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# THE EXISTENCE OF BEST PROXIMITY POINTS FOR GENERALIZED CYCLIC QUASI-CONTRACTIONS IN METRIC SPACES WITH THE $UC$ AND ULTRAMETRIC PROPERTIES

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**Abstract.** In this paper, in the setting of metric spaces we introduce the notions of generalized cyclic quasi-contractions and the ultrametric property as an applied geometric concept. Then we study the existence and uniqueness of best proximity points for such mappings by using this property and the $UC$ property. Also, iterative algorithms are furnished to determine such best proximity points. As a result, we establish a fixed point result and a common fixed point theorem. The presented results extend and improve some recent results in the literature.
**Key Words and Phrases**: Best proximity point, generalized cyclic quasi-contractions, ultrametric property, $UC$ property.
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## 1. Introduction

Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$. The self mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x^* \in A \cup B$ is called a *best proximity point* for $T$ if $d(x^*, Tx^*) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$. If $d(A, B) = 0$, $x^*$ is called a *fixed point* of $T$. In 2006, the cyclic contraction mappings on uniformly convex Banach spaces were introduced and studied by Anthony Eldred and Veeremani [2]. Since then, the problems of the existence of a best proximity point of cyclic mappings, have been extensively studied by many authors; see for instance [4, 5, 6, 9, 10, 11, 12, 13] and references therein. In order to extend the obtained best proximity results in uniformly convex Banach spaces to metric spaces, the $UC$ property were introduced by Suzuki et al. [13]. They also proved the existence of the best proximity points for cyclic contraction type mappings in metric spaces.
In this paper, in the setting of metric spaces we introduce the notions of generalized cyclic quasi-contractions on $A \cup B$ and the ultrametric property as an applied geometric concept. Then we study the existence and uniqueness of best proximity

points for such mappings when $(A, B)$ has the $UC$ property and $(B, A)$ has the ultrametric property. Also, iterative algorithms are furnished to determine such best proximity points. As a result, we establish a fixed point result and a common fixed point theorem. Our results extend and improve some recent results in [3, 8, 12, 13].

## 2. Preliminaries

Here, we recall some definitions and facts will be used in the next section.

**Definition 2.1.** [13] Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$. Then $(A, B)$ is said to satisfies the $UC$ property, if $\{x_n\}$ and $\{x'_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n) = d(A, B),$$

then $\lim_{n\to\infty} d(x_n, x'_n) = 0$.

Suzuki et al. [13] proved that if $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$ such that $A$ is convex, then $(A, B)$ has the $UC$ property.

**Lemma 2.2.** [13] *Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$. Assume that $(A, B)$ has the $UC$ property. Let $\{x_n\}$ and $\{y_n\}$ are sequences in $A$ and $B$ respectively, such that either of the following holds*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B) \text{ or } \lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

*Then $\{x_n\}$ is Cauchy.*

**Definition 2.3.** [5] Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$. Then $(A, B)$ is said to be proximinal if and only if $A = A_0$ and $B = B_0$ that

$$A_0 := \{x \in A : \ d(x, y) = d(A, B) \ for\ some\ y \in B\},$$
$$B_0 := \{y \in B : \ d(x, y) = d(A, B) \ for\ some\ x \in A\}.$$

**Definition 2.4.** [5] Let $A$ and $B$ be nonempty subsets of the metric space $(X, d)$. Then $(A, B)$ is said to be sharp proximinal if and only if, for each $x \in A$ and $y \in B$, there exists a unique $x' \in A$ and $y' \in B$ such that $d(x, y') = d(x', y) = d(A, B)$. The pair $(A, B)$ is said to be a semi-sharp proximinal if and only if, for each $x \in A$ and $y \in B$, there exists at most one point $x' \in A$ and at most one point $y' \in B$ such that $d(x, y') = d(x', y) = d(A, B)$.

Every closed and convex pair $(A, B)$ in a strictly convex Banach space is semi-sharp proximinal [9, Lemma 2.5]. Examples of such pairs are given in [9] for nonstrictly convex Banach spaces.

**Definition 2.5.** [5] A sharp proximinal pair $(A, B)$ is said to have the Pythagorean property if and only if, for each $(x, y) \in A \times B$,

$$d(x, y)^2 = d(x, x')^2 + d(x', y)^2 \text{ and } d(x, y)^2 = d(y, y')^2 + d(y', x)^2,$$

where $x' \in A$ and $y' \in B$ are the unique points such that $d(x, y') = d(x', y) = d(A, B)$.

It is easy to see that if $d(A,B)=0$, then the pair $(A_0,B_0)$ has the Pythagorean property.

**Definition 2.6.** [7] The modulus of convexity of a Banach space $X$ is the function $\delta_X:[0,2]\to[0,1]$ defined by

$$\delta_X(\epsilon)=\inf\big\{1-\|\frac{x+y}{2}\|:\ \|x\|\leq 1,\ \|y\|\leq 1,\ \|x-y\|\geq\epsilon\big\}.$$

Note that for any $\epsilon>0$ the number of $\delta_X(\epsilon)$ is the largest number for which the following implication always holds. For each $x,y\in X$

$$\left.\begin{array}{c}\|x\|\leq 1\\ \|y\|\leq 1\\ \|x-y\|\geq\epsilon\end{array}\right\}\ \Rightarrow\ \|\frac{x+y}{2}\|\leq 1-\delta_X(\epsilon). \tag{2.1}$$

Obviously a space $X$ is uniformly convex if and only if its modulus of convexity satisfies $\delta(\epsilon)>0$ for $\epsilon>0$. We note that (2.1) has the following equivalent formulation. For all $x,y,p\in X$, $R>0$ and $r\in[0,2R]$

$$\left.\begin{array}{c}\|x-p\|\leq R\\ \|y-p\|\leq R\\ \|x-y\|\geq r\end{array}\right\}\ \Rightarrow\ \|p-\frac{x+y}{2}\|\leq\big(1-\delta(\frac{r}{R})\big)R. \tag{2.2}$$

The characteristic (or coefficient) of convexity of Banach space $X$ is the number $\epsilon_0=\epsilon_0(X)=\sup\{\epsilon\geq 0:\ \delta(\epsilon)=0\}$.

## 3. Main results

First, we introduce the geometric concept of the ultrametric property to establish our main results.

**Definition 3.1.** Let $A$ and $B$ be nonempty subsets of the metric space $(X,d)$. Then $(A,B)$ is said to satisfies ultrametric property, if $d(A,B)>0$ then there exists $\epsilon_{(A,B)}>0$ such that for every $0<\epsilon\leq\epsilon_{(A,B)}$, $x,x'\in A$ and $y\in B$ satisfying $\max\{d(x,y),d(x',y)\}\leq\epsilon+d(A,B)$, we have $d(x,x')\leq\max\{d(x,y),d(x',y)\}$.
Equivalently, $(A,B)$ satisfies ultrametric property if either $d(A,B)=0$ or there exists $\epsilon_{(A,B)}>0$ such that for every $0<\epsilon\leq\epsilon_{(A,B)}$, $x,x'\in A$ and $y\in B$

$$\max\{d(x,y),d(x',y)\}\leq\epsilon+d(A,B)\Rightarrow d(x,x')\leq\epsilon+d(A,B).$$

**Example 3.2.** Let $(X,d)$ be a metric space. If the metric $d$ satisfies the ultrametric inequality, that is for all $x,y,z\in X$

$$d(x,y)\leq\max\{d(x,z),d(z,y)\},$$

it is called ultrametric on $X$, and the pair $(X,d)$ is called an ultrametric space. Let $A$ and $B$ be nonempty subsets of an ultrametric space, then $(A,B)$ has the ultrametric property.

**Proposition 3.3.** *Let $X$ be a Banach space with the modulus of covexity $\delta$ and characteristic of convexity $\epsilon_0(X)<1$. Let $A$ and $B$ be nonempty subsets of $X$ such that $A$ is convex. Then $(A,B)$ has the ultrametric property.*

*Proof.* Assume that $d(A,B) \neq 0$. Since $\epsilon_0(X) < 1$ then $\delta(1) > 0$. Choose $\epsilon < \epsilon_{(A,B)}$ sufficiently small such that $\frac{\epsilon}{\epsilon+d(A,B)} < \delta(1)$. Let $x, x' \in A$ and $y \in B$ such that $\|x-y\| \leq \epsilon + d(A,B)$ and $\|x'-y\| \leq \epsilon + d(A,B)$ then we prove $\|x-x'\| \leq \epsilon + d(A,B)$. On the contrary, suppose that $\|x-x'\| > \epsilon + d(A,B)$ then from (2.2) we get

$$\begin{aligned}\left\|\frac{x+x'}{2} - y\right\| &\leq \left(1 - \delta\left(\frac{\epsilon + d(A,B)}{\epsilon + d(A,B)}\right)\right)(\epsilon + d(A,B)) \\ &= \big(1 - \delta(1)\big)(\epsilon + d(A,B)) \\ &< \left(1 - \frac{\epsilon}{\epsilon + d(A,B)}\right)(\epsilon + d(A,B)) \\ &= d(A,B),\end{aligned}$$

since $A$ is convex, this is a contradiction. □

Proposition 3.3 requires that every nonempty pair $(A,B)$ in a uniformly convex Banach space $X$ such that $A$ is convex, has the ultrametric property. This is because $\epsilon_0(X) = 0$. The next proposition gives the relation between the ultrametric property and the $UC$ property.

**Proposition 3.4.** *Let $A$ and $B$ be nonempty subsets of the metric space $(X,d)$ and the pair $(A,B)$ has the $UC$ property. Then $(A,B)$ has the ultrametric property.*

*Proof.* Suppose that $(A,B)$ has not the ultrametric property. So $d(A,B) > 0$ and for every $n \in \mathbb{N}$ there exists $0 < \epsilon_n \leq \frac{1}{n}$, $x_n, x'_n \in A$ and $y_n \in B$ such that $\max\{d(x_n,y_n), d(x'_n,y_n)\} \leq \epsilon_n + d(A,B) \leq \frac{1}{n} + d(A,B)$ and $d(x_n,x'_n) > d(A,B)$. So $\{x_n\}$, $\{x'_n\}$ are sequences in $A$ and $\{y_n\}$ is sequence in $B$ such that

$$\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(x'_n,y_n) = d(A,B)$$

and $\lim_{n\to\infty} d(x_n,x'_n) \neq 0$. So the pair $(A,B)$ has not the $UC$ property. □

Now, we are ready to state our main best proximity point result. The contraction condition that we use is more general than the contraction condition in [13]. The authors of this reference supposed that there exists $c \in [0,1)$ such that

$$d(Tx,Ty) \leq c\max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\} + (1-c)d(A,B),$$

for all $x \in A$ and $y \in B$.

**Theorem 3.5.** *Let $A$ and $B$ be nonempty subsets of the metric space $(X,d)$ such that $A$ is complete, $(A,B)$ has the $UC$ property and $(B,A)$ has the ultrametric property. Let $T : A\cup B \to A\cup B$ be a generalized cyclic quasi-contraction, i. e., for which there exists $c \in [0,1)$ such that*

$$\begin{aligned}d(Tx,Ty) \leq{}& c\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2}\right\} \\ &+ (1-c)d(A,B),\end{aligned} \tag{3.1}$$

*for all $x \in A$ and $y \in B$. Then for every $x_0 \in A$ the sequence $\{T^{2n}x_0\}$ converges to some best proximity point $x^* \in A$. Furthermore, every best proximity point of $T$ in $A$ is a fixed point of $T^2$.*

*Proof.* Take $x_0 \in A$ and consider the sequence $\{x_n\}$ given by $x_{n+1} := Tx_n$ for $n \geq 0$. From (3.1), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_n, x_{n+1}) =& d(Tx_{n-1}, Tx_n) \\ \leq& c \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\ &+ (1-c)d(A, B) \\ \leq& c \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\ &+ (1-c)d(A, B) \\ =& c \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} + (1-c)d(A, B). \end{aligned} \tag{3.2}$$

Assume that for some $n_0 \in \mathbb{N}$,

$$\max\{d(x_{n_0-1}, x_{n_0}), d(x_{n_0}, x_{n_0+1})\} = d(x_{n_0}, x_{n_0+1}),$$

so by (3.2) we get

$$d(x_{n_0-1}, x_{n_0}) = d(x_{n_0}, x_{n_0+1}) = d(A, B),$$

hence

$$\max\{d(x_{n_0-1}, x_{n_0}), d(x_{n_0}, x_{n_0+1})\} = d(x_{n_0-1}, x_{n_0}).$$

Thus, we may assume that for each $n \in \mathbb{N}$,

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Hence, from (3.2) for every $n \in \mathbb{N}$, we obtain

$$d(x_n, x_{n+1}) \leq c\ d(x_{n-1}, x_n) + (1-c)d(A, B).$$

So for every $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq c^n d(x_0, x_1) + (1-c^n)d(A, B). \tag{3.3}$$

Immediately, in the case $d(A, B) = 0$, from (3.3) we get $\{x_n\}$ and so $\{x_{2n}\}$ is a Cauchy sequence. Suppose that $d(A, B) \neq 0$. From (3.3) we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

Since

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = \lim_{n\to\infty} d(x_{2n+2}, x_{2n+1}) = d(A, B),$$

and $(A, B)$ has the $UC$ property, then

$$\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0.$$

Fix $\epsilon > 0$ such that $\epsilon < \min\{\epsilon_{(A,B)}, \epsilon_{(B,A)}\}$. We choose $L \in \mathbb{N}$ satisfying

$$d(x_n, x_{n+1}) < \epsilon + d(A, B) \textit{ and } d(x_{2n}, x_{2n+2}) < \epsilon' = \frac{1-c}{c}\epsilon, \tag{3.4}$$

for all $n \geq L$. Fix $n \in \mathbb{N}$ with $n \geq L$. We shall show that

$$d(x_{2n+1}, x_{2p}) < \epsilon + d(A, B), \tag{3.5}$$

for all $p \geq n$. We assume that

$$d(x_{2n+1}, x_{2m}) < \epsilon + d(A, B), \tag{3.6}$$

holds for some $m \geq n$. Then since $d(x_{2m+1}, x_{2m}) < \epsilon + d(A, B)$ and $(B, A)$ has the ultrametric property, we obtain

$$d(x_{2n+1}, x_{2m+1}) < \epsilon + d(A, B) \tag{3.7}$$

and since $d(x_{2n+1}, x_{2n+2}) < \epsilon + d(A, B)$ and $(A, B)$ has the ultrmetric property we get

$$d(x_{2n+2}, x_{2m}) < \epsilon + d(A, B). \tag{3.8}$$

Also, we have

$$\begin{aligned} d(x_{2n+2}, x_{2m+1}) \leq & c \max\Big\{ d(x_{2n+1}, x_{2m}), d(x_{2n+1}, x_{2n+2}), d(x_{2m}, x_{2m+1}) \\ & , \frac{d(x_{2n+1}, x_{2m+1}) + d(x_{2n+2}, x_{2m})}{2} \Big\} + (1-c)d(A, B). \end{aligned}$$

Now, by relations (3.4), (3.6), (3.7) and (3.8) we obtain

$$d(x_{2n+2}, x_{2m+1}) < c(\epsilon + d(A, B)) + (1-c)d(A, B) < \epsilon + d(A, B). \tag{3.9}$$

Since $d(x_{2m+2}, x_{2m+1}) < \epsilon + d(A, B)$ and $(A, B)$ has the ultrmetric property we obtain

$$d(x_{2n+2}, x_{2m+2}) < \epsilon + d(A, B). \tag{3.10}$$

Hence, we have

$$\begin{aligned} d(x_{2n+1}, x_{2m+2}) \leq & c \max\Big\{ d(x_{2n}, x_{2m+1}), d(x_{2n}, x_{2n+1}), d(x_{2m+1}, x_{2m+2}) \\ & , \frac{d(x_{2n}, x_{2m+2}) + d(x_{2n+1}, x_{2m+1})}{2} \Big\} + (1-c)d(A, B) \\ \leq & c \max\Big\{ d(x_{2n}, x_{2n+2}) + d(x_{2n+2}, x_{2m+1}), d(x_{2n}, x_{2n+1}) \\ & , d(x_{2m+1}, x_{2m+2}) \\ & , \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+2}, x_{2m+2}) + d(x_{2n+1}, x_{2m+1})}{2} \Big\} \\ & + (1-c)d(A, B). \end{aligned}$$

Now, by relations (3.4), (3.7) and (3.9) and (3.10) we obtain

$$\begin{aligned} d(x_{2n}, x_{2m+3}) \leq & c \max\Big\{ \epsilon' + \epsilon + d(A, B), \epsilon + d(A, B), \frac{\epsilon' + 2(\epsilon + d(A, B))}{2} \Big\} \\ & + (1-c)d(A, B) \end{aligned}$$

where $\epsilon' = \frac{1-c}{c}\epsilon$, so we have

$$d(x_{2n+1}, x_{2m+2}) < c(\epsilon' + \epsilon + d(A, B)) + (1-c)d(A, B) = \epsilon + d(A, B).$$

By induction, we obtain (3.5) holds for all $p \geq n$ and so we get

$$\lim_{n\to\infty} \sup_{p\geq n} d(x_{2n+1}, x_{2p}) = d(A, B),$$

that by using the $UC$ property of $(A, B)$ and Lemma 2.2 imply $\{x_{2n}\}$ is a Cauchy sequence.
Hence, in both cases $d(A, B) = 0$ and $d(A, B) \neq 0$, we get the sequence $\{x_{2n}\}$ is Cauchy and so convergent to some $x^* \in A$. But we have

$$\begin{aligned} d(Tx^*, x_{2n}) \leq & c \max\Big\{ d(x^*, x_{2n-1}), d(x^*, Tx^*), d(x_{2n-1}, x_{2n}) \\ & , \frac{d(x^*, x_{2n}) + d(x_{2n-1}, Tx^*)}{2} \Big\} + (1-c)d(A, B) \\ \leq & c \max\Big\{ d(x^*, x_{2n-1}), d(x^*, Tx^*), d(x_{2n-1}, x_{2n}) \\ & , \frac{d(x^*, x_{2n}) + d(x_{2n-1}, x^*) + d(x^*, Tx^*)}{2} \Big\} + (1-c)d(A, B). \end{aligned}$$

Letting $n \to \infty$ and taking lim sup, we obtain

$$d(x^*, Tx^*) \leq cd(x^*, Tx^*) + (1-c)d(A, B),$$

and so $d(x^*, Tx^*) = d(A, B)$. Furthermore, if $z^*$ be an arbitrary best proximity point of $T$ in $A$ then we have

$$\begin{aligned} d(T^2z^*, Tz^*) \leq & c \max\Big\{ d(Tz^*, z^*), d(Tz^*, T^2z^*), \frac{d(Tz^*, Tz^*) + d(z^*, T^2z^*)}{2} \Big\} \\ & + (1-c)d(A, B) \\ \leq & c \max\Big\{ d(z^*, Tz^*), d(Tz^*, T^2z^*), \frac{d(z^*, Tz^*) + d(Tz^*, T^2z^*)}{2} \Big\} \\ & + (1-c)d(A, B) \\ = & c \max\big\{ d(z^*, Tz^*), d(Tz^*, T^2z^*) \big\} + (1-c)d(A, B) \\ = & cd(Tz^*, T^2z^*) + (1-c)d(A, B). \end{aligned}$$

So we obtain $d(T^2z^*, Tz^*) = d(A, B)$, because $d(z^*, Tz^*) = d(A, B)$ and $(A, B)$ has the $UC$ property, we get $T^2z^* = z^*$. □

In the next theorem, we present conditions for uniqueness of best proximity point.

**Theorem 3.6.** *In addition to assumptions of the previous theorem assume that* $(A_0, B_0)$ *has the Pythagorean property and* $(B, A)$ *has the* $UC$ *property. Then* $T$ *has a unique best proximity point* $x^*$ *in* $A$.

*Proof.* Suppose that $x^*$ and $\overline{x}$ are two best proximity points of $T$ in $A$, then

$$T^2x^* = x^* \text{ and } T^2\overline{x} = \overline{x}.$$

Also, we know that $(A_0, B_0)$ has the Pythagorean property and $(\overline{x}, Tx^*) \in (A_0 \times B_0)$, so there exists unique $(u, w) \in (A_0 \times B_0)$ such that $d(\overline{x}, w) = d(u, Tx^*) = d(A, B)$

and

$$d(\overline{x}, Tx^*)^2 = d(\overline{x}, u)^2 + d(u, Tx^*)^2 = d(\overline{x}, w)^2 + d(w, Tx^*)^2.$$

Because $x^*$ and $\overline{x}$ are two best proximity points of $T$ and the pairs $(A, B)$ and $(B, A)$ have the $UC$ property, we obtain $u = x^*$ and $w = T\overline{x}$. So we get

$$d(\overline{x}, Tx^*)^2 = d(\overline{x}, x^*)^2 + d(x^*, Tx^*)^2 = d(\overline{x}, T\overline{x})^2 + d(T\overline{x}, Tx^*)^2.$$

Similarly we have

$$d(x^*, T\overline{x})^2 = d(x^*, \overline{x})^2 + d(\overline{x}, T\overline{x})^2 = d(x^*, Tx^*)^2 + d(Tx^*, T\overline{x})^2.$$

Hence

$$d(x^*, \overline{x}) = d(Tx^*, T\overline{x}) \leq d(\overline{x}, Tx^*) = d(x^*, T\overline{x}). \tag{3.11}$$

From (3.1) and (3.11) we get

$$\begin{aligned}
d(\overline{x}, Tx^*) =& d(T^2\overline{x}, Tx^*) \\
\leq& c \max\Big\{ d(T\overline{x}, x^*), d(T\overline{x}, T^2\overline{x}), d(x^*, Tx^*) \\
& , \frac{d(T\overline{x}, Tx^*) + d(T^2\overline{x}, x^*)}{2} \Big\} + (1-c)d(A, B) \\
=& c \max\big\{ d(T\overline{x}, x^*), d(\overline{x}, x^*) \big\} + (1-c)d(A, B) \\
=& c \max\big\{ d(\overline{x}, Tx^*), d(\overline{x}, x^*) \big\} + (1-c)d(A, B) \\
\leq& c d(\overline{x}, Tx^*) + (1-c)d(A, B),
\end{aligned}$$

and so $d(\overline{x}, Tx^*) = d(A, B)$. Since $d(x^*, Tx^*) = d(A, B)$ and $(A, B)$ has the $UC$ property, we obtain $\overline{x} = x^*$. $\square$

The following example shows that the Pythagorean property of the pair $(A_0, B_0)$ in Theorem 3.6 is necessary to guarantee the uniqueness of best proximity of $T$. Also, it shows that Theorem 3.5 is stronger than Theorem 2 of [13].

**Example 3.7.** Let $\mathbb{R}^2$ equipped with the Euclidian metric and let

$$A = \{a = (0, 0), a' = (1, 1.1)\}$$

and

$$B = \{b = (1, 0), b' = (0, 1.1)\}.$$

We define the cyclic mapping $T : A \cup B \to A \cup B$ by

$$Ta = b, Ta' = b', Tb = a, Tb' = a'.$$

It is straightforward to show that all the assumptions of Theorem 3.5 are satisfied and for all $x \in A$ and $y \in B$ we have

$$\begin{aligned}
d(Tx, Ty) \leq& \frac{3}{4} \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2} \right\} \\
&+ \frac{1}{4} d(A, B),
\end{aligned}$$

and $a, a'$ are two best proximity points of $T$ in $A$. In this example the pair $(A_0, B_0)$ has not the Pythagorean property. Also, note that since

$$\begin{aligned} c\max\left\{d(a',b), d(a',Ta'), d(b,Tb)\right\} + (1-c)d(A,B) &= 1.1c + (1-c) \\ &= 1 + 0.1c < 1.1 \\ &= d(Ta', Tb), \end{aligned}$$

for each $c \in [0,1)$, then we can not invoke Theorem 2 of [13] to show that the existence of best proximity points of $T$ in $A$.

The next example illustrates Theorem 3.6.

**Example 3.8.** Let $\mathbb{R}^2$ with the Euclidian norm, $a = (0,1)$, $a' = (1,0)$, $b = (0,0)$, $b' = (1,1)$, $A = [a,a']$ and $B = [b,b']$. We define the cyclic mapping $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } x \in (a,a'], \\ (0,0) & \text{if } x = a. \end{cases} \quad \textit{and } Ty = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & \text{if } y \in [b,b'), \\ (1,0) & \text{if } y = b'. \end{cases}$$

for all $x \in A$ and $y \in B$ we have

$$\begin{aligned} d(Tx,Ty) \leq & \frac{1}{2}\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2}\right\} \\ & + \frac{1}{2}d(A,B). \end{aligned}$$

It is straightforward to show that all the assumptions of Theorems 3.5 and 3.6 are satisfied and $x^* = (\frac{1}{2}, \frac{1}{2})$ is unique fixed point of $T$ in $A$.

Safari et al. [12] improved Theorem 2.4 in [3]. They showed that if $A$ and $B$ be nonempty, closed and convex subsets of a uniformly convex Banach space $X$ and $T$ be a cyclic mapping on $A \cup B$, then the contraction condition

$$d(T^2x, T^2y) \leq c\ d(x,y) + (1-c)d(A,B),$$

for every $x \in A$, $y \in B$ and some $c \in [0,1)$; is sufficient to prove the existence and uniqueness of best proximity point of $T$.
The following corollary shows that the condition (3.1) is sufficient to prove the existence of best proximity point of $T$ in Theorem 2.4 in [3].

**Corollary 3.9.** *Let $A$ and $B$ be nonempty, closed and convex subsets of a uniformly convex Banach space $X$. Let $T$ be a cyclic mapping on $A \cup B$ such that*

$$\begin{aligned} \|Tx - Ty\| \leq & c\max\left\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \frac{\|x-Ty\| + \|Tx-y\|}{2}\right\} \\ & + (1-c)d(A,B), \end{aligned}$$

*for all $x \in A$ and $y \in B$ where $c \in [0,1)$. Then $T$ has at least a best proximity point $x^*$ in $A$ that is a fixed point of $T^2$.*

Note that when $d(A,B) = 0$, then the pairs $(A,B)$ and $(B,A)$ have the $UC$ property, and $(A_0, B_0)$ has the Pythagorean property. So as a result of Theorems 3.5 and 3.6 we get the following theorem that is the extention of Corollaries 2.3 and 2.10 in [8].

**Theorem 3.10.** *Let $A$ and $B$ be nonempty and closed subsets of a complete metric space $(X, d)$. Let $T$ be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

*for all $x \in A$ and $y \in B$ where $c \in [0, 1)$. Then $T$ has a unique fixed point $x^*$ in $A \cap B$ such that the Picard iteration $\{T^n x_0\}$ converges to $x^*$ for any starting point $x_0 \in A \cup B$.*

*Proof.* Without loss of generality, take $x_0 \in A$ and consider the sequence $\{x_n\}$ given by $x_{n+1} := Tx_n$ for $n \geq 0$. By the proof of Theorem 3.5

$$d(x_n, x_{n+1}) \leq c^n d(x_0, x_1), \ \forall\, n \in \mathbb{N}.$$

Then $\{x_n\}$ is a Cauchy sequence and thus there exists $x^* \in A \cup B$ such that $x_n \to x^*$. Now $\{x_{2n}\}$ is a sequence in $A$ and $\{x_{2n+1}\}$ is a sequence in $B$ and both converges to $x^*$. Since $A$ and $B$ are closed $x^* \in A \cap B$ and by the proof of Theorem 3.5 $x^*$ is a fixed point of $T$. Since $d(A, B) = 0$, from Theorem 3.6 fixed point of $T$ in $A$ and so in $A \cap B$ is unique. □

To illustrate Theorem 3.10, we state the following example.

**Example 3.11.** Let $X = \mathbb{R}^2$ with the Euclidean norm,

$$a = (0, 1),\ b = (0, 0),\ a' = (1, 0),\ b' = (1, 1),\ z = (1, \frac{1}{2}),$$

$$A = \{a, a', z\},\ B = \{b, b', z\}$$

and define the cyclic map $T$ on $A \cup B$ as follows:

$$Ta = b',\ Ta' = z,\ Tz = z,\ Tb = a',\ Tb' = z.$$

It is straightforward to show that

$$d(Tx, Ty) \leq \frac{1}{\sqrt{2}} \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

So from Theorem 3.10 $T$ has a unique fixed point $z \in A \cap B$. □

From Theorem 3.10, we obtain the following common fixed point result which is the extention of Corollary 3.11 in [12], immediatelly.

**Corollary 3.12.** *Let $(X, d)$ be a complete metric space and let $T : X \to X$ and $S : X \to X$ be two mappings satisfying*

$$d(Tx, Sy) \leq c \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\}$$

*for all $x, y \in X$ where $c \in [0, 1)$. Then $T$ and $S$ have a unique common fixed point in $X$.*

In Proposition 3.4 we prove that the ultrametric property is weaker than the $UC$ property. But if we apply the ultrametric property instead of the $UC$ property for the pair $(A, B)$ then the results of Theorem 3.5 need not be true. In fact, we need the $UC$ property to ensure the convergence of sequence $\{T^{2n} x_0\}$. The following examples shows this fact.

**Example 3.13.** Let $\mathbb{R}^3$ equipped with the Euclidian metric and let $A = \{a, a'\}$ and $B = \{b, b'\}$, that $a, a', b$ and $b'$ are four vertices of a regular tringular pyramid made of equilateral triangles of side $d = 1$. We define the cyclic mapping $T : A \cup B \to A \cup B$ by

$$Ta = b,\ Tb = a',\ Ta' = b',\ Tb' = a.$$

It is straightforward to show that the pairs $(A, B)$ and $(B, A)$ have the ultrametric property and $d(Tx, Ty) = 1 = d(A, B)$. Therefore, all the conditions of Theorem 3.5 except the $UC$ property of $(A, B)$ are true. For $x_0 = a$ and every $n \in \mathbb{N}$, we have $x_{4n} = a$, $x_{4n+1} = b$, $x_{4n+2} = a'$, $x_{4n+3} = b'$. This proves that the sequence $\{T^{2n}x_0\}$ is not convergent.

**Example 3.14.** Let $\mathbb{R}$ with the discrete metric. Let $A$ be the set of even counting numbers and $B$ be the set of odd counting numbers. We define the cyclic mapping $T : A \cup B \to A \cup B$ by $Tx = x + 1$. It is obvious that the pairs $(A, B)$ and $(B, A)$ have the ultrametric property and $d(Tx, Ty) = 1 = d(A, B)$. Therefore, all the conditions of Theorem 3.5 except the $UC$ property of $(A, B)$ are true. Then for every $n \in \mathbb{N}$ and $x_0 \in A$ we get $d(T^{2n}x_0, T^{2n+2}x_0) = 1$, so the sequence $\{T^{2n}x_0\}$ is not convergence.

Of course, regardless of the convergence of sequence $\{T^{2n}x_0\}$, the existence of the best proximity point can be examined separately if both the pair $(A, B)$ and $(B, A)$ satisfies the ultrametric property. Here, we present an existence theorem in ultrametric spaces as a special case of metric spaces. In the theorem, the notation $\mathcal{A}(X)$ denotes the family of all admissible subsets of $X$, that is, the family of subsets of $X$ that can be written as the intersection of a family of closed balls centered at points of $X$. The proof of the theorem is similar to Theorem 3.8 in [1].

**Theorem 3.15.** *Suppose the ultrametric space $X$ be spherically complete, i.e., every chain of closed balls in $X$ has nonempty intersection. Let $A$ and $B$ be nonempty subsets of $X$ such that $A \in \mathcal{A}(X)$ and $T : A \cup B \to A \cup B$ be a cyclic map which satisfies the following condition:*

$$\begin{aligned} d(Tx, Ty) \leq & c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\} \\ & + (1 - c)d(A, B), \end{aligned}$$

*for all $x \in A$ and $y \in B$ and for some $c \in [0, 1)$. Then $T$ has a best proximity point.*

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