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ON FIXED POINT THEOREMS AND APPLICATIONS TO PRODUCT OF n-NONLINEAR INTEGRAL OPERATORS IN IDEAL SPACES

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Abstract. The object of the present article is twofold. Firstly, we prove some fixed point theorems for the product of n-given operators need not be Banach algebras, which generalize and extend the existing results. Secondly, we apply the achieved results in proving the existence of solutions for product of n-nonlinear integral equations in ideal spaces (Orlicz spaces and Lebesgue spaces). That results shall be easily applied for numerous problems in different Banach spaces.
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1. Introduction

This article is devoted to solve (via fixed point methods) the problem

$$x(t) = \prod_{i=1}^{n} H_i x(t)$$

in general Banach spaces need not be Banach algebras, where $H_i,\ i = 1, \cdots, n$ are general known operators.

The existing results of such kinds of problems were discussed on Banach algebras ([18]). This method leads to some extra restrictive assumptions on the growth of studied operators. We exceed these difficulties by considering two cases. First, when the operators are contractions concerning to some measures of noncompactness. Second, at least one of the studied operators should be a contraction concerning the measure of uniform integrability c, which is a general condition.

In particular, we apply our fixed points in finding the solutions to the equation

$$x(t) = \prod_{i=1}^{n} \left(g_i(t) + \lambda_i \cdot \int_a^b K_i(t,s) f_i(s, x(s))\, ds \right),\ t \in [a,b] \tag{1.1}$$

in some ideal spaces such as Lebesgue spaces $L_p,\ 1 \le p < \infty$ and Orlicz spaces whose generating functions satisfy Δ_2-condition ([1, 9, 11, 12, 21, 23] for some particular cases).

It is worth noting that, for $n = 2$, equation (1.1) arises in the study of the spread of an infectious disease that does not induce permanent immunity ([6, 17]). In [22] the authors discussed the existence and uniqueness of a continuous solution to the following integral equation

$$x(t) = \prod_{i=1}^{n} \left(g_i(t) + \int_a^t K_i(t, s, x(s))\, ds \right),\ t \in [a, b]$$

and the existence of integrable solution was studied in [5] for the equation

$$x(t) = f(t, x(t)) + \prod_{i=1}^{n} f_i\left(t,\ \int_a^t K_i(t, s, x(s))\, ds\right),\ t > 0.$$

This article is motivated by extending the previous studies by proving some fixed point theorems for the product of n-operators in arbitrary Banach spaces and apply such results to discuss the solvability of equation (1.1) in ideal spaces (Orlicz spaces and Lebesgue spaces).

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^+ = [0, \infty)$, and $I = [a, b] \subset \mathbb{R}$.

We will recall some concepts of ideal spaces (or: Köthe function spaces).

Definition 2.1. [25] A normed space $(X, \|\cdot\|)$ of (classes of) measurable functions $x : I \to U$ (U is a normed space) is called pre-ideal if for each $x \in X$ and each measurable $y : I \to U$ the relation $|y(s)| \le |x(s)|$ (for almost all $s \in I$) implies $y \in X$ and $\|y\| \le \|x\|$. If X is also complete, it is called an ideal space.

Remark 2.2. An ideal normed space X is called regular if all singletons in X have equicontinuous norm, i.e. $\lim_{\delta\to 0} \sup_{meas D \le \delta} \|x \cdot \chi_D\| = 0$, where χ_D is the characteristic function of a measurable set D.

Remark 2.3. The Lebesgue spaces, the Orlicz spaces, or the Lorentz spaces (with suitable norms) are examples of regular ideal spaces. While the space of continuous functions $C(I)$ is not ideal, although it is a closed subspace of an ideal space ([24]).

Next, we will present some concepts of an important example of ideal spaces namely Orlicz spaces ([19]).

A continuous, convex function, $M : \mathbb{R} \to \mathbb{R}^+$, is called N-function if it is even and if it satisfies both $\lim_{u\to 0} \frac{M(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$.

Equivalently, M is N-function if and only if it takes the form

$$M(u) = \int_0^{|u|} p(t)\, dt,\ \forall u \in \mathbb{R},$$

where $p : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right-continuous function and positive for $t > 0$, which satisfies the conditions $p(0) = 0,\ \lim_{t\to\infty} p(t) = \infty$.

If $q : \mathbb{R}^+ \to \mathbb{R}^+$ be the right-inverse of p, that is, if $q(s) = \sup\{t : p(t) \leq s\}$, $\forall s \in \mathbb{R}^+$, then $N : \mathbb{R} \to \mathbb{R}^+$ given by $N(v) = \int_0^{|v|} q(s)\, ds$, $\forall v \in \mathbb{R}$, is also N-function, and M and N are called mutually complementary.

The Orlicz class, denoted by $\mathcal{O}_M$, contains measurable functions $x : I \to \mathbb{R}$ for which

$$\rho(x; M) = \int_I M(x(t))dt < \infty.$$

Denote by $L_M(I)$ the *Orlicz space* of all measurable functions $x : I \to \mathbb{R}$ for which

$$\|x\|_M = \inf_{\epsilon > 0} \left\{ \int_I M\left(\frac{x(s)}{\epsilon}\right)\, ds \leq 1 \right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Moreover, E_M spaces be a class of functions from L_M having absolutely continuous norms.

Note that $E_M \subseteq L_M \subseteq \mathcal{O}_M$. The inclusion $L_M \subset L_P$ holds if, and only if, there exists positive constants u_0 and a such that $P(u) \leq aM(u)$ for $u \geq u_0$.

Moreover, we have $E_M = L_M = \mathcal{O}_M$ if M satisfies the Δ_2-condition, i.e.

Definition 2.4. [19] The N-function M is said to satisfy Δ_2-condition if there exist $\omega,\ t_0 \geq 0$ such that for $t \geq t_0$, we have $M(2t) \leq \omega M(t)$.

The N-function $M(u) = \exp u^2 - 1$ satisfies this condition, while the function $M(u) = \exp|u| - |u| - 1$ does not.

Definition 2.5. [19] Assume that a function $f :\ I \times \mathbb{R} \ \to\ \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F_f(x)(t) \ = \ f(t, x(t)),\ t \ \in\ I.$$

The operator F_f in such a way is called the superposition operator generated by the function f.

Lemma 2.6. [19, Theorem 17.6] *Assume that a function $f :\ I \times \mathbb{R} \ \to\ \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F_f maps $E_{M_1}(I) \to L_{M_2}(I) = E_{M_2}(I)$ is continuous and bounded if and only if*

$$|f(s,x)| \leq a(s) + bM_2^{-1}\big(M_1(x)\big),$$

where $b \geq 0$ and $a \in E_{M_2}(I)$ in which the N-function $M_2(x)$ satisfies the Δ_2-condition.

We will say that a set T in an ideal space E is compact in measure if it is compact in the topology of convergence in measure, i.e. as a subset of the space of all measurable functions $L^0(I)$ (see [15, 8]).

Lemma 2.7. [10] *Assume, that a bounded set U is a subset of the regular ideal space E of real-valued functions over a bounded interval I such that all the functions from U are a.e. monotonic. Then this set is compact in measure in the space E.*

For Orlicz spaces $L_M(I)$ we have the following:

Lemma 2.8. [11] *Let X be a bounded subset of $L_M(I)$. Assume that, there is a family of subsets $(\Omega_c)_{0\leq c\leq b-a}$ of the interval I such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$,*

$$x(t_1) \geq x(t_2),\ (t_1 \in \Omega_c,\ t_2 \notin \Omega_c).$$

Then X is compact in measure in $L_M(I)$.

Lemma 2.9. [20, Theorem 6.2] *The operator $K_0x(t) = \int_I K(t,s)x(s)\,ds$ preserves the monotonicity of functions if and only if*

$$\int_0^l K(t_1,s)\,ds \geq \int_0^l K(t_2,s)\,ds$$

for $t_1 < t_2$, $t_1, t_2 \in I$ and for any $l \in I$.

For the product of n-operators in Orlicz spaces, we have the following theorem:

Lemma 2.10. [16, Theorem 2.1] *Let $n \geq 2$. If φ and φ_i are arbitrary N-functions for $i = 1, 2, \cdots n$, then the following statements are equivalent:*

(1) *For every $u_i \in L_{\varphi_i}(I)$, then $\prod_{i=1}^n u_i \in L_\varphi(I)$.*
(2) *There exists a constant $k > 0$ such that*

$$\left\| \prod_{i=1}^n u_i \right\|_\varphi \leq k \prod_{i=1}^n \|u_i\|_{\varphi_i},$$

for every $u_i \in L_{\varphi_i}(I),\ i = 1, 2, \cdots n$.

(3) *There exists a constant $C > 0$ such that*

$$\prod_{i=1}^n \varphi_i^{-1}(t) \leq C\varphi^{-1}(t)$$

for every $t \geq 0$.

(4) *There exists a constant $C > 0$ such that for all $t_i \geq 0,\ i = 1, \cdots n$,*

$$\varphi\left(\frac{\prod_{i=1}^n t_i}{C}\right) \leq \sum_{i=1}^m \varphi_i(t_i).$$

The Lebesgue spaces $L_p(I)$ can be treated as Orlicz spaces $L_{M_p}(I)$ with N-function $M_p = \frac{t^p}{p}$, which satisfies the Δ_2-condition. Further, $L_p(I),\ 1 \leq p < \infty$ represent a regular ideal space and we have the following corollary.

Corollary 2.11. [7, 16] *Let $n \geq 2$. If $1 \leq p, p_i < \infty$ for $i = 1, \cdots, n$, then the following statements are equivalent:*

(1) $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$.

(2) $\left\| \prod_{i=1}^n u_i \right\|_p \leq \prod_{i=1}^n \|u_i\|_{p_i}$ *for every $u_i \in L_{p_i}(I),\ i = 1, \cdots, n$.*

(3) *For every $u_i \in L_{p_i}(I)$, then $\prod_{i=1}^n u_i \in L_p(I)$.*

3. Measure of noncompactness

Assume that $(E, \|\cdot\|)$ be arbitrary Banach spaces with zero element θ. Denote by B_r the closed ball with radius r and centered at θ. The symbol $B_r(E)$ is to point out the space. Moreover, by $\mathcal{M}_E$ we denote the family of all nonempty and bounded subsets of the Banach space E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets. If $X \subset E$, then $\bar{X}$ and $convX$ indicate the closure and convex closure of X, respectively.

Definition 3.1. [3] A mapping $\mu : \mathcal{M}_E \iff [0, \infty)$ is called a measure of noncompactness in E if it satisfies:
(i) $\mu(X) = 0 \Rightarrow X \in \mathcal{N}_E$.
(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(iii) $\mu(\bar{X}) = \mu(convX) = \mu(X)$.
(iv) $\mu(\lambda X) = |\lambda|\, \mu(X),\ for\ \lambda \in \mathbb{R}$.
(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
(vi) $\mu(X \bigcup Y) = \max\{\mu(X),\ \mu(Y)\}$.
(vii) If X_l is a sequence of nonempty, bounded, closed subsets of E such that $X_{l+1} \subset X_l$, $l = 1, 2, 3, \cdots$, and $\lim_{l\to\infty} \mu(X_l) = 0$, then the set $X_\infty = \bigcap_{l=1}^{\infty} X_l$ is nonempty.

The kernel of the measure μ i.e. "$ker\ \mu$" is the family of sets A with $\mu(A) = 0$. Let us give an example:

Definition 3.2. [3] The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as follows

$\beta_H(X) = inf\{r > 0 :$ there exists a finite subset Y of E such that $X \subset Y + B_r\}$,

where X is an arbitrary nonempty and bounded subset of E.

Let c denote the measure of uniform integrability of the set X in an ideal function space E on the compact interval I (introduced in [2], see also [25, Definition 3.9] or [14]):

$$c(X) = \limsup_{\varepsilon\to 0} \sup_{mesD\leq\varepsilon} \sup_{x\in X} \|x \cdot \chi_D\|_E, \tag{3.1}$$

where χ_D denotes the characteristic function of a measurable subset $D \subset I$.

Proposition 3.3. [14, Theorem 1] *Let X be a nonempty, bounded, and compact in measure subset of an ideal regular space E. Then*

$$\beta_H(X) = c(X).$$

Theorem 3.4. [3] *Let Q be a nonempty, bounded, closed, and convex subset of E and let $V : Q \to Q$ be a continuous transformation which is a contraction concerning to the measure of noncompactness μ, i.e. there exists $k \in [0, 1)$ such that*

$$\mu(V(X)) \leq k\mu(X),$$

for any nonempty subset X of E. Then V has at least one fixed point in the set Q and the set $FixV$ of all fixed points of V satisfies $\mu(FixV) = 0$.

Definition 3.5. [4, 18] Let μ be a measure of noncompactness in E. We say that μ satisfies condition (m) if

$$\mu_E(N_1N_2) \leq \|N_2\| \cdot \mu_{E_1}(N_1) + \|N_1\| \cdot \mu_{E_2}(N_2),\ N_1 \subset E_1,\ N_2 \subset E_2.$$

This definition was extended to n-product of operators as follows:

Definition 3.6. [18] Let μ be a measure of noncompactness in E satisfying condition (m). Let $\{N_i\}_{i=1,\ldots,n}$ be a finite sequence in $\mathcal{M}_E, n \geq 2$. Then

$$\mu_E\left(\prod_{i=1}^{n} N_i\right) \leq \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \|N_j\| \cdot \mu_{E_i}(N_i). \tag{3.2}$$

4. Main results

First we prove some fixed points for product of n-operators. Let us consider appropriate types of measures of noncompactness μ_E on E, μ_{E_i} on $E_i,\ i = 1, \cdots n$ satisfying the axioms from Definition 3.1, where $E, E_i,\ i = 1, \cdots n$ are arbitrary Banach spaces not necessary Banach algebras. We will assume, that the internal operators have values in some intermediate spaces $E_i,\ i = 1, \cdots n$ and then the product will return to the target space E.

4.1. **Fixed point theorems.** We discuss the existence of fixed point $x \in Q$ of the problem

$$x = Hx = \prod_{i=1}^{n} H_i x, \tag{4.1}$$

for $Q \neq \phi$ and $H_i : Q \to E_i,\ i = 1, \cdots, n,\ n \geq 1$ are given operators.
We have the following fixed point results.

Theorem 4.1. *Let $E, E_i,\ i = 1, \cdots n$ be Banach spaces. Assume that Q is nonempty, bounded, closed, and convex subset of the Banach space E. Moreover, assume that the operators $H_i : E \to E_i,\ i = 1, \cdots n$ and that:*

- **(A1):** *H_i transforms continuously the set Q into $Q_i \subset E_i$ and H_iQ is bounded in E_i, for $i = 1, \cdots n$.*
- **(A2):** *$HQ \subset Q$, where $H = \prod_{i=1}^{n} H_i$.*
- **(A3):** *There exists a constant k such that for arbitrary $x_i \in E_i$, the product $\prod_{i=1}^{n} x_i \in E$ and*
$$\left\| \prod_{i=1}^{n} x_i \right\|_E \leq k\ \prod_{i=1}^{n} \|x_i\|_{E_i}.$$
- **(A4):** *There exist constants $k_i > 0$ such that H_i satisfies the inequality:*
$$\mu_{E_i}(H_i(U)) \leq k_i\ \mu_E(U),\ i = 1, \cdots n$$
for arbitrary bounded subset U of E,
- **(A5):** *$\sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|H_jQ\|_{E_j} < 1.$*

Then problem 4.1 has at least one solution in Q and that the set of all fixed points of H i.e. Fix H is relatively compact in E.

Proof. First, since the operators $H_i : E \to E_i$ for $i = 1, \cdots n$, then by using assumption (A3) and Lemma 2.10, we have that the operator $H = \prod_{i=1}^{n} H_i : E \to E$. Moreover, by assumptions (A1) and (A2), the operator $H : Q \to Q$ is bounded.
Let (x_k) be an arbitrary sequence in Q convergent to $x \in Q$, then we have

$$\begin{aligned}
\|H(x_k) \;-\; H(x)\|_E = & \left\| \prod_{i=1}^{n} H_i(x_k) \;-\; \prod_{i=1}^{n} H_i x \right\|_E \\
\leq & \left\| H_1(x_k) \cdots H_n(x_k) \;-\; H_1(x_k) \cdots H_{n-1}(x_k) H_n x \right\|_E \\
+ & \left\| H_1(x_k) \cdots H_{n-1}(x_k) H_n x \;-\; H_1(x_k) \cdots H_{n-2}(x_k) H_{n-1} x H_n x \right\|_E \\
+ & \cdots + \left\| H_1(x_k) H_2 x \cdots H_n x \;-\; H_1 x H_2 x \cdots H_n x \right\|_E \\
\leq & \; k \prod_{i=1}^{n-1} \|H_i(x_k)\|_{E_i} \|H_n(x_k) \;-\; H_n x\|_{E_n} \\
+ & \; k \prod_{i=1}^{n-2} \|H_i(x_k)\|_{E_i} \|H_n x\|_{E_n} \|H_{n-1}(x_k) \;-\; H_{n-1} x\|_{E_{n-1}} \\
+ & \cdots + k \prod_{i=2}^{n} \|H_i x\|_{E_i} \|H_1(x_k) \;-\; H_1 x\|_{E_1}.
\end{aligned}$$

From our assumptions, it follows that H is sequentially continuous, so it is continuous from Q into E.
Now, we will investigate the contraction property for the measure of noncompactness $\mu_E(X)$.
Assume that $\phi \neq X \subset Q$ and fix an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned}
\mu_E(HX) &= \mu_E\left(\prod_{i=1}^{n} H_i X \right) \\
&\leq \sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \|H_j X\|_{E_j} \mu_{E_i}(H_i X) \\
&\leq \sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|H_j X\|_{E_j} \mu_E(X) \\
&\leq \left(\sum_{i=1}^{n} k_i \prod_{j=1, j\neq i}^{n} \|H_j Q\|_{E_j} \right) \mu_E(X).
\end{aligned}$$

Then we can apply Theorem 3.4. It follows that $\mu_E(FixH) = 0$, hence, $FixH$ is relatively compact. This accomplishes the proof. □

Remark 4.2. **(1):** If $n = 1$, then Theorem 4.1 is reduced to Theorem 3.4.

(2): If $n = 2$, these results reduced to the results introduced in [4] in Banach algebras i.e. $E = E_1 = E_2 = C(I) = C(I, \mathbb{R})$ $(k = 1)$ (see also [13], for instance).

(3): If $n = 2$, these results reduced to the results introduced in [10, 8] for arbitrary Banach spaces E, E_1, E_2 need not be Banach algebra.

(4): If $n = 1, \cdots, n,\ n \geq 2$, these results were discussed in [18] in the case of Banach algebra $E = E_i = C(I, \mathbb{R})$.

Next, we generalize the above theorem by assuming that at least one of the studied operators is a contraction concerning the measure of uniform integrability c as in definition (3.1).

Theorem 4.3. *Let $E,\ E_i,\ i = 1, \cdots, n$, be regular ideal spaces. Assume that Q is nonempty, bounded, closed, convex and compact in measure subset of E, and the operators $H_i : E \to E_i,\ i = 1, \cdots, n$, and that:*

(B1): *H_i transforms continuously the set Q into $Q_i \subset E_i$ and H_iQ is bounded in $E_i,\ i = 1, \cdots, n$.*

(B2): *$HQ \subset Q$, where $H = \prod_{i=1}^{n} H_i$.*

(B3): *There exists a constant k such that for arbitrary $x_i \in E_i$, the product $\prod_{i=1}^{n} x_i \in E$ and*

$$\left\| \prod_{i=1}^{n} x_i \right\|_E \leq k \prod_{i=1}^{n} \|x_i\|_{E_i}.$$

(B4): *There exist constants $k_i > 0$ such that H_i satisfies the inequality:*

$$c_{E_i}(H_i(U)) \leq k_i\ c_E(U),\ i = 1, \cdots, n$$

for arbitrary bounded $U \subset E$.

(B5): $k \cdot k_n \prod_{i=1}^{n-1} k_i \cdot r^i < 1,\ r > 0.$

Then there exists at least one fixed point for the operator H in the set Q and the set of all fixed points of H, i.e. $Fix\ H$ is relatively compact in E.

Proof. It is obvious that by assumption (B3) the operator H is well defined and by assumptions (B1), (B2) it maps Q into itself.

The proof of the continuity of H is as in Theorem 4.1.

Now, we will investigate the contraction property for the measure of noncompactness $c_E(X)$.

Assume that $\phi \neq X \subset Q$ and fix an arbitrary $\varepsilon > 0$. Then for any $x \in X$ and for a set $D \subset I$, meas$D \leq \varepsilon$ we obtain

$$\|(Hx) \cdot \chi_D\|_E = \left\| \prod_{i=1}^{n} (H_i x) \cdot \chi_D \right\|_E \leq k \prod_{i=1}^{n} \|(H_i x) \cdot \chi_D\|_{E_i}.$$

Since for any non-negative real-valued functions $f = \prod_{i=1}^{n} f_i$, we have $\sup_I\ f \leq \prod_{i=1}^{n} \sup_I f_i$, by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $measD \leq \varepsilon$ we get

$$\sup_{measD \leq \varepsilon} \sup_{x \in X} \|H(x) \cdot \chi_D\|_E \leq k \cdot \prod_{i=1}^{n} \sup_{measD \leq \varepsilon} \sup_{x \in X} \|(H_i x) \cdot \chi_D\|_{E_i}.$$

Therefore,

$$c_E(HX) \leq k \cdot \prod_{i=1}^{n} c_{E_i}(H_i X) \leq \Big(k \cdot \prod_{i=1}^{n} k_i\Big) c_E^n(X).$$

As in [25, p. 66], for any $x \in X$, we have

$$\|x\chi_D\|_E \leq \|x\|_E \cdot \|\chi_D\|_\infty = \|x\|_E \leq r,\ r > 0,$$

then we have

$$c_E(HX) \leq \Big(k \cdot k_n \prod_{i=1}^{n-1} k_i \cdot r^i\Big) c_E(X).$$

Recall that under our assumptions, the operator H maps set Q being compact in measure into itself. Because $\phi \neq X \subset Q$ is a nonempty, bounded and compact in measure subset of the regular ideal space E, we can use Proposition 3.3 and then

$$\beta_H(HX) \leq \Big(k \cdot k_n \prod_{i=1}^{n-1} k_i \cdot r^i\Big) \beta_H(X).$$

The inequality obtained above together with the properties of the operator H and the set Q established before, allow us to apply Theorem 3.4. This accomplishes the proof. □

4.2. **Applications to n-product of Hammerstein integral equations.** In this section we will discuss the existence of solutions of equation (1.1) in $L_p(I),\ 1 \leq p < \infty$ and in Orlicz spaces when their generating functions satisfy Δ_2-conditions i.e. in (regular ideal spaces).

Rewrite equation (1.1) in the form

$$x(t) = Hx(t) = \prod_{i=1}^{n} H_i x(t),$$

where

$$H_i(x) = g_i(t) + \lambda_i \cdot A_i(x) \text{ and } A_i(x)(t) = \int_a^b K_i(t,s) f_i(s, x(s))\, ds.$$

We shall stress on the assumptions of the considered functions to nominate the intermediate spaces, in which our results are in the target spaces $L_\varphi(I)$ or $L_p(I)$.

4.2.1. *The case of Orlicz spaces.* We will characterize the case, which permits us to get more general growth conditions on the studied functions.

Theorem 4.4. *Let $i = 1, \cdots n$, and assume, that φ, φ_i, are N-functions and that M_i and N_i are complementary N-functions. Moreover, put the following set of assumptions:*

(N1) *There exists a constant $k > 0$ such that for $v_i \in L_{\varphi_i}(I),\ i = 1, \cdots, n$, we have $\| \prod_{i=1}^{n} v_i\|_\varphi \leq k \prod_{i=1}^{n} \|v_i\|_{\varphi_i}$.*

(C1) *$g_i \in E_{\varphi_i}(I),\ i = 1, \cdots, n$, are a.e. nondecreasing on I,*

(C2) *$f_i :\ I \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions and $f_i(t,x),\ i = 1, \cdots, n$, are assumed to be nondecreasing with respect to both variables t and x separately,* (C3) $|f_i(t,x)| \le b_i(t) + d_i N_i^{-1}\Big(\varphi(x)\Big)$ *for* $t \in I$ *and* $x \in \mathbb{R}$*, where* $b_i \in E_{N_i}(I)$ *and* $d_i \ge 0$ *in which the N-functions* $N_i(x)$ *satisfy the* Δ_2*-condition for* $i = 1, \cdots, n$.

(K1) $s \to K_i(t,s) \in L_{M_i}(I)$ *for a.e.* $t \in I$ *and* $p_i(t) = \|K_i(t,\cdot)\|_{M_i} \in E_{\varphi_i}(I)$, $i = 1, \cdots, n$.

(K2) $\int_a^b K_i(t_1,s)\,ds \ge \int_a^b K_i(t_2,s)\,ds$, $i = 1, \cdots, n$, *for* $t_1, t_2 \in [a,b]$ *with* $t_1 < t_2$.

(K3) *Assume that for some* $q_i > 0$*, there exists* $r > 0$ *on the interval* I *such that*

$$k\prod_{i=1}^{n}\int_I \varphi_i\Bigg(|g_i(t)| + \ q_i \cdot |p_i(t)|\Bigg(\|b_i\|_{N_i} \ + \ d_i \cdot r\Bigg)\Bigg)\,dt \le r.$$

If $\prod_{i=1}^n \Bigg(d_i|\lambda_i| \cdot \|p_i\|_{\varphi_i}\Bigg) < \frac{1}{kr^{n-1}}$*, then there exist numbers* $\rho_i > 0$ *such that for all* $\lambda_i \in \mathbb{R}$ *with* $|\lambda_i| < \rho_i$*,* $i = 1, \cdots, n$*, there exists a solution* $x \in E_\varphi(I)$ *of (1.1) which is a.e. nondecreasing on* I.

Proof. **Step I.** Let $i = 1, \cdots, n$. Assumptions (C2), (C3) and Lemma 2.6 imply that the operators F_{f_i} map continuously $B_1(E_\varphi(I)) \to E_{N_i}(I)$. The operators A_i are continuous mappings from the unit ball $B_1(E_\varphi(I))$ into $E_{\varphi_i}(I)$ by using [19, Lemma 16.3 and Theorem 16.3] (with $M_1 = N_i, M_2 = \varphi_i$ and $N_1 = M_i$). By assumption (C1) the operators $H_i : \ B_1(E_\varphi(I)) \ \to \ E_{\varphi_i}(I)$ are continuous. Finally, by assumption (N1) we can deduce that the operator $H : \ B_1(E_\varphi(I)) \ \to \ E_\varphi(I)$ is continuous.

Step II. We will construct an invariant set $V \subset B_1(E_\varphi(I))$ for the operator H is bounded in $L_\varphi(I)$.

Fix $\lambda_i \in \mathbb{R}$ with $|\lambda_i| < \rho_i$ and let $\rho_i = \sup Q$, where Q is the set of all positive numbers q_i for which there exists $r > 0$ such that

$$k\prod_{i=1}^{n}\int_I \varphi_i\Bigg(|g_i(t)| + \ q_i \cdot |p_i(t)|\Bigg(\|b_i\|_{N_i} \ + d_i \cdot r\Bigg)\Bigg)\,dt \ \le r.$$

Let V denote the closure of the set $\{x \in E_\varphi(I) : \int_a^b \varphi(|x(s)|)\,ds \le r - 1\}$. Clearly V is not a ball in $E_\varphi(I)$, but $V \subset B_r(E_\varphi(I))$ ([19, p. 222]). Notice that $\overline{V}$ is a bounded closed and convex subset of $E_\varphi(I)$.

Take an arbitrary $x \in V$. By using ([19, Theorem 10.5 with $k = 1$]), we obtain that for any $t \in I$

$$\Bigg\|N_i^{-1}\big(\varphi\left(|x|\right)\big)\Bigg\|_{N_i} \le 1 + \int_a^b \varphi\left(|x(s)|\right)\,ds \tag{4.2}$$

and then by the Hölder inequality and our assumptions we get

$$|A_i(x)(t)| \le |p_i(t)|\Bigg(\|b_i\|_{N_i} \ + \ d_i\Bigg\|N_i^{-1}\big(\varphi(|x|)\big)\Bigg\|_{N_i}\Bigg).$$

Thus for any measurable subset T of I. For arbitrary $x \in V$ and $t \in I$, we have

$$\begin{aligned}|H_i(x)(t)| &\le |g_i(t)| + |\lambda_i| \cdot |A_i(x)(t)| \\ &\le |g_i(t)| + |\lambda_i| \cdot |p_i(t)| \left(\|b_i\|_{N_i} + d_i \left\| N_i^{-1}\big(\varphi(|x|)\big) \right\|_{N_i} \right) \\ &\le |g_i(t)| + |\lambda_i| \cdot |p_i(t)| \left(\|b_i\|_{N_i} + d_i + d_i \int_a^b \varphi\left(|x(s)|\right) \, ds \right) \\ &\le |g_i(t)| + |\lambda_i| \cdot |p_i(t)| \Big(\|b_i\|_{N_i} + d_i + d_i(r-1) \Big).\end{aligned}$$

Therefore, by using assumption (N1) and see also [16], we have

$$\begin{aligned}\int_I \varphi\big(H(x)(t)\big) \, dt \le k \prod_{i=1}^n \int_I \varphi_i\big(H_i(x)(t)\big) \, dt \le k \prod_{i=1}^n \int_I \varphi_i \Big(&|g_i(t)| \\ &+ |\lambda_i| \cdot |p_i(t)| \big(\|b_i\|_{N_i} + d_i \cdot r \big) \Big) \, dt.\end{aligned}$$

By the definition of r we get $\int_I \varphi(H(x)(t)) \, dt \le r$ and then $H(V) \subset V$. Consequently $H(\overline{V}) \subset \overline{H(V)} \subset \overline{V} = V$, which implies $H : V \to V$ is continuous on $V \subset B_r(E_\varphi(I))$.

Step III. Let Q_r stands for the subset of V consisting of all functions which are a.e. nondecreasing on I. Similarly as claimed in [9] this set is nonempty, bounded, convex and closed set in $L_\varphi(I)$. Moreover, in view of Lemma 2.8 the set Q_r is compact in measure.

Step IV. Now, we show, that H preserve the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nondecreasing on I and consequently F_{f_i} is also of the same type in virtue of the assumption (C2). Further, $A_i(x)$ is a.e. nondecreasing on I thanks for the assumption (K2). Assumption (C1) permits us to deduce that H_i is also a.e. nondecreasing on I. Then, by assumption (N1) we have $H : Q_r \to Q_r$ is continuous.

Step V. Next, we prove that H is a contraction concerning the measure of noncompactness. Recall that for $x \in B_1(E_\varphi(I))$ we have

$$\int_I N_i \left(N_i^{-1} \big[\varphi\big(x(s)\big) \big] \right) ds = \int_I \varphi\big(x(s)\big) \, ds \le \|x\|_\varphi.$$

Assume that $X \subset Q_r$ is a nonempty and let $\varepsilon > 0$ be arbitrary fixed constant. Then for an arbitrary $x \in X$ and for a set $D \subset I$, meas $D \le \varepsilon$, we obtain

$$\begin{aligned}\|H_i(x) \cdot \chi_D\|_\varphi &\le \|g_i \cdot \chi_D\|_{\varphi_i} + |\lambda_i| \cdot \|A_i(x) \cdot \chi_D\|_{\varphi_i} \\ &\le \|g_i \cdot \chi_D\|_{\varphi_i} + |\lambda_i| \cdot \|p_i\|_{\varphi_i} \left(\|b_i \cdot \chi_D\|_{N_i} + d_i \left\| N_i\big(\varphi(|x(s)|)\big) \cdot \chi_D \right\|_{N_i} \right) \\ &\le \|g_i \cdot \chi_D\|_{\varphi_i} + |\lambda_i| \cdot \|p_i\|_{\varphi_i} \Big(\|b_i \cdot \chi_D\|_{N_i} + d_i \|x \cdot \chi_D\|_\varphi \Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\|H(x)\cdot\chi_D\|_\varphi &\leq k\prod_{i=1}^{n}\|H_i(x)\cdot\chi_D\|_{\varphi_i}\leq k\prod_{i=1}^{n}\Bigg(\|g_i\chi_D\|_{\varphi_i}\\ &+|\lambda_i|\cdot\|p_i\|_{\varphi_i}\Big(\|b_i\cdot\chi_D\|_{N_i}+d_i\|x\cdot\chi_D\|_\varphi\Big)\Bigg).\end{aligned}$$

Hence, taking into account that $g_i\in E_{\varphi_i}$, $b_i\in E_{N_i}$, then

$$\lim_{\varepsilon\to 0}\left\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|g_i\chi_D\|_{\varphi_i}\}]\right\}=0 \text{ and } \lim_{\varepsilon\to 0}\left\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|b_i\chi_D\|_{N_i}\}]\right\}=0.$$

Thus by definition of $c(x)$ and by taking the supremum over all $x\in X$ and all measurable subsets D with $measD\leq\varepsilon$, we get

$$c(H(X))\leq k\prod_{i=1}^{n}\Big(d_i|\lambda_i|\cdot\|p_i\|_{\varphi_i}\Big)r^{n-1}c(X).$$

Since $X\subset Q_r$ is a nonempty, bounded and compact in measure subset of an ideal regular space E_φ, we can use Proposition 3.3 and get

$$\beta_H(B(X))\leq kr^{n-1}\prod_{i=1}^{n}\Big(d_i|\lambda_i|\cdot\|p_i\|_{\varphi_i}\Big)\beta_H(X).$$

The above inequality with $\prod_{i=1}^{n}\Big(d_i|\lambda_i|\cdot\|p_i\|_{\varphi_i}\Big)<\frac{1}{kr^{n-1}}$ allow us to apply Theorem 4.3, which fulfills the proof. □

4.2.2. *The case of Lebesgue spaces.* The Lebesgue spaces are interesting example of ideal spaces which is discussed in many different monographs. Moreover, the L_p-solutions still represent general solutions than those discussed in previous studies.

Assume that $\frac{1}{p}=\sum_{i=1}^{n}\frac{1}{p_i}$ and consider the following conditions:

(i) $g_i\in L_{p_i}(I),\ i=1,\cdots,n$ be a.e. nondecreasing functions on I.

(ii) The functions $f_i:\ I\times\mathbb{R}\ \to\ \mathbb{R}$ satisfy Carathéodory conditions and there exist positive constants d_i and functions $b_i\in L_{p_i}(I)$ such that

$$|f_i(t,x)|\leq b_i(t)+d_i|x|^{\frac{p}{p_i}},\ i=1,\cdots,n.$$

Moreover, $f_i(t,x),\ i=1,\cdots,n$ are assumed to be nondecreasing with respect to both variables t and x separately.

(iii) The linear integral operators $K_{0_i}x(t)=\int_a^b K_i(t,s)x(s)\ ds$ map $L_{p_i}(I)\to L_{p_i}(I),\ i=1,\cdots,n$.

(iv) $\int_a^b K_i(t_1,s)\ ds\geq\int_a^b K_i(t_2,s)\ ds,\ i=1,\cdots,n$, for $t_1,t_2\in[a,b]$ with $t_1<t_2$.

Remark 4.5. Let us stress, that the condition (iii) implies that the kernels $K_i(t,s)$ should be of Hille-Tamarkin classes i.e. $\big\|\|K_i(t,\cdot)\|_{p_i'}\big\|_{p_i}$, which it is sufficient to suppose that they are finite and being the upper bounds for $\|K_{0i}\|_{p_i',p_i}$, where $\frac{1}{p_i}+\frac{1}{p_i'}=1$.

Theorem 4.6. *Let the assumptions (i) - (iv) be satisfied. If*

$$\prod_{i=1}^{n}\Big(d_i|\lambda_i|\cdot\|K_{0_i}\|_{p'_i,p_i}\Big)<1,$$

then there exist numbers $\rho_i>0$ such that for all $\lambda_i\in\mathbb{R}$ with $|\lambda_i|<\rho_i,\ i=1,\cdots,n$, there exists a solution $x\in L_p(I)$ of (1.1) which is a.e. nondecreasing on I.

Proof. **Step I'.** Let $i=1,\cdots,n$. Assumption (ii) implies that F_{f_i} map $L_p(I)$ into $L_{p_i}(I)$ continuously. The operators A_i map $L_p(I)$ into $L_{p_i}(I)$ continuously (thanks to assumption (iii)). Assumption (i) gives us that the operators $H_i(x)$ map continuously $L_p(I)$ into $L_{p_i}(I)$. By using Corollary 2.11, we can deduce that, the operator

$$H=\prod_{i=1}^{n}H_i$$

maps continuously $L_p(I)$ into itself.

Step II'. Fix $\lambda_i\in\mathbb{R}$ with $|\lambda_i|<\rho_i$, where

$$\rho_i=\frac{\alpha^{\frac{1}{n}}-\|g_i\|_{p_i}}{\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\alpha^{\frac{p}{p_i}}\ \big)},\ \ i=1,\cdots,n.$$

Let $B_R=\{x\in L_p(I):\ \ \|x\|_p\le R\}$, where R is a positive number satisfying the inequality

$$\prod_{i=1}^{n}\Big(\|g_i\|_{p_i}+|\lambda_i|\cdot\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\cdot R^{\frac{p}{p_i}}\ \big)\Big)\le R.$$

Now, for $x\in L_p(I)$ and by using assumptions (i) - (iii) with $\frac{1}{p_i}+\frac{1}{p'_i}=1$, we have

$$\begin{aligned}
\|H_i(x)\|_{p_i} &\le \|g_i\|_{p_i}+|\lambda_i|\cdot\|A_ix\|_{p_i}\\
&\le \|g_i\|_{p_i}+|\lambda_i|\cdot\left\|\int_a^b K_i(t,s)\big(\ b_i(s)+d_i\cdot|x(s)|^{\frac{p}{p_i}}\ \big)\ ds\right\|_{p_i}\\
&\le \|g_i\|_{p_i}+|\lambda_i|\cdot\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\cdot\left\|x^{\frac{p}{p_i}}\right\|_{p_i}\ \big)\\
&\le \|g_i\|_{p_i}+\rho_i\cdot\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\cdot\|x\|_p^{\frac{p}{p_i}}\ \big),
\end{aligned}$$

where $\|x^{\frac{p}{p_i}}\|_{p_i}=\|x\|_p^{\frac{p}{p_i}}$. By using Corollary 2.11, we have

$$\begin{aligned}
\|H(x)\|_p &= \left\|\prod_{i=1}^{n}H_i(x)\right\|_p\\
&\le \prod_{i=1}^{n}\Big(\|g_i\|_{p_i}+\rho_i\cdot\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\cdot\|x\|_p^{\frac{p}{p_i}}\ \big)\Big)\\
&\le \prod_{i=1}^{n}\Big(\|g_i\|_{p_i}+\rho_i\cdot\|K_{0_i}\|_{p'_i,p_i}\big(\ \|b_i\|_{p_i}+d_i\cdot R^{\frac{p}{p_i}}\ \big)\Big)\le R.
\end{aligned}$$

Then, we can deduce that $H:B_R\to B_R$ is continuous.

Step III' and **Step IV'** are similar to those from Theorem 4.4 with $Q_R \subset B_R(L_p)$.

Step V'. Assume that $X \subset Q_R$ is nonempty set and let $\varepsilon > 0$ be arbitrary fixed constant. Then for an arbitrary $x \in X \subset Q_R$ and for a set $D \subset I$, meas $D \leq \varepsilon$, we obtain

$$\begin{aligned}\|H_i(x)\cdot\chi_D\|_{p_i} &\leq \|g_i\cdot\chi_D\|_{p_i} + |\lambda_i|\cdot\|A_i(x)\cdot\chi_D\|_{p_i}\\ &\leq \|g_i\cdot\chi_D\|_{p_i} + |\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\left(\|b_i\cdot\chi_D\|_{p_i} + d_i\cdot\|x^{\frac{p}{p_i}}\cdot\chi_D\|_{p_i}\right)\\ &\leq \|g_i\cdot\chi_D\|_{p_i} + |\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\left(\|b_i\cdot\chi_D\|_{p_i} + d_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{p_i}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|H(x)\cdot\chi_D\|_p &\leq \prod_{i=1}^n \|H_i(x)\cdot\chi_D\|_{p_i}\\ &\leq \prod_{i=1}^n\left(\|g_i\cdot\chi_D\|_{p_i} + |\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\left(\|b_i\cdot\chi_D\|_{p_i} + d_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{p_i}}\right)\right).\end{aligned}$$

Hence, taking into account that $g_i, b_i \in L_{p_i}$, then

$$\lim_{\varepsilon\to 0}\left\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|g_i\cdot\chi_D\|_{p_i}\}]\right\} = 0 \text{ and } \lim_{\varepsilon\to 0}\left\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|b_i\cdot\chi_D\|_{p_i}\}]\right\} = 0.$$

Thus by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $measD \leq \varepsilon$, we get

$$c(H(X)) \leq \prod_{i=1}^n \left(d_i|\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\right)c(X).$$

Since $X \subset Q_R$ is a nonempty, bounded and compact in measure subset of an ideal regular space L_p, we can use Proposition 3.3 and get

$$\beta_H(B(X)) \leq \prod_{i=1}^n \left(d_i|\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\right)\beta_H(X).$$

The above inequality with $\prod_{i=1}^n \left(d_i|\lambda_i|\cdot\|K_{0_i}\|_{p_i',p_i}\right) < 1$ allow us to apply Theorem 4.3. This fulfills the proof. □

4.2.3. *Example.* Finally, we illustrate an example to show the applicability of our results.

Example 4.7. For $t \in I = [0,1]$, considers the following product of integral equations in $L_2(I)$:

$$x(t) = \left(e^{\frac{t}{4}} + \int_0^1 e^{\frac{3(s-t)}{4}}\left(s^{\frac{1}{4}} + d_1|x(s)|^{\frac{2}{4}}\right)\right)\cdot\left(e^{\frac{t}{8}} + \int_0^1 e^{\frac{7(s-t)}{8}}\left(s^{\frac{1}{8}} + d_2|x(s)|^{\frac{2}{8}}\right)\right)^2. \quad (4.3)$$

Let $p_1 = 4,\ p_2 = p_3 = 8$, then we have

- $g_1(t) = e^{\frac{t}{4}}, g_2(t) = g_3(t) = e^{\frac{t}{8}}$, with
$$\|g_1\|_4 = \sqrt[4]{e-1},\ \|g_2\|_8 = \|g_3\|_8 = \sqrt[8]{e-1}.$$
- $b_1(t) = t^{\frac{1}{4}}, b_2(t) = b_3(t) = t^{\frac{1}{8}}$, with $\|b_1\|_4 = \sqrt[4]{\frac{1}{2}}, \|b_2\|_8 = \|b_3\|_8 = \sqrt[8]{\frac{1}{2}}$.
- $K_1(t,s) = e^{\frac{3(s-t)}{4}}$ and $K_2(t,s) = K_3(t,s) = e^{\frac{7(s-t)}{8}}$, with
$$\left\|\|K_{0_1}\|_{\frac{4}{3}}\right\|_4 = \sqrt[4]{\frac{(e-1)^3(1-e^{-3})}{3}},$$
$$\left\|\|K_{0_2}\|_{\frac{8}{7}}\right\|_8 = \left\|\|K_{0_3}\|_{\frac{8}{7}}\right\|_8 = \sqrt[8]{\frac{(e-1)^7(1-e^{-7})}{7}}.$$
- Moreover, one can choose the constants $d_i \geq 0,\ i = 1,2,3$, such that
$$\prod_{i=1}^{3}\left(d_i \cdot \|K_{0_i}\|_{p_i',p_i}\right) < 1.$$

Hence, Theorem 4.6 implies that equation (4.3) has a solution $x \in L_2(I)$ which is a.e. nondecreasing on I.

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