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FIXED POINT THEOREMS FOR MAPPINGS WITH A CONTRACTIVE ITERATE AT A POINT IN MODULAR METRIC SPACES

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Abstract. In this manuscript, we introduce two new types of contraction, namely, w-contraction and strong Sehgal w-contraction, in the framework of modular metric spaces. We indicate that under certain assumptions, such mappings possess a unique fixed point. For the sake of completeness, we consider examples and an application to matrix equations.
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1. Introduction and preliminaries

Modular metric spaces are the metric spaces generated by modular and their theory was developed by Chistyakov [7, 8, 9, 10] as an extension of the theory of modular for linear spaces founded by Nakano [16], which generalizes Lebesgue, Riesz, and Orlicz spaces of integrable functions. In spite of the fact that Orlicz and modular linear spaces have many applications in nonlinear functional analysis, they are restricted to certain situations [10]. Theory of modular on arbitrary sets is consistent with the theories of metric spaces and modular linear spaces and is important in problems of multivalued analysis such as the definition of metric functional spaces, characterization of set-valued superposition operators and existence of regular selections of multifunctions [10].

In 1969, V. M. Sehgal proved that in a complete metric space, continuous self mappings with a contractive iterate at each point of the space have a unique fixed point, see [18]. Guseman [11] generalized the result of Sehgal to mappings having a contractive iterate at each point in a subset of the space. After that, numerous generalizations have been introduced by many authors, see for example [15, 17, 19, 2, 12, 14, 3, 13]

The aim of this paper is to extend the fixed point results for mappings with a contractive iterate at a point in the settings of modular metric spaces. In the following we give some preliminary results on metric modular, modular spaces and existence of fixed points of mappings with a contractive iterate at a point in metric spaces.

We begin with the basic notion of metric modular and modular metric space introduced by Chistyakov [7, 8, 9] and some of their properties.

A function $w : (0,\infty) \times \mathcal{X} \times \mathcal{X} \to [0,\infty]$, $w_\lambda(u,z) = w(\lambda,u,z)$, is called a **modular metric** on a nonempty set $\mathcal{X}$ if the following axioms are satisfied:

(m_1) $u = z$ if and only if $w_\lambda(u,z) = 0$ for all $\lambda > 0$;
(m_2) $w_\lambda(u,z) = w_\lambda(z,u)$, for all $u,z \in \mathcal{X}$ and $\lambda > 0$;
(m_3) $w_{\lambda+\mu}(u,z) \leq w_\lambda(u,v) + w_\mu(v,z)$, for all $u,v,z \in \mathcal{X}$ and $\lambda,\mu > 0$.

The function $w : (0,\infty) \times \mathcal{X} \times \mathcal{X} \to (0,\infty)$ is said to be a **pseudomodular metric** on $\mathcal{X}$ in case that instead of (m_1),

(m_1') $w_\lambda(u,u) = 0$ for all $\lambda > 0$

is satisfied. Likewise, if the axiom (m_1) is replaced by

(m_1^*) there exists a $\lambda > 0$, such that if $u = z$ then $w_\lambda(u,z) = 0$,

then the modular metric w on $\mathcal{X}$ is called **strict**. If w is either modular or pseudomodular metric on $\mathcal{X}$, for every $\lambda,\mu > 0$, with $\lambda > \mu$, by (m_3) we have for all $u,z \in \mathcal{X}$,

$$w_\lambda(u,z) \leq w_{\lambda-\mu}(u,u) + w_\mu(u,z) = w_\mu(u,z), \tag{1.1}$$

which means that the function $\lambda \to w_\lambda(u,z)$ is non-increasing on $(0,\infty)$. A modular w on $\mathcal{X}$ is called **convex** if for all $\lambda,\mu > 0$ and $u,z,v \in \mathcal{X}$ the following inequality is satisfied

$$w_{\lambda+\mu}(u,z) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(u,v) + \frac{\mu}{\lambda+\mu} w_\mu(v,z).$$

Let u_0 be fixed in $\mathcal{X}$. In [7, 8] the following sets, named **modular spaces (around** u_0**)** were introduced as follows.

$$\begin{aligned} \mathcal{X}_w &= \mathcal{X}_w(u_0) = \{u \in \mathcal{X} : w_\lambda(u,u_0) \to 0 \text{ as } \lambda \to \infty\} \\ \mathcal{X}_w^* &= \mathcal{X}_w^*(u_0) = \{u \in \mathcal{X} : \exists \lambda = \lambda(u) \text{ such that } w_\lambda(u,u_0) < \infty\}. \end{aligned}$$

It was shown that the modular space $\mathcal{X}_w$ can be endowed with a metric d_w, where

$$d_w(u,z) = \inf\{\lambda > 0 : w_\lambda(u,z) \leq \lambda\},$$

for $u,z \in \mathcal{X}_w$. Moreover, according to ([7, 8]), if the modular w on $\mathcal{X}$ is convex, then $\mathcal{X}_w = \mathcal{X}_w^*$ and we can equip this set with the metric defined by

$$d_w^*(u,z) = \inf\{\lambda > 0 : w_\lambda(u,z) \leq 1\}$$

for any $u,z \in \mathcal{X}_w$.

On a modular metric space $\mathcal{X}_w$, a sequence $\{u_n\}$ is

(c) convergent to $u \in \mathcal{X}$ if $\lim_{n\to\infty} w_\lambda(u_n, u) = 0$, for all $\lambda > 0$;
(by (m_3), it follows that the limit of convergent sequence in $\mathcal{X}_w$ is unique.)

(C) Cauchy if $\lim_{n\to\infty} w_\lambda(u_n, u_{n+m}) = 0$, for all $m > 0$ and $\lambda > 0$.

A modular metric space is complete if every Cauchy sequence is convergent. If in the above assertions, we assume that the conditions (c) and (C) hold only for some $\lambda > 0$, not for all, we say that the sequence $\{u_n\}$ is w-convergent, respectively w-Cauchy, and if any w-Cauchy sequence is convergent, the modular space is said to be w-complete.

The contractive mapping definition and the Banach fixed point theorem is generalized to the setting of modular metric spaces by Chistyakov [9, 10].

Definition 1.1. Let $\mathcal{X}$ be a nonempty set and w be a metric modular on $\mathcal{X}$.

(i) A map $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is said to be w-contractive provided that there exist $0 < \kappa < 1$ and $\lambda_0 > 0$ depending on κ such that

$$w_{\kappa\lambda}(\mathcal{T}u, \mathcal{T}z) \leq w_\lambda(u, z) \tag{1.2}$$

for all $0 < \lambda < \lambda_0$ and $u, z \in \mathcal{X}_w^*$.

(ii) A map $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is said to be strong w-contractive provided that there exist $0 < \kappa < 1$ and $\lambda_0 > 0$ depending on κ such that

$$w_{\kappa\lambda}(\mathcal{T}u, \mathcal{T}z) \leq \kappa w_\lambda(u, z) \tag{1.3}$$

for all $0 < \lambda < \lambda_0$ and $u, z \in \mathcal{X}_w^*$.

Theorem 1.2. *Let $\mathcal{X}$ be a nonempty set and w be a strict convex metric modular on $\mathcal{X}$. Let $\mathcal{X}_w^*$ be a complete modular metric space induced by w and $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ be a w-contractive self mapping.*

If for every $\lambda > 0$ and all $u \in \mathcal{X}_w^*$ we have $w_\lambda(u, \mathcal{T}u) < \infty$, then the mapping $\mathcal{T}$ has a fixed point in $\mathcal{X}_w^*$.

If in addition $w_\lambda(u, z) < \infty$ for all $u, z \in \mathcal{X}_w^*$ and every $\lambda > 0$, then the fixed point of $\mathcal{T}$ is unique.

Theorem 1.3. *Let $\mathcal{X}$ be a nonempty set and w be a strict metric modular on $\mathcal{X}$. Let $\mathcal{X}_w^*$ be a complete modular metric space induced by w and $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ be a strong w-contractive self mapping.*

If for every $\lambda > 0$ and all $u \in \mathcal{X}_w^*$ we have $w_\lambda(u, \mathcal{T}u) < \infty$ then the mapping $\mathcal{T}$ has a fixed point in $\mathcal{X}_w^*$.

If in addition $w_\lambda(u, z) < \infty$ for all $u, z \in \mathcal{X}_w^*$ and every $\lambda > 0$, then the fixed point of $\mathcal{T}$ is unique.

The following variants of Palais's inequality for modular contractive mappings are proved in [1].

Proposition 1.4. *Let $\mu_1, \mu_2 \geq 0$ be chosen such that $\mu_1 + \mu_2 = (1 - \kappa)\lambda$, where $0 < \kappa < 1$ and $0 < \lambda < \lambda_0$.*

(i) *(Fundamental modular contraction inequality). Let $\mathcal{X}$ be a non-empty set and w be a convex modular in $\mathcal{X}$. If $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is a w-contraction, i.e.* (1.2) *holds for $0 < \lambda < \lambda_0$, then for every $u, z \in \mathcal{X}_w^*$*

$$w_\lambda(u, z) \leq \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}z)}{\lambda(1 - \kappa)}. \tag{1.4}$$

(ii) *(Fundamental strong modular contraction inequality). Let $\mathcal{X}$ be a non-empty set and w be a modular in $\mathcal{X}$. If $\mathcal{T} : \mathcal{X}^*_w \to \mathcal{X}^*_w$ is a strong w-contraction, i.e.* (1.3) *holds for $0 < \lambda < \lambda_0$, then for every $u, z \in \mathcal{X}^*_w$*

$$w_\lambda(u, z) \leq \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}z)}{1 - \kappa}. \tag{1.5}$$

In the following, we review some fixed point results for iterative mappings in metric spaces which will be generalized in the setting of modular metric spaces in the next section.

Let $(\mathcal{X}^*, d)$ be a complete metric space. In the following theorem, Bryant [6] proved an analog of Banach's fixed point theorem in which not the mapping itself but one of its iterates satisfies the contractive condition.

Theorem 1.5. [6] *A self-mapping $\mathcal{T}$ on $(\mathcal{X}^*, d)$ admits a unique fixed point $\xi \in \mathcal{X}$, if there exist $\kappa \in [0, 1)$ and $m \in \mathbb{N}$ so that*

$$d(\mathcal{T}^m u, \mathcal{T}^m z) \leq \kappa d(u, z), \tag{1.6}$$

for all $u, z \in \mathcal{X}$.

Sehgal [18] improved this result by taking not a fixed but variable iterate of the mapping under consideration.

Theorem 1.6. [18] *A continuous self-mapping $\mathcal{T}$ on $(\mathcal{X}^*, d)$ admits a unique fixed point $\xi \in \mathcal{X}$, if it satisfies the condition: there exists a constant $\kappa \in [0, 1)$ such that for each $u \in \mathcal{X}$, there is a positive integer $p(u)$ such that*

$$d(\mathcal{T}^{p(u)} u, \mathcal{T}^{p(u)} z) \leq \kappa d(u, z), \tag{1.7}$$

for all $z \in \mathcal{X}$.

Guseman [11] on the other hand, has shown that the condition of continuity of the mapping was unnecessary.

2. Fixed point theorems for contractive iterative mappings in modular metric spaces

This section is devoted to the extension of the existence of fixed point results given in [9] for mappings $\mathcal{T}$ with the property that some iterate of $\mathcal{T}$ satisfies one of the following generalized versions of the definitions of contractions in modular metric spaces.

Definition 2.1. Let w be a metric modular on a non-empty set $\mathcal{X}$.

(i) A mapping $\mathcal{T} : \mathcal{X}_w \to \mathcal{X}_w$ is called a **Sehgal w-contraction** if there exist constants $\kappa \in (0, 1)$ and $\lambda_0 > 0$ satisfying the condition: for each $u \in \mathcal{X}_w$ there is a positive integer $p(u)$ such that

$$w_{\kappa\lambda}(\mathcal{T}^{p(u)} u, \mathcal{T}^{p(u)} z) \leq w_\lambda(u, z) \tag{2.1}$$

for all $z \in \mathcal{X}_w$ and $0 < \lambda < \lambda_0$.

(ii) A mapping $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is called a **strong Sehgal w-contraction** if there exist constants $\kappa \in (0,1)$ and $\lambda_0 > 0$ satisfying the condition: for each $u \in \mathcal{X}_w^*$ there is a positive integer $p(u)$ such that

$$w_{\kappa\lambda}(\mathcal{T}^{p(u)}u, \mathcal{T}^{p(u)}z) \le \kappa w_\lambda(u,z) \tag{2.2}$$

for all $z \in \mathcal{X}_w^*$ and $0 < \lambda < \lambda_0$.

We introduce the following analogues of inequalities given in Proposition 1.4 for Sehgal w-contraction maps, which will be employed in the subsequent fixed point theorem.

Proposition 2.2. *Let $\mu_1 + \mu_2 = (1-\kappa)\lambda$, for some $\mu_1, \mu_2 \ge 0$ with $0 < \lambda < \lambda_0$.*

(i) *Let $\mathcal{X}$ be a non-empty set and w be a convex modular in $\mathcal{X}$. If $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is a Sehgal w-contraction satisfying* (2.1) *for $0 < \lambda < \lambda_0$, then for every $u, z \in \mathcal{X}_w^*$*

$$w_\lambda(u,z) \le \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}^{p(u)}z)}{\lambda(1-\kappa)}. \tag{2.3}$$

(ii) *Let $\mathcal{X}$ be a non-empty set and w be a modular in $\mathcal{X}$. If $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ is a strong Sehgal w-contraction satisfying* (2.2) *for $0 < \lambda < \lambda_0$, then for every $u, z \in \mathcal{X}_w^*$*

$$w_\lambda(u,z) \le \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}^{p(u)}z)}{1-\kappa}. \tag{2.4}$$

Proof. Let $\mu_1, \mu_2 \ge 0$, such that $\mu_1 + \mu_2 = (1-\kappa)\lambda$ and $p(u)$ be a positive integer.

(i) Convexity of w implies that

$$w_{\mu_1+\kappa\lambda+\mu_2}(u,z) \le \frac{\mu_1}{\lambda} w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \kappa w_{\kappa\lambda}(\mathcal{T}^{p(u)}u, \mathcal{T}^{p(u)}z) + \frac{\mu_2}{\lambda} w_{\mu_2}(\mathcal{T}^{p(u)}z, z)$$

holds. Since $\mathcal{T}$ is a Sehgal w-contraction and $\lambda = \mu_1 + \kappa\lambda + \mu_2$, we get

$$w_\lambda(u,z) \le \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}^{p(u)}z)}{\lambda(1-\kappa)}. \tag{2.5}$$

(ii) The axiom (m_3) assures that

$$w_{\mu_1+\kappa\lambda+\mu_2}(u,z) \le \mu_1 w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \kappa\lambda w_{\kappa\lambda}(\mathcal{T}^{p(u)}u, \mathcal{T}^{p(u)}z) + \mu_2 w_{\mu_2}(\mathcal{T}^{p(u)}z, z)$$

holds. Since $\mathcal{T}$ is a strong Sehgal w-contraction and $\lambda = \mu_1 + \kappa\lambda + \mu_2$, we get

$$w_\lambda(u,z) \le \frac{\mu_1 w_{\mu_1}(u, \mathcal{T}^{p(u)}u) + \mu_2 w_{\mu_2}(z, \mathcal{T}^{p(u)}z)}{1-\kappa}. \tag{2.6}$$

Theorem 2.3. *Let $\mathcal{X}$ be a nonempty set, w be a strict convex metric modular on $\mathcal{X}$ and $\mathcal{X}_w$ be a complete modular metric space induced by w. If $\mathcal{T} : \mathcal{X}_w \to \mathcal{X}_w$ is a Sehgal w-contraction, then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}_w$.*

Proof. Let u_0 be an arbitrary fixed point in $\mathcal{X}_w$ and the sequence $\{u_n\}$ be defined by

$$u_1 = \mathcal{T}^{p_0}u_0, u_2 = \mathcal{T}^{p_1}u_1, \ldots, u_n = \mathcal{T}^{p_{n-1}}u_{n-1},$$

where $p_i = p(u_i)$. Since the mapping $\mathcal{T}$ is a Sehgal w-contraction, by (2.1), for $u = u_{n-1}$ and $z = u_n$ we have

$$\begin{aligned} w_\lambda(u_n, u_{n+1}) &= w_\lambda(\mathcal{T}^{p_{n-1}}u_{n-1}, \mathcal{T}^{p_n}u_n) = w_\lambda(\mathcal{T}^{p_{n-1}}u_{n-1}, \mathcal{T}^{p_{n-1}}(\mathcal{T}^{p_n}u_{n-1})) \\ &\leq w_{\frac{\lambda}{\kappa}}(u_{n-1}, \mathcal{T}^{p_n}u_{n-1}) \leq \ldots \leq w_{\frac{\lambda}{\kappa^n}}(u_0, \mathcal{T}^{p_n}u_0). \end{aligned} \tag{2.7}$$

The definition of the modular space $\mathcal{X}_w$ implies $\lim_{n\to\infty} w_{\frac{\lambda}{\kappa^n}}(u_0, \mathcal{T}^{p_n}u_0) = 0$ and according to the above inequality (2.7) we get

$$\lim_{n\to\infty} w_\lambda(u_n, u_{n+1}) = 0. \tag{2.8}$$

Let n, m be positive integers and replace u by u_n and z by u_{n+m} in the inequality (2.3). Then by (2.8) we have

$$\begin{aligned} w_\lambda(u_n, u_{n+m}) &\leq \frac{\mu_1 w_{\mu_1}(u_n, \mathcal{T}^{p_n}u_n) + \mu_2 w_{\mu_2}(u_{n+m}, \mathcal{T}^{p_n}u_{n+m})}{(1-\kappa)\lambda} \\ &= \frac{\mu_1 w_{\mu_1}(u_n, u_{n+1}) + \mu_2 w_{\mu_2}(\mathcal{T}^{p_{n+m-1}}u_{n+m-1}, \mathcal{T}^{p_{n+m-1}}(\mathcal{T}^{p_n}u_{n+m-1}))}{(1-\kappa)\lambda} \\ &\leq \frac{\mu_1 w_{\mu_1}(u_n, u_{n+1}) + \mu_2 w_{\frac{\mu_2}{\kappa}}(u_{n+m-1}, \mathcal{T}^{p_n}u_{n+m-1})}{(1-\kappa)\lambda} \\ &\leq \ldots \\ &\leq \frac{\mu_1 w_{\mu_1}(u_n, u_{n+1}) + \mu_2 w_{\frac{\mu_2}{\kappa^m}}(u_n, u_{n+1})}{(1-\kappa)\lambda} \\ &\to 0, \end{aligned}$$

as $n \to \infty$. Thus, the sequence $\{u_n\}$ is w-Cauchy, hence w-convergent since the space $\mathcal{X}_w$ is w-complete. Let $u^* \in \mathcal{X}_w$ be the limit of the sequence $\{u_n\}$, that is

$$\lim_{n\to\infty} w_\lambda(u_n, u^*) = 0. \tag{2.9}$$

On one hand, by (2.1) we have

$$\begin{aligned} w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) &= w_\lambda(\mathcal{T}^{p_{n-1}}(\mathcal{T}^{p(u^*)}u_{n-1}), \mathcal{T}^{p_{n-1}}u_{n-1}) \\ &\leq w_{\lambda/\kappa}((\mathcal{T}^{p_{u^*}}u_{n-1}, u_{n-1})) \\ &\ldots \\ &\leq w_{\lambda/\kappa^n}(\mathcal{T}^{p_{u^*}}u_0, u_0) \end{aligned}$$

and since $\lambda/\kappa^n \to \infty$ as $n \to \infty$ by the above inequality we get

$$\lim_{n\to\infty} w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) = 0. \tag{2.10}$$

On the other hand,

$$\begin{aligned} w_{(\kappa+2)\lambda}(u^*, \mathcal{T}^{p(u^*)}u^*) &\leq w_\lambda(u^*, u_n) + w_\lambda(u_n, \mathcal{T}^{p(u^*)}u_n) + w_{\kappa\lambda}(\mathcal{T}^{p(u^*)}u_n, \mathcal{T}^{p(u^*)}u^*) \\ &\leq w_\lambda(u^*, u_n) + w_\lambda(u_n, \mathcal{T}^{p(u^*)}u_n) + w_\lambda(u_n, u^*), \end{aligned}$$

and together with (2.9) and (2.10) we get that $\lim_{n\to\infty} w_{(\kappa+2)\lambda}(u^*, \mathcal{T}^{p(u^*)}u^*) = 0$, and since the modular w is strict, we can conclude that $\mathcal{T}^{p(u^*)}u^* = u^*$. Finally, because

$$\mathcal{T}^{p(u^*)}(\mathcal{T}u^*) = \mathcal{T}(\mathcal{T}^{p(u^*)}u^*) = \mathcal{T}(u^*),$$

it follows that $\mathcal{T}u^*$ is a fixed point of $\mathcal{T}^{p(u^*)}$. As a consequence of (2.3) we note that, if there exists a fixed point u^* of $\mathcal{T}^{p(u^*)}$, it is unique. Hence, we conclude that $\mathcal{T}u^* = u^*$.

The following result will be used in our second fixed point theorem.

Lemma 2.4. *Let $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ be a strong Sehgal w-contraction. If there exists a point $u_0 \in \mathcal{X}_w$ such that $w_\lambda(u_0, \mathcal{T}^i u_0) < \infty$ for $i \in \{1, 2, \dots p(u_0)\}$ and for all $\lambda > 0$ then every element of the set $\Sigma(u_0) = \{w_\lambda(u_0, \mathcal{T}^m u_0) : m \in \mathbb{N}\}$ is finite for each $\lambda > 0$.*

Proof. Let $u_0 \in \mathcal{X}_w^*$ such that $w_\lambda(u_0, \mathcal{T}^{p(u_0)} u_0) < \infty$. We set $p_0 = p(u_0)$ and $\sigma_\lambda(u_0) = w_\lambda(u_0, \mathcal{T}^{p_0} u_0)$ and we consider a positive integer $m \in \mathbb{N}$, where $m \geq p_0$. Then, m can be written $m = jp_0 + l$, where $j \in \mathbb{N}, j \geq 1$ and $l \in \{0, 1, 2, ..., p_0 - 1\}$. Using the fact that $\mathcal{T}$ is a strong Sehgal w-contraction, the property (1.1) and the condition (m_3) iteratively, we have

$$\begin{aligned}
w_\lambda(u_0, \mathcal{T}^m u_0) &\leq w_{\frac{\lambda}{2}}(u_0, \mathcal{T}^{p_0} u_0) + w_{\frac{\lambda}{2}}(\mathcal{T}^{p_0} u_0, \mathcal{T}^m u_0) \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + w_{\kappa\frac{\lambda}{2}}(\mathcal{T}^{p_0} u_0, \mathcal{T}^{p_0}(\mathcal{T}^{m-p_0} u_0)) \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + \kappa w_{\frac{\lambda}{2}}(u_0, \mathcal{T}^{m-p_0}(u_0)) \\
&= \sigma_{\frac{\lambda}{2}}(u_0) + \kappa w_{\frac{\lambda}{2}}(u_0, \mathcal{T}^{(j-1)p_0+l} u_0) \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + \kappa \left[w_{\frac{\lambda}{2^2}}(u_0, \mathcal{T}^{p_0} u_0) + w_{\frac{\lambda}{2^2}}(\mathcal{T}^{p_0} u_0, \mathcal{T}^{(j-1)p_0+l} u_0) \right] \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + \kappa \sigma_{\frac{\lambda}{2^2}}(u_0) + \kappa w_{\kappa\frac{\lambda}{2^2}}(\mathcal{T}^{p_0} u_0, \mathcal{T}^{p_0}(\mathcal{T}^{(j-2)p_0+l} u_0)) \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + \kappa \sigma_{\frac{\lambda}{2^2}}(u_0) + \kappa^2 w_{\frac{\lambda}{2^2}}(u_0, \mathcal{T}^{(j-2)p_0+l} u_0) \\
&\dots \\
&\leq \sigma_{\frac{\lambda}{2}}(u_0) + \kappa \sigma_{\frac{\lambda}{2^2}}(u_0) + \kappa^2 \sigma_{\frac{\lambda}{2^3}}(u_0) + \dots \\
&+ \kappa^j \sigma_{\frac{\lambda}{2^{j+1}}}(u_0) + \kappa^j w_{\frac{\lambda}{2^{j+1}}}(u_0, \mathcal{T}^l u_0) \\
&\leq (1 + \kappa + \dots + \kappa^j) \sigma_{\frac{\lambda}{2^{j+1}}}(u_0) + \kappa^j w_{\frac{\lambda}{2^{j+1}}}(u_0, \mathcal{T}^l u_0) \\
&= \frac{1 - \kappa^{j+1}}{1 - \kappa} \sigma_{\frac{\lambda}{2^{j+1}}}(u_0) + \kappa^j w_{\frac{\lambda}{2^{j+1}}}(u_0, \mathcal{T}^l u_0) < \infty,
\end{aligned}$$

since both $\sigma_{\frac{\lambda}{2^{j+1}}}(u_0) = w_{\frac{\lambda}{2^{j+1}}}(u_0, \mathcal{T}^{p_0} u_0)$ and $w_{\frac{\lambda}{2^{j+1}}}(u_0, \mathcal{T}^l u_0)$ are finite by the assumption. This proves that all elements of the set $\Sigma(u_0)$ are finite for each $\lambda > 0$.

In the following fixed point theorem, the convexity assumption on the modular is replaced by the strongly contractive mapping condition.

Theorem 2.5. *Let $(\mathcal{X}, w)$ be a complete modular metric space. A strong Sehgal w-contraction, $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ admits a fixed point $u^* \in \mathcal{X}_w^*$, presuming that there exists a point $u_0 \in \mathcal{X}_w^*$ such that $w_\lambda(u_0, \mathcal{T}^i u_0) < \infty$ for all $i \in \{1, 2 \dots p(u_0)\}$ and all $\lambda > 0$. If in addition, we assume that $w_\lambda(u^*, z) < \infty$ for any $z \in \mathcal{X}_w^*$, $\lambda > 0$, then the fixed point of $\mathcal{T}$ is unique.*

Proof. Let $u_0 \in \mathcal{X}_w^*$ such that $w_\lambda(u_0, \mathcal{T}^i u_0) < \infty$ for all $i \in \{1, 2 \dots p(u_0)\}$ and all $\lambda > 0$. Set $\sigma_\lambda(u_0) = w_\lambda(u_0, \mathcal{T}^{p(u_0)} u_0)$ which is also finite.

Starting with this point u_0, we build a sequence, named $\{u_n\}$ as follows:

$$u_1 = \mathcal{T}^{p_0} u_0, u_2 = \mathcal{T}^{p_1} u_1, ..., u_{n+1} = \mathcal{T}^{p_n} u_n, \tag{2.11}$$

where we use the notation $p_i = p(u_i)$, for every $i \in \mathbb{N}$. Moreover, by (2.11) we have that

$$u_n = \mathcal{T}^{p_{n-1}+p_{n-2}+\ldots+p_0} u_0.$$

In addition to this, we have $u_{n+m} = \mathcal{T}^s u_n$, where $s = p_{n+m-1} + \ldots + p_{n+1} + p_n$ and $n, m \in \mathbb{N}$. If we replace u with u_{n-1} and z with u_n in (2.2) and taking into account (2.11), by (m_3) we have

$$\begin{aligned} w_{\kappa\lambda}(u_n, u_{n+1}) &= w_{\kappa\lambda}(\mathcal{T}^{p_{n-1}} u_{n-1}, \mathcal{T}^{p_n} u_n) = w_{\kappa\lambda}(\mathcal{T}^{p_{n-1}} u_{n-1}, \mathcal{T}^{p_{n-1}}(\mathcal{T}^{p_n} u_{n-1})) \\ &\leq \kappa w_\lambda(u_{n-1}, \mathcal{T}^{p_n} u_{n-1}). \end{aligned}$$

But, since for $\kappa \in (0,1)$ we have $0 < \kappa\lambda < \lambda$, and then

$$\begin{aligned} w_{\kappa\lambda}(u_n, u_{n+1}) &\leq \kappa w_\lambda(u_{n-1}, \mathcal{T}^{p_n} u_{n-1}) \leq \kappa w_{\kappa\lambda}(u_{n-1}, \mathcal{T}^{p_n} u_{n-1}) \\ &= \kappa w_{\kappa\lambda}(\mathcal{T}^{p_{n-2}} u_{n-2}, \mathcal{T}^{p_{n-2}} \mathcal{T}^{p_n} u_{n-2}) \\ &\leq \kappa^2 w_\lambda(u_{n-2}, \mathcal{T}^{p_n} u_{n-2}) \\ &\ldots \\ &\leq \kappa^n w_\lambda(u_0, \mathcal{T}^{p_n} u_0). \end{aligned}$$

By the Lemma 2.4, $w_\lambda(u_0, \mathcal{T}^{p_n} u_0)$ is bounded for each $n \in \mathbb{N}_0$ and $\lambda > 0$. Hence, there exists a positive number M_λ such that

$$w_\lambda(u_0, \mathcal{T}^{p_n} u_0) \leq M_\lambda, \tag{2.12}$$

for each $n \in \mathbb{N}_0$ and $\lambda > 0$. Therefore, we get

$$w_\lambda(u_n, u_{n+1}) \leq w_{\kappa\lambda}(u_n, u_{n+1}) \leq \kappa^n M_\lambda, \tag{2.13}$$

and letting $n \to \infty$ we get

$$\lim_{n\to\infty} w_\lambda(u_n, u_{n+1}) = 0, \text{ for } \lambda > 0. \tag{2.14}$$

Let $m \geq 1$. According to (2.13) and using (m_3) we have

$$\begin{aligned} w_\lambda(u_n, u_{n+m}) &\leq w_{\frac{\lambda}{m}}(u_n, u_{n+1}) + w_{\frac{\lambda}{m}}(u_{n+1}, u_{n+2}) + \ldots + w_{\frac{\lambda}{m}}(u_{n+m-1}, u_{n+m}) \\ &\leq \kappa^n M_{\frac{\lambda}{m}} + \kappa^{n+1} M_{\frac{\lambda}{m}} + \ldots + \kappa^{n+m-1} M_{\frac{\lambda}{m}} \\ &= \kappa^n \tfrac{1-\kappa^m}{1-\kappa} M_{\frac{\lambda}{m}}. \end{aligned}$$

Thus, $\lim_{n,m\to\infty} w_\lambda(u_n, u_{n+m}) = 0$, for all $\lambda > 0$, which shows that the sequence $\{u_n\}$ is Cauchy in $\mathcal{X}_w^*$. Since this space is complete, there exists a limit, say u^*, of the sequence $\{u_n\}$. We claim that u^* is a fixed point of $\mathcal{T}^{p(u^*)}$. Indeed, we have

$$\begin{aligned} w_\lambda(\mathcal{T}^{p(u^*)} u^*, u^*) &\leq w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u^*, \mathcal{T}^{p(u^*)} u_n) + w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u_n, u_n) + w_{\frac{\lambda}{3}}(u_n, u^*) \\ &\leq w_{\frac{\kappa\lambda}{3}}(u^*, u_n) + w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u_n, u_n) + w_{\frac{\lambda}{3}}(u_n, u^*). \end{aligned} \tag{2.15}$$

From (2.2), we have

$$\begin{aligned} w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u_n, u_n) &\leq w_{\frac{\kappa\lambda}{3}}(\mathcal{T}^{p(u^*)} u_n, u_n) = w_{\frac{\kappa\lambda}{3}}(\mathcal{T}^{p_{n-1}}(\mathcal{T}^{p(u^*)} u_{n-1}), \mathcal{T}^{p_{n-1}} u_{n-1}) \\ &\leq \kappa w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u_{n-1}, u_{n-1}) \\ &\ldots \\ &\leq \kappa^n w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)} u_0, u_0) \end{aligned}$$

By the (2.12) we know that $w_{\frac{\lambda}{3}}(\mathcal{T}^{p(u^*)}u_0, u_0) \le M_{\frac{\lambda}{3}}$ and hence, from (2.15), we get

$$w_\lambda(\mathcal{T}^{p(u^*)}u^*, u^*) \le w_{\kappa\frac{\lambda}{3}}(u^*, u_n) + \kappa^n M_{\frac{\lambda}{3}} + w_{\frac{\lambda}{3}}(u^*, u_n).$$

Letting $n \to \infty$ in the above inequality, we have $w_\lambda(\mathcal{T}^{p(u^*)}u^*, u^*) = 0$, thus u^* is a fixed point of $\mathcal{T}^{p(u^*)}$. Let $z^* \in \mathcal{X}_w^*$ be another fixed point of $\mathcal{T}^{p(u^*)}$ with $u^* \ne z^*$. Then we have

$$w_\lambda(u^*, z^*) \le w_{\kappa\lambda}(\mathcal{T}^{p(u^*)}u^*, \mathcal{T}^{p(u^*)}z^*) \le \kappa w_\lambda(u^*, z^*) < w_\lambda(u^*, z^*) < \infty.$$

Therefore, $w_\lambda(u^*, z^*) = 0$ and by (m_1), we have $u^* = z^*$. Thus, u^* is the unique fixed point of $\mathcal{T}^{p(u^*)}$. As a final step, since

$$\mathcal{T}^{p(u^*)}(\mathcal{T}u^*) = \mathcal{T}(\mathcal{T}^{p(u^*)}u^*) = \mathcal{T}u^*,$$

taking into account the uniqueness of the fixed point of $\mathcal{T}^{p(u^*)}$, we get that $\mathcal{T}u^* = u^*$.

If we replace the completeness of $\mathcal{X}_w^*$ with w-completeness we can state the next result.

Theorem 2.6. *Let w be a strict modular on a non-empty set $\mathcal{X}$ such that $\mathcal{X}_w^*$ is w-complete. A strong Sehgal w-contraction $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$ admits a unique fixed point presuming that there exists a point $u_0 \in \mathcal{X}_w$ such that such that $w_\lambda(u_0, \mathcal{T}^i u_0) < \infty$ for all $i \in \{1, 2 \dots p(u_0)\}$ and all $0 < \lambda < \lambda_0$.*

Proof. Let the sequence $\{u_n\}$ be defined as in Theorem 2.5. Then from (2.4) with $u = u_n$ and $z = u_{n+m}$ we get

$$\begin{aligned}
w_\lambda(u_n, u_{n+m}) &\le \frac{w_{\mu_1}(u_n, \mathcal{T}^{p_n}u_n) + w_{\mu_2}(u_{n+m}, \mathcal{T}^{p_n}u_{n+m})}{1-\kappa} \\
&= \frac{w_{\mu_1}(u_n, u_{n+1}) + w_{\mu_2}(\mathcal{T}^{p_{n+m-1}}u_{n+m-1}, \mathcal{T}^{p_{n+m-1}}(\mathcal{T}^{p_n}u_{n+m-1}))}{1-\kappa} \\
&\le \frac{w_{\mu_1}(u_n, u_{n+1}) + \kappa w_{\frac{\mu_2}{\kappa}}(u_{n+m-1}, \mathcal{T}^{p_n}u_{n+m-1})}{1-\kappa} \\
&\le \dots \\
&\le \frac{w_{\mu_1}(u_n, u_{n+1}) + \kappa^m w_{\frac{\mu_2}{\kappa^m}}(u_n, \mathcal{T}^{p_n}u_n)}{1-\kappa} \\
&\le \frac{w_{\mu_1}(u_n, u_{n+1}) + \kappa^m w_{\frac{\mu_2}{\kappa^m}}(u_n, u_{n+1})}{1-\kappa} \\
&\le \frac{\kappa^n w_{\frac{\mu_1}{\kappa^n}}(u_0, \mathcal{T}^{p_n}u_0) + \kappa^{m+n} w_{\frac{\mu_2}{\kappa^{m+n}}}(u_0, \mathcal{T}^{p_n}u_0)}{1-\kappa}.
\end{aligned}$$

We can easily see that $\kappa^{-n}\mu_i \to \infty$, for $i = 1, 2$ and so, there exists $n_0 \in \mathbb{N}$, big enough, such that $\kappa^{-n}\mu_i > \lambda$. Thus,

$$\begin{aligned}
w_{\kappa^{-n}\mu_1}(u_0, \mathcal{T}^{p_n}u_0) &\le w_\lambda(u_0, \mathcal{T}^{p_n}u_0) \le M_\lambda, \\
w_{\kappa^{-n-m}\mu_2}(u_0, \mathcal{T}^{p_n}u_0) &\le w_\lambda(u_0, \mathcal{T}^{p_n}u_0) \le M_\lambda,
\end{aligned}$$

and hence,

$$w_\lambda(u_n, u_{n+m}) \le \frac{\kappa^n(1+\kappa^m)M_\lambda}{1-\kappa} \to 0, \text{ as } n \to \infty.$$

This proves that the sequence $\{u_n\}$ is w-Cauchy. Since by the hypotheses $\mathcal{X}_w^*$ is w-complete and the modular w is strict the sequence $\{u_n\}$ is w-convergent to some u^* and this limit is unique.

As in the previous Theorem, in order to prove that u^* is the unique fixed point of $\mathcal{T}$, the following steps are required.
First of all, we will show that $\lim_{n\to\infty} w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) = 0$. By (2.2) and taking into account (w_3) we have

$$\begin{aligned} w_{\kappa^n\lambda}(\mathcal{T}^{p(u^*)}u_n, u_n) &= w_{\kappa^n\lambda}(\mathcal{T}^{p_{n-1}}(\mathcal{T}^{p(u^*)}u_{n-1}), \mathcal{T}^{p_{n-1}}u_{n-1}) \\ &\le \kappa w_{\kappa^{n-1}\lambda}(\mathcal{T}^{p(u^*)}u_{n-1}, u_{n-1}) \\ &\dots \\ &\le \kappa^n w_\lambda(\mathcal{T}^{p(u^*)}u_0, u_0) \\ &\le \kappa^n M_\lambda \to 0, \text{ as } n \to \infty, \end{aligned}$$

so that,

$$\lim_{n\to\infty} w_{\kappa^n\lambda}(\mathcal{T}^{p(u^*)}u_n, u_n) = 0.$$

Furthermore, since $0 < \kappa^n\lambda < \lambda$ we have $w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) \le w_{\kappa^n\lambda}(\mathcal{T}^{p(u^*)}u_n, u_n)$ and letting $n \to \infty$,

$$\lim_{n\to\infty} w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) = 0. \tag{2.16}$$

Next, we claim that u^* is a fixed point of $\mathcal{T}^{p(u^*)}$. Indeed, by (m_3) we have

$$\begin{aligned} w_{(\kappa+2)\lambda}(\mathcal{T}^{p(u^*)}u^*, u^*) &\le w_{\kappa\lambda}(\mathcal{T}^{p(u^*)}u^*, \mathcal{T}^{p(u^*)}u_n) + w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) + w_\lambda(u_n, u^*) \\ &\le \kappa w(u^*, u_n) + w_\lambda(\mathcal{T}^{p(u^*)}u_n, u_n) + w_\lambda(u_n, u^*). \end{aligned}$$

Letting $n \to \infty$ and keeping in mind the fact that the sequence $\{u_n\}$ is w-convergent and (2.16) we get

$$w_{(k+2)\lambda}(\mathcal{T}^{p(u^*)}u^*, u^*) = 0.$$

Therefore, since w is strict we have $\mathcal{T}^{p(u^*)}u^* = u^*$.
We claim now, that u^* is the only fixed point of $\mathcal{T}^{p(u^*)}$. We assume that, in the contrary, there is another point $z^* \in \mathcal{X}^*_w$ such that $\mathcal{T}^{p(u^*)}z^* = z^* \ne u^*$. Considering in (2.4), $u = u^*$ and $z = z^*$ we have

$$w_\lambda(u^*, z^*) \le \frac{w_{\mu_1}(u^*, \mathcal{T}^{p(u^*)}u^*) + w_{\mu_2}(z^*, \mathcal{T}^{p(u^*)}z^*)}{1-\kappa} = \frac{w_{\mu_1}(u^*, u^*) + w_{\mu_2}(z^*, z^*)}{1-\kappa} = 0.$$

Thus, $w_\lambda(u^*, z^*) = 0$, that is $u^* = z^*$, because w is strict. Moreover, as we showed in Theorem 2.5, due to the uniqueness of the fixed point of $\mathcal{T}^{p(u^*)}$, we obtain that u^* is the unique fixed point of $\mathcal{T}$.

3. Examples and an application on matrix equations

In this section we first provide two examples of mappings which are not w-contraction or strong w-contraction, satisfying the conditions of Theorems (2.5) and (2.6) and thus have fixed points in the corresponding modular spaces.

Example 3.1. Let the set $\mathcal{X} = [0,1] \cup [2, +\infty)$ and the function $w : (0,\infty) \times \mathcal{X} \times \mathcal{X} \to [0,\infty)$ defined by $w_\lambda(u, z) = \frac{|u-z|}{\lambda}$. It can be seen that, w is a strict modular metric on $\mathcal{X}$. Let the mapping $\mathcal{T} : \mathcal{X}^*_w \to \mathcal{X}^*_w$, where

$$\mathcal{T}u = \begin{cases} \frac{3}{4}u, & \text{for } u \in [0,1] \\ \frac{5}{2}, & \text{for } u = 2 \\ 0, & \text{for } u > 2. \end{cases}$$

It is easy to observe that, $\mathcal{T}$ is not a w-contraction. Indeed, choosing $u = 2$ and $z = \frac{17}{8}$ we have $w_\lambda(2, \frac{17}{8}) = \frac{1}{8\lambda}$ and $w_\lambda(\mathcal{T}2, \mathcal{T}\frac{17}{8}) = w_\lambda(\frac{5}{2}, 0) = \frac{5}{2\lambda}$ and then

$$w_{\kappa\lambda}\left(\mathcal{T}2, \mathcal{T}\frac{17}{8}\right) = \frac{5}{2\kappa\lambda} > \frac{1}{8\lambda} = w_\lambda\left(2, \frac{17}{8}\right)$$

for every $\kappa \in (0, 1)$ and $\lambda > 0$. On the other hand,

$$\mathcal{T}^3(u) = \begin{cases} \left(\frac{3}{4}\right)^3 u, & \text{for } u \in [0,1] \\ 0, & \text{for } u \geq 2 \end{cases}$$

and letting $\kappa = \frac{3}{4}$ we have:

- For $u, z \in [0, 1]$ and $p(u) = 3$:

$$w_{\kappa\lambda}(\mathcal{T}^3 u, \mathcal{T}^3 z) = \frac{\left(\frac{3}{4}\right)^3 |u-z|}{\frac{3}{4}\lambda} = \left(\frac{3}{4}\right)^2 \frac{|u-z|}{\lambda} \leq \frac{|u-z|}{\lambda} = w_\lambda(u, z).$$

- For $u \in [0, 1], z \in [2, +\infty)$ and $p(u) = 3$:

$$w_{\kappa\lambda}(\mathcal{T}^3 u, \mathcal{T}^3 z) = \left(\frac{3}{4}\right)^3 \frac{u}{\frac{3}{4}\lambda} = \left(\frac{3}{4}\right)^2 \frac{u}{\lambda} \leq \frac{z-u}{\lambda} = w_\lambda(u, z).$$

Thus, $\mathcal{T}$ is strong Sehgal w-contraction and Theorem (2.6) implies that $\mathcal{T}$ admits a fixed point.

Example 3.2. Let the set $\mathcal{X} = [0, +\infty]$ and the modular metric $w : \mathbb{R}^+ \times \mathcal{X} \times \mathcal{X}$ given by $w_\lambda(u, z) = \frac{|u-z|}{\lambda}$. Let the mapping $\mathcal{T} : \mathcal{X}_w^* \to \mathcal{X}_w^*$, where

$$\mathcal{T}u = \begin{cases} \frac{3}{4}, & \text{for } u \in [0,1] \\ \frac{u}{4}, & \text{for } u \in (1,4] \\ \frac{2u^2+3u+1}{u^2-3}, & \text{for } u \in (4, +\infty). \end{cases}$$

The mapping $\mathcal{T}$ is not a strong w-contraction, since for $u = 1$ and $z = \frac{5}{4}$ we have $w_\lambda(1, \frac{5}{4}) = \frac{1}{4\lambda}$ and $w_\lambda(\mathcal{T}1, \mathcal{T}\frac{5}{4}) = w_\lambda(\frac{3}{4}, \frac{5}{16}) = \frac{7}{16\lambda}$ and then

$$w_{\kappa\lambda}\left(\mathcal{T}1, \mathcal{T}\frac{5}{4}\right) = \frac{7}{16\kappa\lambda} > \kappa\frac{1}{4\lambda} = \kappa w_\lambda\left(1, \frac{5}{4}\right)$$

for every $\kappa \in (0, 1)$ and $\lambda > 0$. Now, we have

$$\mathcal{T}^2(u) = \begin{cases} \frac{3}{4}, & \text{for } u \in [0,4] \\ \frac{2u^2+3u+1}{4u^2-12}, & \text{for } u \in (4, +\infty) \end{cases}$$

and $\mathcal{T}^3 u = \frac{3}{4}$ for every $u \in \mathcal{X}$. Consequently, $w_{\kappa\lambda}(\mathcal{T}^3 u, \mathcal{T}^3 z) = 0$ for any $u, z \in \mathcal{X}$ and $\kappa \in (0, 1)$. Then the assumptions of Theorem (2.5) are satisfied and then $\mathcal{T}$ has a unique fixed point, that is $u = 0$.

Finally, as an application of our results, we consider the following matrix equation

$$AU = B, \tag{3.1}$$

where

$$A=\begin{pmatrix} a-1 & 0 & -a \\ b & -1 & b \\ a & 0 & -a-1 \end{pmatrix}, \quad B=\begin{pmatrix} 2a-1 & 1 & -1 \\ 1 & -2b & 2b \\ 2a+1 & 1 & -1 \end{pmatrix} \text{ with } a,b\in\mathbb{R}, a\geq 1$$

and

$$U\in\mathcal{X}=\left\{\begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix} : u_i\in\mathbb{R}, i=1,2,\dots,9.\right\}$$

Let $w:(0,+\infty)\times\mathcal{X}\times\mathcal{X}\to[0,+\infty)$ be the modular metric given by

$$w_\lambda(U,Z)=\begin{cases} \frac{1}{\lambda}\max\limits_{1\leq i\leq 9}|u_i-z_i|, & \text{if } u_i\neq z_i \\ 0, & \text{if } u_i=z_i \end{cases}$$

Because we can write

$$A=M-I_3, \qquad \text{where } M=\begin{pmatrix} a & 0 & -a \\ b & 0 & b \\ a & 0 & -a \end{pmatrix},$$

the equation 3.1) can be rewritten as $MU-B=U$. Denoting $\mathcal{T}U=MU-B$, where $\mathcal{T}:\mathcal{X}_w^*\to\mathcal{X}_w^*$, we see that solving the equation (3.1) is equivalent to finding the fixed point of $\mathcal{T}$.

First of all, we can see that, choosing for example

$$U=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } Z=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$MU=\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ a & 0 & 0 \end{pmatrix} \text{ and } MZ=\begin{pmatrix} 0 & a & 0 \\ 0 & b & 0 \\ 0 & a & 0 \end{pmatrix}.$$

In this case, since $a>1$,

$$w_{\kappa\lambda}(\mathcal{T}U,\mathcal{T}Z)=\frac{1}{\kappa\lambda}\max\{a,b\}>\frac{1}{\lambda}=w_\lambda(U,Z)$$

which show us that $\mathcal{T}$ is not a w-contraction. By calculation we get $M^3=O_3$ and since

$$\mathcal{T}^2U=M^2U-MB-B, \quad \mathcal{T}^3U=M^3U-M^2B-MB-B$$

we obtain that $\mathcal{T}^3U=-M^2B-MB-B$. For these reasons, for any $\kappa\in(0,1)$ we have

$$w_\kappa\lambda(\mathcal{T}^3U,\mathcal{T}^3Z)=0\leq w_\lambda(U,Z),$$

for all $U,Z\in\mathcal{X}$. Thus, the mapping $\mathcal{T}$ has a fixed point,

$$U_0=\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

and this matrix is the unique solution of equation (3.1).

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