0(\bar{y})-\theta_0(y)\|_E^{1/2}\|u\|_E^{1/2}+\|\theta_0\|_{X_1}^{1/2}\|u-\bar{u}\|_E^{1/2}\right]. \end{aligned} \tag{4.28}$$

Thus

$$\begin{aligned} &R_2=\|h\|_{X_1}\textstyle\int_0^1\left|\left|\dfrac{u(s)}{\theta_0(y;s)}\right|^{1/2}-\left|\dfrac{\bar{u}(s)}{\theta_0(\bar{y};s)}\right|^{1/2}\right|ds\\ &\le e\|h\|_{X_1}\left[\|\theta_0(\bar{y})-\theta_0(y)\|_E^{1/2}\|u\|_E^{1/2}+\|\theta_0\|_{X_1}^{1/2}\|u-\bar{u}\|_E^{1/2}\right]\to 0 \end{aligned} \tag{4.29}$$

as $|\bar{y}-y|+\|u-\bar{u}\|_E\to 0$.

Estimating R_3. Similarly

$$\begin{aligned} &\left|\tfrac{v(s)}{D_1D_2\theta_0(y;s)}-\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right|\\ &\le e^2\left[\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E\|v\|_E+\|\theta_0\|_{X_1}\|v-\bar{v}\|_E\right]. \end{aligned} \tag{4.30}$$

Applying the following inequalities

$$\begin{aligned} &\left||a|^{q-1}a-|b|^{q-1}b\right|\le 2^{1-q}|a-b|^q \ \ \forall a,b\in\mathbb{R},\ \forall q\in(0,1],\\ &(a+b)^q\le a^q+b^q \ \forall a,b\ge 0,\ \forall q\in(0,1], \end{aligned} \tag{4.31}$$

we obtain

$$\begin{aligned} &\left|\left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3}-\left(\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right)^{1/3}\right|\\ &=\left|\left|\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right|^{-2/3}\left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right)-\left|\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right|^{-2/3}\left(\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right)\right|\\ &\le 2^{2/3}\left|\tfrac{v(s)}{D_1D_2\theta_0(y;s)}-\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right|^{1/3}\\ &\le 2^{2/3}e^{2/3}\left[\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E\|v\|_E+\|\theta_0\|_{X_1}\|v-\bar{v}\|_E\right]^{1/3}\\ &\le 2^{2/3}e^{2/3}\left[\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E^{1/3}\|v\|_E^{1/3}+\|\theta_0\|_{X_1}^{1/3}\|v-\bar{v}\|_E^{1/3}\right]. \end{aligned} \tag{4.32}$$

It implies that

$$\begin{aligned} &R_3=\|h\|_{X_1}\textstyle\int_0^1\left|\left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3}-\left(\tfrac{\bar{v}(s)}{D_1D_2\theta_0(\bar{y};s)}\right)^{1/3}\right|ds\\ &\le 2^{2/3}e^{2/3}\|h\|_{X_1}\left[\|D_1D_2\theta_0(\bar{y})-D_1D_2\theta_0(y)\|_E^{1/3}\|v\|_E^{1/3}+\|\theta_0\|_{X_1}^{1/3}\|v-\bar{v}\|_E^{1/3}\right]\to 0, \end{aligned} \tag{4.33}$$

as $|\bar{y}-y|+\|v-\bar{v}\|_E\to 0$.

Estimating R_4. Similarly

$$\begin{aligned}
&R_4 = \|h\|_{X_1} \int_0^1 \left| \left(\frac{w(s)}{D_2D_1\theta_0(y;s)} \right)^{1/5} - \left(\frac{\bar{w}(s)}{D_2D_1\theta_0(\bar{y};s)} \right)^{1/5} \right| ds \\
&\leq 2^{4/5}e^{2/5} \|h\|_{X_1} \left[\|D_2D_1\theta_0(\bar{y}) - D_2D_1\theta_0(y)\|_E^{1/5} \|w\|_E^{1/5} + \|\theta_0\|_{X_1}^{1/5} \|w-\bar{w}\|_E^{1/5} \right] \\
&\to 0, \quad as\, |\bar{y}-y| + \|w-\bar{w}\|_E \to 0.
\end{aligned} \tag{4.34}$$

It follows from (4.25), (4.29), (4.33), (4.34) that

$$\|H(x,y;u,v,w) - H(\bar{x},\bar{y};\bar{u},\bar{v},\bar{w})\|_E \leq \sum\nolimits_{i=1}^4 R_i \to 0, \tag{4.35}$$

as $|x-\bar{x}| + |\bar{y}-y| + \|u-\bar{u}\|_E + \|v-\bar{v}\|_E + \|w-\bar{w}\|_E \to 0$, and the continuity of H is proved.

Similarly, we also have $D_1H,\ D_2H,\ D_1D_2H,\ D_2D_1H : \Omega \times \Omega \times E^3 \to E$ are continuous.

Now, $(A_3, (i), (ii), (iii), (iv))$ holds by the following.

Applying the inequality $a^q \leq 1 + a\ \forall a \geq 0,\ \forall q \in (0,1]$, we obtain

$$\begin{aligned}
&|H(x,y;u,v,w)(t)| \\
&\leq \|h(x)\|_E \Big[(y_1y_2)^{\bar{\alpha}_1} e^{1/2} \int_0^1 |u(s)|^{1/2} ds + (y_1y_2)^{\bar{\alpha}_2} e^{1/3} \int_0^1 |v(s)|^{1/3} ds \\
&\qquad + (y_1y_2)^{\bar{\alpha}_3} e^{1/5} \int_0^1 |w(s)|^{1/5} ds \Big] \\
&\leq e^{1/2} \|h(x)\|_E \left[(y_1y_2)^{\bar{\alpha}_1} (1+\|u\|_E) + (y_1y_2)^{\bar{\alpha}_2} (1+\|v\|_E) + (y_1y_2)^{\bar{\alpha}_3} (1+\|w\|_E) \right] \\
&\leq e^{1/2} \|h(x)\|_E \sum\nolimits_{i=1}^3 (y_1y_2)^{\bar{\alpha}_i} \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right).
\end{aligned} \tag{4.36}$$

It leads to

$$\|H(x,y;u,v,w)\|_E \leq \bar{h}_0(x,y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right), \tag{4.37}$$

in which

$$\bar{h}_0(x,y) = e^{1/2} \|h(x)\|_E \sum\nolimits_{i=1}^3 (y_1y_2)^{\bar{\alpha}_i}. \tag{4.38}$$

Similarly,

$$\begin{aligned}
&\|D_iH(x,y;u,v,w)\|_E \leq \bar{h}_i(x,y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right),\ i = 1,2, \\
&\|D_iD_jH(x,y;u,v,w)\|_E \leq \bar{h}_{ij}(x,y) \left(1 + \|u\|_E + \|v\|_E + \|w\|_E\right), \\
&(i,j) \in \{(1,2),(2,1)\},
\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}
&\bar{h}_i(x,y) = e^{1/2} \|D_ih(x)\|_E \sum\nolimits_{i=1}^3 (y_1y_2)^{\bar{\alpha}_i},\ i = 1,2, \\
&\bar{h}_{ij}(x,y) = e^{1/2} \|D_iD_jh(x)\|_E \sum\nolimits_{i=1}^3 (y_1y_2)^{\bar{\alpha}_i},\ (i,j) \in \{(1,2),\ (2,1)\}.
\end{aligned} \tag{4.40}$$

We have

$$\begin{aligned}
&\int_\Omega k_0(x,y)dy \leq 4\pi e \left(2 + e^2\right) \sum\nolimits_{i=1}^3 \tfrac{1}{(1+\alpha_i)^2}; \\
&\int_\Omega k_i(x,y)dy \leq 4\pi e \left(\tilde{\gamma}_i + e^2\right) \sum\nolimits_{i=1}^3 \tfrac{1}{(1+\alpha_i)^2},\ i = 1,2; \\
&\int_\Omega k_{ij}(x,y)dy \leq 4\pi e^3 \sum\nolimits_{i=1}^3 \tfrac{1}{(1+\alpha_i)^2},\ (i,j) \in \{(1,2),\ (2,1)\};
\end{aligned} \tag{4.41}$$

$$\begin{aligned}&\int_\Omega \bar{h}_0(x,y)dy \le e^{1/2}\left(2+e^2\right)\textstyle\sum_{i=1}^3 \frac{1}{(1+\bar{\alpha}_i)^2};\\&\int_\Omega \bar{h}_i(x,y)dy \le e^{1/2}\left(\bar{\gamma}_i+e^2\right)\textstyle\sum_{i=1}^3 \frac{1}{(1+\bar{\alpha}_i)^2},\ i=1,2;\\&\int_\Omega \bar{h}_{ij}(x,y)dy \le e^{5/2}\textstyle\sum_{i=1}^3 \frac{1}{(1+\bar{\alpha}_i)^2}, (i,j)\in\{(1,2),\ (2,1)\}.\end{aligned}\tag{4.42}$$

It is easy to see that

$$\begin{aligned}\beta_1^* &= \sum_{i=0}^2 \sup_{x\in\Omega}\int_\Omega k_i(x,y)dy + \sup_{x\in\Omega}\int_\Omega k_{21}(x,y)dy + \sup_{x\in\Omega}\int_\Omega k_{12}(x,y)dy\\&\le 4\pi e\left(2+\tilde{\gamma}_1+\tilde{\gamma}_2+5e^2\right)\sum_{i=1}^3 \tfrac{1}{(1+\alpha_i)^2},\\ \beta_2^* &= \sum_{i=0}^2 \sup_{x\in\Omega}\int_\Omega \bar{h}_i(x,y)dy + \sup_{x\in\Omega}\int_\Omega \bar{h}_{21}(x,y)dy + \sup_{x\in\Omega}\int_\Omega \bar{h}_{12}(x,y)dy\\&\le e^{1/2}\left(2+\bar{\gamma}_1+\bar{\gamma}_2+5e^2\right)\sum_{i=1}^3 \tfrac{1}{(1+\bar{\alpha}_i)^2}.\end{aligned}$$

Hence

$$\begin{aligned}&\beta_1^*+\beta_2^* \le 4\pi e\left(2+\tilde{\gamma}_1+\tilde{\gamma}_2+5e^2\right)\textstyle\sum_{i=1}^3 \frac{1}{(1+\alpha_i)^2}\\&+e^{1/2}\left(2+\bar{\gamma}_1+\bar{\gamma}_2+5e^2\right)\textstyle\sum_{i=1}^3 \frac{1}{(1+\bar{\alpha}_i)^2} < 1.\end{aligned}\tag{4.43}$$

Thus, assumption (A_5) holds. Assumption (A_4) also holds, the proof is as below.
(a) Prove $H:\Omega\times\Omega\times E^3\to E$ is completely continuous.
By $H\in C(\Omega\times\Omega\times E^3;E)$, we have to prove that $H:\Omega\times\Omega\times E^3\to E$ is compact. Let B be bounded in $\Omega\times\Omega\times E^3$. We have

$$\begin{aligned}&\|H(x,y;u,v,w)\|_E \le \bar{h}_0(x,y)\left(1+\|u\|_E+\|v\|_E+\|w\|_E\right)\\&\le \sup_{(x,y;u,v,w)\in B}\bar{h}_0(x,y)\left(1+\|u\|_E+\|v\|_E+\|w\|_E\right)\\&\le 3e^{1/2}\|h\|_{X_1}\sup_{(x,y;u,v,w)\in B}\left(1+\|u\|_E+\|v\|_E+\|w\|_E\right)\equiv M_1.\end{aligned}\tag{4.44}$$

for all $(x,y;u,v,w)\in B$, which implies that $H(B)$ is uniformly bounded in E.
For all $t,\bar{t}\in[0,1]$, for all $(x,y;u,v,w)\in B$, put

$$\begin{aligned}&\tilde{H}(y;u,v,w)(t) = (y_1y_2)^{\bar{\alpha}_1}\int_0^t\left|\tfrac{u(s)}{\theta_0(y;s)}\right|^{1/2}ds + (y_1y_2)^{\bar{\alpha}_2}\int_0^t\left(\tfrac{v(s)}{D_1D_2\theta_0(y;s)}\right)^{1/3}ds\\&+(y_1y_2)^{\bar{\alpha}_3}\int_0^t\left(\tfrac{w(s)}{D_2D_1\theta_0(y;s)}\right)^{1/5}ds,\end{aligned}\tag{4.45}$$

we have

$$\begin{aligned}&|H(x,y;u,v,w)(t)-H(x,y;u,v,w)(\bar{t})|\\&=\left|h(x;t)\tilde{H}(y;u,v,w)(t)-h(x;\bar{t})\tilde{H}(y;u,v,w)(\bar{t})\right|\\&\le|h(x;t)-h(x;\bar{t})||\tilde{H}(y;u,v,w)(t)|\\&+|h(x;\bar{t})||\tilde{H}(y;u,v,w)(t)-\tilde{H}(y;u,v,w)(\bar{t})|.\end{aligned}\tag{4.46}$$

On the other hand

$$
\begin{aligned}
&|h(x;\bar{t})| \leq \left(2+e^2\right);\\
&|h(x;t)-h(x;\bar{t})| \leq \left(2+e^2\right)|t-\bar{t}|;\\
&\left|\tilde{H}(y;u,v,w)(t)-\tilde{H}(y;u,v,w)(\bar{t})\right| \leq e^{1/2}\left(\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}\right)|t-\bar{t}|;\\
&\left|\tilde{H}(y;u,v,w)(t)\right| \leq e^{1/2}\left(\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}\right).
\end{aligned}
\tag{4.47}
$$

Thus

$$
\begin{aligned}
&|H(x,y;u,v,w)(t)-H(x,y;u,v,w)(\bar{t})|\\
&\leq 2\left(2+e^2\right)e^{1/2}\left(\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}\right)|t-\bar{t}|\\
&\leq C\,|t-\bar{t}| \text{ for all } (x,y;u,v,w)\in B \text{ and } t,\ \bar{t}\in[0,1].
\end{aligned}
\tag{4.48}
$$

Consequently, $H(B)$ is equicontinuous.

(b) Similarly, we also have $D_1H,\ D_2H,\ D_1D_2H,\ D_2D_1H : \Omega\times\Omega\times E^3 \to E$ are completely continuous.

(c) Finally, for all bounded subset J of E^3, for all $\varepsilon>0$, there exists $\delta>0$ such that

$$
\begin{aligned}
\forall x,\ \bar{x}\in\Omega,\ |x-\bar{x}|<\delta \Longrightarrow\ &\|H(x,y;u,v,w)-H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1H(x,y;u,v,w)-D_1H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2H(x,y;u,v,w)-D_2H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1D_2H(x,y;u,v,w)-D_1D_2H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2D_1H(x,y;u,v,w)-D_2D_1H(\bar{x},y;u,v,w)\|_E<\varepsilon,\\
\forall y\in\Omega, \forall(u,v,w)\in J.&
\end{aligned}
\tag{4.49}
$$

Indeed, we get the above property since

$$
\begin{aligned}
&\|H(x,y;u,v,w)-H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1H(x,y;u,v,w)-D_1H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2H(x,y;u,v,w)-D_2H(\bar{x},y;u,v,w)\|_E\\
&+\|D_1D_2H(x,y;u,v,w)-D_1D_2H(\bar{x},y;u,v,w)\|_E\\
&+\|D_2D_1H(x,y;u,v,w)-D_2D_1H(\bar{x},y;u,v,w)\|_E\\
&\leq e^{1/2}\left(\|u\|_E^{1/2}+\|v\|_E^{1/3}+\|w\|_E^{1/5}\right)[\ \|h(x)-h(\bar{x})\|_E\\
&\qquad+\|D_1h(x)-D_1h(\bar{x})\|_E+\|D_2h(x)-D_2h(\bar{x})\|_E\\
&+\|D_1D_2h(x)-D_1D_2h(\bar{x})\|_E+\|D_2D_1h(x)-D_2D_1h(\bar{x})\|_E\ ]\\
&\leq C\left[\|h(x)-h(\bar{x})\|_E+\|D_1h(x)-D_1h(\bar{x})\|_E+\|D_2h(x)-D_2h(\bar{x})\|_E\right.\\
&\left.+\|D_1D_2h(x)-D_1D_2h(\bar{x})\|_E+\|D_2D_1h(x)-D_2D_1h(\bar{x})\|_E\right],
\end{aligned}
\tag{4.50}
$$

$\forall y\in\Omega$, $\forall(u,v,w)\in J$, $\forall x,\ \bar{x}\in\Omega$, where $h,\ D_1h,\ D_2h,\ D_1D_2h,\ D_2D_1h:\Omega\to E$ are uniformly continuous on Ω.

The assumptions from Theorem 3.1 are fulfilled and we see that $\theta_0\in X_1$ is a solution of the corresponding integral equation (1.1). □

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