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EXISTENCE OF NONTRIVIAL RADIAL SOLUTION FOR SEMILINEAR EQUATION WITH CRITICAL OR SUPERCRITICAL VARIABLE EXPONENT

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Abstract. This paper is devoted to study a class of semilinear elliptic equations with critical or supercritical variable exponents. By means of the mountain pass lemma, the existence of a nontrivial radial solution to this problem is obtained.
Key Words and Phrases: Semilinear elliptic equation, variable exponent, nontrivial radial solution, mountain pass lemma.
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1. Introduction and main results

Let $B \subset \mathbb{R}^N (N \geq 3)$ denote the unit ball, this paper considers the following semilinear elliptic equation with variable exponent

$$\begin{cases} -\Delta u = u^{2^*+f(|x|)-1}, & \text{in } B, \\ u > 0, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $2^* = \frac{2N}{N-2}$, $f \in C([0,1], \mathbb{R})$ satisfies condition:

(f) $f(0) = 0$, $f(t) \geq 0$ for $0 \leq t \leq 1$ and there exists $0 < \alpha < \min\left\{N-2, \frac{N}{2}\right\}$ such that $f(t) = O(t^\alpha)$ for $t \to 0^+$.

In 2008, Kurata and Shioji in [8] posed the following problem: if a variable exponent $q(\cdot)$ satisfies $2 < \inf\limits_{x\in\Omega} q(x) \leq \sup\limits_{x\in\Omega} q(x) \leq 2^*$ and $q(\cdot)$ is equal to 2^* at a point, then does the equation

$$\begin{cases} -\Delta u = |u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

have a positive solution? They showed that if there exist $x_0 \in \Omega$, $C_0 > 0$, $\eta > 0$ and $0 < l < 1$ such that $\sup\limits_{\Omega\setminus B_\eta(x_0)} q(x) < 2^*$ and $q(x) \leq 2^* - \frac{C_0}{|\log|x-x_0||^l}$ for a.e. $x \in \Omega \cap B_\eta(x_0)$, then the embedding from $H_0^1(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and problem (1.2) has a positive solution. Subsequently, many researchers studied this type of equation involving critical variable exponent(see [1, 3, 7, 10, 11]). Recently, for $q(x) = 2^* + |x|^\alpha$ and $\Omega = B$, Marcos do Ó et al. in [9] showed that the embedding from $H^1_{0,rad}(B)$ to $L^{q(x)}(B)$ is continuous and obtained a particular solution of problem (1.2). Moreover, Cao et al. in [6] considered multiple nodal solutions in the same situation.

In fact, they required the measure size of the criticality set $\{x \in \Omega : q(x) = 2^*\}$ is "small". A natural and interesting question is whether or not we can obtain existence results of nontrivial solution for problem (1.1) when the constraint of the criticality set is violated? Motivated by [8] and [9], our aim in this paper is to obtain the existence of nontrivial solutions for problem (1.1) with critical or supercritical exponent. The main result of this paper reads as follows.

Theorem 1.1. *Assume that $f \in C([0,1], \mathbb{R})$ satisfies the condition (f). Then problem* (1.1) *has a nontrivial radial solution.*

Remark 1.2. Our results extend the results of Theorem 1.5 in [9]. There are many functions satisfying the condition (f). In addition, it is worth mentioning that we do not require the measure size of the criticality set and the strictly supercritical growth except in the origin.

Throughout this paper, we use $\|\cdot\|$ and $|\cdot|_s$ to denote the usual norms of $H^1_0(B)$ and $L^s(B)$ for $s \geq 1$, respectively. The letter C and C_i stand for positive constants which may take different values at different places.

2. Preliminaries and proof of the results

Let $H^1_{0,rad}(B)$ be the subspace of $H^1_0(B)$ consisting of radially symmetric functions and the variable exponent Lebesgue space $L^{p(x)}(B)$ is defined by

$$L^{p(x)}(B) = \left\{ u \mid u : B \to \mathbb{R} \text{ is measurable, } \int_B |u|^{p(x)}\, dx < \infty \right\}$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \;\middle|\; \int_B \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

It follows from assumption (f) that there exist $A > 0$ and $\delta \in (0,1)$ such that

$$\frac{A}{2} t^\alpha < f(t) < \frac{3A}{2} t^\alpha, \quad \text{for } t \in (0, \delta). \tag{2.1}$$

Moreover, $f(t) \leq M t^\alpha$ for $t \in [\delta, 1]$, where $M = \max\limits_{\delta \leq t \leq 1} \frac{f(t)}{t^\alpha}$. Set $C = \max\{\frac{3A}{2}, M\}$, we have

$$0 \leq f(t) \leq C t^\alpha, \quad \text{for } t \in [0,1]. \tag{2.2}$$

According to (2.2), it is easy to see that $f(t)$ satisfies the conditions (f_2) and (f_3) in [9]. In addition, according to the proof of Theorem 2.1 in [9], their condition (f_1) can be reduced to $f(t) \geq 0$. Therefore, similar to Theorem 2.1 in [9], we have the following lemma.

Lemma 2.1. *Let $q(x) = 2^* + f(|x|)$ and $f \in C([0,1], \mathbb{R})$ satisfies the condition (f). Then the imbedding from $H^1_{0,rad}(B)$ to $L^{q(x)}(B)$ is continuous.*

Define the energy functional on $H^1_{0,rad}(B)$ corresponding to problem (1.1)

$$I(u) = \frac{1}{2}\|u\|^2 - \int_B \frac{|u|^{2^*+f(|x|)}}{2^* + f(|x|)}\,dx.$$

Due to Lemma 2.1, we have that the functional I is well-defined and of class C^1. It is well known that the critical points of the functional I in $H^1_{0,rad}(B)$ are solutions of problem (1.1). Define the best Sobolev constant

$$S = \inf_{u\in H(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}}. \tag{2.3}$$

From ([5]), we know that S is attained by functions $v_\varepsilon(x) = \frac{(N(N-2)\varepsilon)^{\frac{N-2}{2}}}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}}}$.

Lemma 2.2. *Assume that $f \in C([0,1],\mathbb{R})$ satisfies the condition (f), then the functional I satisfies the $(PS)_c$ condition with $c \in (0, \frac{1}{N}S^{\frac{N}{2}})$ in $H^1_{0,rad}(B)$.*

Proof. Let $\{u_n\} \subset H^1_{0,rad}(B)$ be a $(PS)_c$ sequence of I with $c \in (0, \frac{1}{N}S^{\frac{N}{2}})$. Then

$$|I(u_n)| \le c, \qquad I'(u_n) \to 0 \text{ as } \; n \to \infty. \tag{2.4}$$

It follows from the condition (f) that

$$I(u_n) - \frac{1}{2^*}\langle I'(u_n), u_n\rangle = \frac{1}{N}\|u_n\|^2 + \int_B \frac{f(|x|)|u|^{2^*+f(|x|)}}{2^*(2^*+f(|x|))}\,dx \ge \frac{1}{N}\|u_n\|^2,$$

which implies that $\frac{1}{N}\|u_n\|^2 \le c + o(\|u_n\|)$. Thus $\{u_n\}$ is a bounded sequence in $H^1_{0,rad}(B)$. Up to a subsequence, there exists $u \in H^1_{0,rad}(B)$ such that u_n converges to u weakly in $H^1_{0,rad}(B)$. Let $v_n = u_n - u$, then we see that v_n converges to 0 weakly in $H^1_{0,rad}(B)$. Since $H^1_{rad}([\theta,1]) \hookrightarrow\hookrightarrow L^p([\theta,1])$ for any $\theta \in (0,1)$ and $p \ge 1$. It implies from (2.2) that

$$\int_\theta^1 (|v_n|^{2^*+f(r)} - |v_n|^{2^*})r^{N-1}\,dr \to 0, \quad \text{as} \quad n \to \infty.$$

Similar to (3.9) in [9], for any $\varepsilon > 0$, we obtain that there exists $\theta = \theta(\varepsilon) > 0$ such that

$$\int_0^\theta (|v_n|^{2^*+f(r)} - |v_n|^{2^*})r^{N-1}\,dr \le \varepsilon.$$

Therefore, we have

$$\int_0^1 |v_n|^{2^*+f(r)} r^{N-1}\,dr = \int_0^1 |v_n|^{2^*} r^{N-1}\,dr + o(1). \tag{2.5}$$

By the Brezis-Lieb lemma(see [4]), one has

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1),$$

and

$$\int_B \frac{|u_n|^{2^*+f(|x|)}}{2^*+f(|x|)}\,dx = \int_B \frac{|v_n|^{2^*+f(|x|)}}{2^*+f(|x|)}\,dx + \int_B \frac{|u|^{2^*+f(|x|)}}{2^*+f(|x|)}\,dx + o(1).$$

Since $I(u_n) = c + o(1)$, we obtain

$$\frac{1}{2}\|v_n\|^2 - \int_B \frac{|v_n|^{2^*+f(|x|)}}{2^* + f(|x|)}\,dx = c - I(u) + o(1). \tag{2.6}$$

According to $I'(u_n) = o(1)$ and $\langle I'(u), u\rangle = 0$, we get

$$\|v_n\|^2 - \int_B |v_n|^{2^*+f(|x|)}\,dx = o(1). \tag{2.7}$$

Assume that $\|v_n\| \to l$, we have $\int_B |v_n|^{2^*+f(|x|)}\,dx \to l^2$. By (2.5), one has

$$\int_B |v_n|^{2^*}\,dx \to l^2.$$

It follows from (2.3)

$$\|v_n\|^{2^*} \geq S^{\frac{2^*}{2}} \int_{\mathbb{R}^N} |v_n|^{2^*} dx.$$

As $n \to \infty$, we obtain $l \geq S^{\frac{N}{4}}$. Note that $I(u) \geq 0$, it implies from (2.6) and $f(t) \geq 0$ that

$$c \geq \frac{1}{N} l^2 + I(u) \geq \frac{1}{N} S^{\frac{N}{2}},$$

which contradicts the fact $c < \frac{1}{N}S^{\frac{N}{2}}$. Therefore, we have $l = 0$, which implies that $u_n \to u$ strongly in $H^1_{0,rad}(B)$. Hence I satisfies the $(PS)_c$ condition with $c \in (0, \frac{1}{N}S^{\frac{N}{2}})$. □

Let $\phi(x) \in C_0^\infty(B)$ be a cut-off function such that $0 \leq \phi(x) \leq 1$ in B, $\phi(x) \equiv 1$ for $|x| \leq \rho$ and $\phi(x) \equiv 0$ for $|x| \geq 2\rho$, where $0 < \rho < \frac{1}{2}$ is a constant. Define $u_\varepsilon(x) = \phi(x)v_\varepsilon(x)$, it is known ([5]) that

$$\|u_\varepsilon\|^2 = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \qquad \int_B |u_\varepsilon|^{2^*}\,dx = S^{\frac{N}{2}} + O(\varepsilon^N). \tag{2.8}$$

Lemma 2.3. *Assume that $f \in C([0,1],\mathbb{R})$ satisfies the condition (f), then $\sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}}$ for $\varepsilon > 0$ sufficiently small.*

Proof. Define

$$h(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int_B |u_\varepsilon|^{2^*}\,dx,$$

and

$$I(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2 - \int_B \frac{t^{2^*+f(|x|)}}{2^* + f(|x|)}|u_\varepsilon|^{2^*+f(|x|)}\,dx.$$

By (2.8), after a straightforward calculation, we have

$$\sup_{t\geq 0} h(t) = \frac{1}{N}S^{\frac{N}{2}} + O(\varepsilon^{N-2}). \tag{2.9}$$

It follows from (f) and (2.8) that $\lim_{t\to+\infty} I(tu_\varepsilon) = -\infty$. Note that $I(0) = 0$ and $I(tu_\varepsilon) > 0$ for $t \to 0^+$, so $\sup_{t\geq 0} I(tu_\varepsilon)$ attains for some $t_\varepsilon > 0$ and there exist $R,\ \tau > 0$

such that $\tau \le t_\varepsilon \le R$. Let $a_\varepsilon = \left(N(N-2)\varepsilon t_\varepsilon^{\frac{2}{N-2}} - \varepsilon^2\right)^{\frac{1}{2}}$. Choose ε small enough, we have $|t_\varepsilon u_\varepsilon| \ge 1$ for $r \le a_\varepsilon$ and $|t_\varepsilon u_\varepsilon| \le 1$ for $r \ge a_\varepsilon$. Moreover, there exists $C_1 > 0$ such that

$$\varepsilon \le C_1 \varepsilon^{\frac{1}{2}} \le \left(N(N-2)\tau^{\frac{2}{N-2}}\varepsilon - \varepsilon^2\right)^{\frac{1}{2}} \le a_\varepsilon \le \left(N(N-2)R^{\frac{2}{N-2}}\varepsilon - \varepsilon^2\right)^{\frac{1}{2}} < 1. \tag{2.10}$$

Define

$$A_\varepsilon = \int_B \frac{|t_\varepsilon u_\varepsilon(x)|^{2^*+f(|x|)}}{2^*+f(|x|)}\,dx - \int_B \frac{|t_\varepsilon u_\varepsilon(x)|^{2^*}}{2^*+f(|x|)}\,dx$$

and

$$B_\varepsilon = \int_B \left(\frac{1}{2^*} - \frac{1}{2^*+f(|x|)}\right)|t_\varepsilon u_\varepsilon(x)|^{2^*}\,dx.$$

Then

$$\sup_{t\ge 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon) = h(t_\varepsilon) - A_\varepsilon + B_\varepsilon \le \sup_{t\ge 0} h(t) - A_\varepsilon + B_\varepsilon. \tag{2.11}$$

Using the definition of u_ε and assumption (f), it follows from (2.1) and (2.10) that

$$\begin{aligned}
A_\varepsilon &= N\omega_N \int_0^1 \frac{|t_\varepsilon u_\varepsilon(r)|^{2^*}}{2^*+f(r)}(|t_\varepsilon u_\varepsilon(r)|^{f(r)} - 1)r^{N-1}\,dr \\
&\ge N\omega_N \int_0^\varepsilon \frac{|t_\varepsilon u_\varepsilon(r)|^{2^*}}{2^*+f(r)}(|t_\varepsilon u_\varepsilon(r)|^{f(r)} - 1)r^{N-1}\,dr - N\omega_N \int_{a_\varepsilon}^1 \frac{|t_\varepsilon u_\varepsilon(r)|^{2^*}}{2^*+f(r)} r^{N-1}\,dr \\
&\ge C_2 \int_0^\varepsilon \varepsilon^{-N}|\log\varepsilon| r^{N-1} f(r)\,dr - C_3 \int_{a_\varepsilon}^1 \frac{\varepsilon^N}{r^{2N}} r^{N-1}\,dr \\
&\ge \frac{C_2 A}{2}\int_0^\varepsilon \varepsilon^{-N}|\log\varepsilon| r^{N-1+\alpha}\,dr - C_3\varepsilon^N(a_\varepsilon^{-N} - 1) \\
&\ge \frac{C_2 A}{2}\varepsilon^\alpha|\log\varepsilon| - \frac{C_3}{C_1^N}\varepsilon^{\frac{N}{2}}
\end{aligned} \tag{2.12}$$

where $\omega_N = (2\pi^{N/2})/(N\Gamma(N/2))$ denotes the volume of the unit ball B. In addition, by (2.2), we have

$$\begin{aligned}
B_\varepsilon &= N\omega_N \int_0^1 \frac{f(r)}{2^*(2^*+f(r))}|t_\varepsilon u_\varepsilon(r)|^{2^*} r^{N-1}\,dr \\
&\le C_4 \int_0^\varepsilon \varepsilon^{-N} r^{N-1} f(r)\,dr + C_4 \int_\varepsilon^1 \frac{\varepsilon^N}{r^{2N}} r^{N-1} f(r)\,dr \\
&\le C_5 \int_0^\varepsilon \varepsilon^{-N} r^{N-1} r^\alpha\,dr + C_5 \int_\varepsilon^1 \frac{\varepsilon^N}{r^{2N}} r^{N-1} r^\alpha\,dr \\
&= C_5\varepsilon^\alpha + C_5(\varepsilon^\alpha - \varepsilon^N) \\
&\le 2C_5\varepsilon^\alpha.
\end{aligned} \tag{2.13}$$

Note that $0 < \alpha < \min\{N-2, \frac{N}{2}\}$, from (2.11), (2.12) and (2.13), we obtain

$$\sup_{t\geq 0} I(tu_\varepsilon) \leq \frac{1}{N} S^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \frac{C_2 A}{2}\varepsilon^{\alpha}|\log\varepsilon| + \frac{C_3}{C_1^N}\varepsilon^{\frac{N}{2}} + 2C_5\varepsilon^{\alpha} < \frac{1}{N} S^{\frac{N}{2}}$$

for $\varepsilon > 0$ sufficiently small. The proof is complete. □

Proof of Theorem 1.1. According to (2.8), we know that the functional I has the mountain pass geometry and there exists $t_0 > 0$ such that $I(t_0 u_\varepsilon) < 0$. Define

$$\Gamma = \{\gamma \in C([0,1], X_0) |\ \gamma(0) = 0,\ \gamma(1) = t_0 u_\varepsilon\}, \quad \tilde{c} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)).$$

From Lemma 2.3, we have $\tilde{c} \leq \sup_{t\geq 0} I(tu_\varepsilon) < \frac{1}{N} S^{\frac{N}{2}}$. Applying Lemma 2.2, we know that I satisfies the $(PS)_{\tilde{c}}$ condition. By the mountain pass theorem(see [2]), we obtain that problem (1.1) has a nontrival solution. The proof is complete. □

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