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MEIR-KEELER S TYPE CONTRACTIONS AND CONTRACTIONS WITH F CONTROL FUNCTIONS ON S-METRIC SPACES

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Abstract. In this paper, some fixed point results under the Meir-Keeler S type contraction conditions on S-metric spaces and some fixed point theorems with F control functions on S-metric spaces are proved. Moreover, the relationships among G-metric and S-metric are investigated and an example of discontinuity at a fixed point is given.
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1. Introduction

In 1992, Dhage introduced the D-metric in his Ph.D. thesis [4]. In 2003, Mustafa however demonstrated that most claims concerning the fundamental topological properties of D-metric were incorrect and he instead introduced the G-metric [14]. Since then, many authors studied fixed points and common fixed points on G-metric spaces [27, 6]. In 2007, Sedghi et al. gave some definitions of D^*-metric [22] and in 2012, he also defined an S-metric [21] as a generalization of D^*-metric. From then on many authors studied the S-metric [24, 23, 15, 16, 25, 17, 5, 18, 26]. In this paper we investigate the relationships between the G-metric and S-metric, and we introduce the concept of Meir-Keeler S type contraction on S-metric space. We also obtain new fixed point theorems with F control functions on S-metric spaces.

In this section, we recall some definitions that will be used in the remainder of this paper.

Definition 1.1. [14] Let X be a nonempty set, $G : X \times X \times X \longrightarrow [0, +\infty]$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, $a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. [14] A G-metric space (X, G) is symmetric if $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

Definition 1.3. [21] Let X be a nonempty set and $S : X \times X \times X \longrightarrow R^+$ be a function satisfying the following properties:

(1) $S(x, y, z) = 0$ iff $x = y = z$;
(2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $a, x, y, z \in X$.

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Remark 1.4. [21] $S(x, x, y) = S(y, y, x)$.

Example 1.5. [21] $S(x, y, z) =\mid x - z \mid + \mid y - z \mid$ is an S-metric on R.

Example 1.6. [20] $S(x, y, z) =\mid x - z \mid + \mid y + z - 2x \mid$ is an S-metric on R.

Remark 1.7. G-metric and S-metric can not contain each other. Because $G(x, x, y)$ is not always equal $G(y, y, x)$, but $S(x, x, y) = S(y, y, x)$. And $G(x, x, y) \leq G(x, y, z)$ for $z \neq y$, but $S(x, x, y)$ is not always less than $S(x, y, z)$ even $z \neq y$.

Example 1.8. Let $X = \{a, b\}$ and define G by
$G(a, a, a) = G(b, b, b) = 0$,
$G(a, a, b) = G(a, b, a) = G(b, a, a) = 2$,
$G(b, b, a) = G(b, a, b) = G(a, b, b) = 4$.
Then G is a G-metric, but $G(a, a, b) \neq G(b, b, a)$.

Example 1.9. Let $X = \{a, b, c\}$ and define S by
$S(a, a, a) = S(b, b, b) = S(c, c, c) = 0$,
$S(a, a, b) = S(b, b, a) = 1$, $S(a, a, c) = S(c, c, a) = 2$,
$S(b, b, c) = S(c, c, b) = 1$, $S(a, b, c) = 1$.
Assume $S(x, y, z) = S(y, x, z) = S(z, y, x) = \cdots$ (symmetry in all three variables).
Then S is an S-metric, but $S(a, b, c) < S(a, a, c)$.

Definition 1.10. [3] Let X be a nonempty set. A function $d : X \to [0, +\infty)$ is said to be a b-metric if there exists $b \geq 1$ such that for all $x, y, z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \leq b(d(x, z) + d(z, y))$.

In this case, the pair (X, d) is called a b-metric space.

Remark 1.11. [24] $d(x,y) := S(x,x,y)$ is a b-metric, and

$$d(x,y) \le \frac{3}{2}(d(x,z)+d(z,y)).$$

Definition 1.12. [21] Let (X,S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X is said to S-converge to a point $x \in X$ if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, $S(x_n,x_n,x) < \varepsilon$.
(2) A sequence $\{x_n\}$ in X is called an S-Cauchy sequence if, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \ge n_0$.
(3) The S-metric space (X,S) is said to be S-complete if every S-Cauchy sequence is S-convergent.

Lemma 1.13. [21] *Let (X,S) be an S-metric space. If a sequence $\{x_n\}$ in X S-converges to x, then x is unique.*

Definition 1.14. [8] Let $T,g : X \to X$. If $Tx = gx$ implies $Tgx = gTx$ for all $x \in X$, then the pair (T,g) is said to be weakly compatible.

2. Meir-Keeler S type contraction on S-metric spaces

Meir-Keeler's result [12], proved in 1969, plays a fundamental role in the fixed point theory for metric spaces [2, 9]. Z. Mustafa generalized Meir-Keeler type contraction on G-metric spaces [13]. Here we generalize that contraction on S-metric spaces.

Definition 2.1. Let (X,S) be an S-metric space and T be a self-mapping of X. Then T is called a Meir-Keeler S type contraction whenever for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < M(x,y,z) < \varepsilon + \delta \Rightarrow S(Tx,Ty,Tz) \le \varepsilon,$$

where

$$\begin{aligned} M(x,y,z) = \max\Big\{ & S(x,y,z), S(Tx,Tx,x), S(Ty,Ty,y), S(Tz,Tz,z), \\ & \frac{S(Tx,Tx,y)+S(Ty,Ty,x)}{3}, \frac{S(Tx,Tx,z)+S(Tz,Tz,x)}{3}, \\ & \frac{S(Tz,Tz,y)+S(Ty,Ty,z)}{3} \Big\}. \end{aligned}$$

Remark 2.2. Note that if T is a Meir-Keeler S type contraction and $M(x,y,z) > 0$, we have $S(Tx,Ty,Tz) < M(x,y,z)$.

Proposition 2.3. *Let (X,S) be an S-metric space and $T : X \to X$ be a Meir-Keeler S type contraction. Then*

$$\lim_{n\to\infty} S(T^{n+1}x, T^{n+1}x, T^n x) = 0 \text{ and } \lim_{n\to\infty} S(T^n x, T^n x, T^{n+1}x) = 0$$

for all $x \in X$.

Proof. Let $x_0 \in X$. We define an iterative sequence $\{x_n\}$ as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0$$

for all $n \geq 0$. If some $x_{n_0+1} = x_{n_0}$ for some $n_0 \geq 0$, then x_{n_0} is a fixed point of T. In this case, $S(T^{n+1}x, T^{n+1}x, T^n x) = 0$, for $n \geq n_0$, then the proposition follows. Throughout the proof, we assume that $x_{k+1} \neq x_k$ for all $k \in N$. Since

$$\begin{aligned}
&M(x_{n+1}, x_n, x_n) \\
&= \max\Big\{S(x_{n+1}, x_{n+1}, x_n), S(Tx_{n+1}, Tx_{n+1}, x_{n+1}), S(Tx_n, Tx_n, x_n), S(Tx_n, Tx_n, x_n), \\
&\quad \frac{S(Tx_{n+1}, Tx_{n+1}, x_n) + S(Tx_n, Tx_n, x_{n+1})}{3}, \\
&\quad \frac{S(Tx_{n+1}, Tx_{n+1}, x_n) + S(Tx_n, Tx_n, x_{n+1})}{3}, \\
&\quad \frac{S(Tx_n, Tx_n, x_n) + S(Tx_n, Tx_n, x_n)}{3}\Big\} \\
&= \max\left\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1}), \frac{S(x_{n+2}, x_{n+2}, x_n)}{3}\right\} \\
&\leq \max\Big\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1}), \\
&\quad \frac{2S(x_{n+2}, x_{n+2}, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_n)}{3}\Big\} \\
&= \max\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1})\}.
\end{aligned}$$

So $M(x_{n+1}, x_{n+1}, x_n) = \max\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1})\} > 0$. Since T is a Meir-Keeler S type contraction,

$$S(x_{n+2}, x_{n+2}, x_{n+1}) = S(Tx_{n+1}, Tx_{n+1}, Tx_n) < M(x_{n+1}, x_{n+1}, x_n).$$

Then it is impossible that

$$\max\{S(x_{n+1}, x_{n+1}, x_n), S(x_{n+2}, x_{n+2}, x_{n+1})\} = S(x_{n+2}, x_{n+2}, x_{n+1}).$$

Hence we derive that

$$S(x_{n+2}, x_{n+2}, x_{n+1}) < M(x_{n+1}, x_{n+1}, x_n) = S(x_{n+1}, x_{n+1}, x_n)$$

for every n. Thus $\{S(x_{n+1}, x_{n+1}, x_n)\}_{n=0}^{\infty}$ is a decreasing sequence, hence converges to some $\varepsilon \in [0, \infty)$, that is

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = \varepsilon.$$

In particular, we have

$$\lim_{n\to\infty} M(x_{n+1}, x_{n+1}, x_n) = \varepsilon. \tag{2.1}$$

Notice that $\varepsilon = \inf\{S(x_{n+1}, x_{n+1}, x_n) : n \in N\}$.
We claim that $\varepsilon = 0$. Suppose to the contrary that $\varepsilon > 0$. Regarding (2.1) together with the assumption that T is a Meir-Keeler S type contraction, for this $\varepsilon > 0$, there

exists $\delta > 0$ and a natural number m such that $\varepsilon < M(x_{m+1}, x_{m+1}, x_m) < \varepsilon + \delta$, then we have

$$S(Tx_{m+1}, Tx_{m+1}, Tx_m) = S(x_{m+2}, x_{m+2}, x_{m+1}) \leq \varepsilon$$

which is a contradiction, because

$$\varepsilon = \inf\{S(x_{n+1}, x_{n+1}, x_n) : n \in N\}$$

and

$$\{S(x_{n+1}, x_{n+1}, x_n)\}_{n=0}^{\infty}$$

is a strictly decreasing sequence. So we get

$$\lim_{n\to\infty} S(T^{n+1}x, T^{n+1}x, T^n x) = 0.$$

Since

$$S(x_n, x_n, x_{n+1}) = S(x_{n+1}, x_{n+1}, x_n),$$

we also obtain $\lim_{n\to\infty} S(T^{n+1}x, T^{n+1}x, T^n x) = 0$. □

Theorem 2.4. *Let (X, S) be a complete S-metric space. Let $T : X \to X$ be an orbitally continuous mapping and a Meir-Keeler S type contraction. Then T has a unique fixed point, say $w \in X$. Moreover, $\lim\limits_{n\to\infty} S(T^{n+1}x, T^{n+1}x, w) = 0$ for all $x \in X$.*

Proof. Let $x_0 \in X$. We define an iterative sequence $\{x_n\}$ as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0$$

for all $n \geq 0$. We claim that $\lim_{m,n\to\infty} S(x_n, x_n, x_m) = 0$. If this is not the case, then there exists a $\varepsilon > 0$ and a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ such that

$$S(x_{n(i)}, x_{n(i)}, x_{n(i+1)}) > 2\varepsilon. \tag{2.2}$$

For the same $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < M(x, y, z) < \varepsilon + \delta$ which implies $S(Tx, Ty, Tz) \leq \varepsilon$. Set $r = \min\{\varepsilon, \delta\}$. By Proposition 2.3, one can choose a natural number n_0 such that

$$S(x_{n+1}, x_{n+1}, x_n) < \frac{r}{8}, \quad S(x_n, x_n, x_{n+1}) < \frac{r}{8} \tag{2.3}$$

for all $n \geq n_0$. Let $n(i) > n_0$, we have $n(i) \leq n(i+1) - 1$. Because

$$\begin{aligned} S(x_{n(i)}, x_{n(i)}, x_{n(i+1)-1}) &\geq S(x_{n(i)}, x_{n(i)}, x_{n(i+1)}) - 2S(x_{n(i+1)-1}, x_{n(i+1)-1}, x_{n(i+1)}) \\ &\geq 2\varepsilon - \frac{r}{4} \geq 2\varepsilon - \frac{\varepsilon}{4} = \varepsilon + \frac{3\varepsilon}{4} \geq \varepsilon + \frac{r}{2} \end{aligned}$$

and

$$S(x_{n(i)}, x_{n(i)}, x_{n(i)+1}) < \frac{r}{4} < \varepsilon + \frac{r}{2},$$

it follows that the value of $S(x_{n(i)}, x_{n(i)}, x_k)$ changes from less than $\varepsilon + \frac{r}{2}$ to no less than $\varepsilon + \frac{r}{2}$ when k increases from $n(i) + 1$ to $n(i+1) - 1$. We can choose the smallest integer k with $n(i)+2 \le k \le n(i+1)-1$ such that $S(x_{n(i)}, x_{n(i)}, x_k) \ge \varepsilon + \frac{r}{2}$, and $S(x_{n(i)}, x_{n(i)}, x_{k-1}) < \varepsilon + \frac{r}{2}$. We then get

$$\begin{aligned} S(x_{n(i)}, x_{n(i)}, x_k) &\le S(x_{n(i)}, x_{n(i)}, x_{k-1}) + 2S(x_{k-1}, x_{k-1}, x_k) \\ &< \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4}. \end{aligned} \tag{2.4}$$

Therefore, we obtain the inequalities

$$\varepsilon + \frac{r}{2} \le S(x_{n(i)}, x_{n(i)}, x_k) < \varepsilon + r, \tag{2.5}$$

$$S(x_{n(i)+1}, x_{n(i)+1}, x_{n(i)}) < \frac{r}{8} < \varepsilon + r, \tag{2.6}$$

$$S(x_{k+1}, x_{k+1}, x_k) < \frac{r}{8} < \varepsilon + r, \tag{2.7}$$

$$\begin{aligned} &\frac{S(x_{n(i)+1}, x_{n(i)+1}, x_k) + S(x_{k+1}, x_{k+1}, x_{n(i)})}{3} \\ \le &\frac{2S(x_{n(i)+1}, x_{n(i)+1}, x_{n(i)}) + S(x_{n(i)}, x_{n(i)}, x_k) + 2S(x_{k+1}, x_{k+1}, x_k) + S(x_k, x_k, x_{n(i)})}{3} \\ < &\frac{\frac{r}{4} + \varepsilon + \frac{3r}{4} + \frac{r}{4} + \varepsilon + \frac{3r}{4}}{3} = \varepsilon + r. \end{aligned} \tag{2.8}$$

By (2.5)-(2.8), we get that $\varepsilon < M(x_{n(i)}, x_k, x_k) < \varepsilon + r$. Since T is a Meir-Keeler S type contraction, we derive $S(x_{n(i)+1}, x_{k+1}, x_{k+1}) \le \varepsilon$. But

$$\begin{aligned} S(x_{n(i)+1}, x_{n(i)+1}, x_{k+1}) &\ge S(x_{n(i)}, x_{n(i)}, x_k) - 2S(x_{n(i)}, x_{n(i)}, x_{n(i)+1}) \\ &\quad - 2S(x_{k+1}, x_{k+1}, x_k) \\ &> \varepsilon + \frac{r}{2} - \frac{r}{4} - \frac{r}{4} = \varepsilon. \end{aligned}$$

This is a contradiction. Therefore, our claim is proved. So $\{x_n\}$ is an S-Cauchy sequence. Since (X, S) is S-complete, the sequence $\{x_n\}$ S-converges to some $w \in X$, we have

$$\lim_{n\to\infty} S(T^n x_0, T^n x_0, w) = \lim_{n\to\infty} S(w, w, T^n x_0) = 0.$$

Since T is orbitally continuous and $\lim_{n\to\infty} S(T^n x_0, w, w) = 0$, we get

$$\lim_{n\to\infty} S(TT^n x_0, TT^n x_0, Tw) = 0,$$

that is,

$$\lim_{n\to\infty} S(T^{n+1} x_0, T^{n+1} x_0, Tw) = \lim_{n\to\infty} S(x_{n+1}, x_{n+1}, Tw) = 0.$$

Thus, $\{x_{n+1}\}$ converges to Tw in (X, S). By the uniqueness of limit, we get $Tw = w$. Finally, we show that T has a unique fixed point. If there exists $u \in X$ such that $Tu = u$ and $S(u, u, w) > 0$,

$$\begin{aligned} &M(u, u, w) \\ &= \max\left\{ S(u, u, w), S(Tu, Tu, u), S(Tw, w, w), \frac{S(Tu, Tu, w) + S(Tw, Tw, u)}{3} \right\} \\ &= S(u, u, w) > 0. \end{aligned}$$

Since T is a Meir-Keeler S type contraction, we derive

$$M(u,u,w) > S(Tu,Tu,Tw) = S(u,u,w),$$

which is a contradiction. Thus, we find that $S(u,u,w) = 0$. So we conclude that $u = w$. T has a unique fixed point. □

Bisht and Pant [1] gave a solution to the question of the existence of a contractive mapping that has a fixed point which is discontinuous at the fixed point. The following theorem shows some Meir-Keeler S type contractions on S-metric space have a fixed point but the mapping need not be continuous at the fixed point.

Theorem 2.5. *Let (X,S) be a complete S-metric space. Let $T : X \to X$ be a Meir-Keeler S type contraction. Assume T^2 is an orbitally continuous mapping. Then, T has a unique fixed point, say $w \in X$. And, $\lim_{n\to\infty} S(T^{n+1}x,w,w) = 0$ for all $x \in X$. Moreover, T is continuous at w iff $\lim_{x\to w} M(x,w,w) = 0$.*

Proof. Because T is a Meir-Keeler S type contraction, Proposition 2.3 is still correct. And just like the proof of Theorem 2.4, we can also define an iterative sequence $\{x_n = T^n x_0\}$, where $x_0 \in X$ is arbitrary; and also we can show the sequence is an S-Cauchy sequence. Since X is S-complete, there exists a point $w \in X$ such that

$$\lim_{n\to\infty} S(T^n x_0, T^n x_0, w) = 0.$$

Also

$$\lim_{n\to\infty} S(T^2T^n x_0, T^2T^n x_0, w) = \lim_{n\to\infty} S(x_{n+2}, x_{n+2}, w) = 0.$$

By the orbital continuity of T^2, we have

$$\lim_{n\to\infty} S(T^2T^n x_0, T^2T^n x_0, T^2w) = 0.$$

By the uniqueness of limit, we get $T^2w = w$. We claim that $Tw = w$. If $w \neq Tw$, then

$$\begin{aligned} M(Tw,Tw,w) =& \max\Big\{S(Tw,Tw,w), S(Tw,Tw,Tw), S(Tw,Tw,w), \\ & \frac{S(T^2w,T^2w,w) + S(Tw,Tw,Tw)}{3}, \\ & \frac{S(T^2w,T^2w,w) + S(Tw,Tw,Tw)}{3}, \\ & \frac{S(Tw,Tw,w) + S(Tw,Tw,w)}{3}\Big\} \\ =& \max\{S(Tw,Tw,w), S(Tw,Tw,Tw)\} \\ =& S(w,w,Tw) > 0, \end{aligned}$$

and since T is a Meir-Keeler S type contraction,

$$S(w,w,Tw) = S(T^2w,T^2w,Tw) < M(Tw,Tw,w) = S(w,w,Tw).$$

which is a contradiction. Thus, w is a fixed point of T. The uniqueness of fixed point we can also get just as the proof of Theorem 2.4.

Finally, we show that T is continuous at w iff $\lim_{x\to w} M(x,x,w)=0$.

Let T be continuous at the fixed point w and let a sequence $\{y_n\}$ in X converge to w, i.e., $\lim_{n\to\infty} S(y_n,y_n,w)=\lim_{n\to\infty} S(w,w,y_n)=0$, and $\lim_{n\to\infty} S(Ty_n,Ty_n,Tw)=0$. Since

$$\begin{aligned}
&M(y_n,y_n,w)\\
=&\max\left\{S(y_n,y_n,w),S(Ty_n,Ty_n,y_n),\frac{S(Ty_n,Ty_n,w)+S(Tw,Tw,y_n)}{3}\right\}\\
\le&\max\left\{S(y_n,y_n,w),S(Ty_n,Ty_n,w)+S(w,w,y_n),\frac{S(Ty_n,Ty_n,Tw)+S(w,w,y_n)}{3}\right\},
\end{aligned}$$

we get $\lim_{n\to\infty} M(y_n,y_n,w)=0$.
On the other hand, if

$$\lim_{n\to\infty} M(y_n,y_n,w)=0$$

and

$$\lim_{n\to\infty} S(y_n,y_n,w)=\lim_{n\to\infty} S(w,w,y_n)=0,$$

and since

$$\begin{aligned}
\frac{S(Ty_n,Ty_n,w)}{3}&\le M(y_n,y_n,w)\\
=\max&\left\{S(y_n,y_n,w),S(Ty_n,Ty_n,y_n),\frac{S(Ty_n,Ty_n,w)+S(Tw,Tw,y_n)}{3}\right\},
\end{aligned}$$

we get $\lim_{n\to\infty} S(Ty_n,Ty_n,Tw)=0$, that is, T is continuous at w. □

Example 2.6. Let $X=[0,2]$ and $S(x,y,z)=\max\{|x-y|,|x-z|\}$ for all $x,y,z\in X$. Define $T:X\to X$ by

$$T(x)=\begin{cases}1, & \text{if } x\le 1,\\ 0, & \text{if } x>1.\end{cases}$$

We shall show that T is a Meir-Keeler S type contraction. Without loss of generality, take $z\le y\le x$. We have the following cases:

Case 1: $0\le z\le y\le x\le 1$. Here we have $S(Tx,Ty,Tz)=S(1,1,1)=0$ and

$$\begin{aligned}
M(x,y,z)=&\max\Big\{S(x,y,z),S(1,1,x),S(1,1,y),S(1,1,z),\\
&\frac{S(1,1,y)+S(1,1,x)}{3},\frac{S(1,1,z)+S(1,1,x)}{3},\frac{S(1,1,y)+S(1,1,z)}{3}\Big\}\\
=&1-z;
\end{aligned}$$

Case 2: $0\le z\le y\le 1$ and $1<x\le 2$. Here we have $S(Tx,Ty,Tz)=1$ and $M(x,y,z)=x$;

Case 3: $0\le z\le 1$ and $1<y\le x\le 2$. Here we have $S(Tx,Ty,Tz)=1$ and $M(x,y,z)=x$;

Case 4: $1<z\le y\le x\le 2$. Here we have $S(Tx,Ty,Tz)=0$ and $M(x,y,z)=x$.

When $\varepsilon \geq 1$ the case 2, case 3, case 4 probably satisfy $\varepsilon < M(x,y,z)$, here $S(Tx,Ty,Tz) \leq \varepsilon$ and the $\delta(\varepsilon)$ can be any positive number. When $0 < \varepsilon < 1$ the case 1 probably satisfy $\varepsilon < M(x,y,z)$, and here we let $\delta(\varepsilon) = 1 - \varepsilon$ to limit only the case 1 occur, and the $S(Tx,Ty,Tz) \leq \varepsilon$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < M(x,y,z) < \varepsilon + \delta \Rightarrow S(Tx,Ty,Tz) \leq \varepsilon.$$

So T is a Meir-Keeler S type contraction. T^2 is continuous, since $T^2(x) = 1$ for all $x \in X$. Then T satisfies the condition of Theorem 2.5 and has a unique fixed point $x = 1$. It can also be seen that

$$\begin{aligned}\lim_{x\to 1^+} M(1,x,x) &= \lim_{x\to 1^+} \max\Big\{S(1,1,x), S(1,1,1), S(0,0,x), S(0,0,x), \\ &\qquad \frac{S(1,1,x)+S(0,0,1)}{3}, \frac{S(0,0,x)+S(0,0,x)}{3}\Big\} \\ &= \lim_{x\to 1^+} \max\Big\{x-1, 0, x, x, \frac{x-1+1}{3}, \frac{x+x}{3}\Big\} \\ &= \lim_{x\to 1^+} x = 1 \neq 0,\end{aligned}$$

and T is discontinuous at the fixed point $x = 1$.

3. Contraction by F control function on S-metric spaces

This section is inspired by [11, 7]. In the papers [11, 7] using contraction through F control function on metric-like space to get common fixed point and coupled common fixed point. We generalize the contraction with F control function on S-metric spaces.

Definition 3.1. [19] Let $\phi, \varphi : [0,+\infty) \to [0,+\infty)$ be two functions. If they satisfy the following conditions:

(1) if $\phi(u) \leq \varphi(v)$, then $u \leq v$;
(2) for $u_n, v_n \in [0,+\infty)$ with $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = w$, if $\phi(u_n) \leq \varphi(v_n)$ for all n $n \in \mathbb{N}$, then $w = 0$,

then (ϕ, φ) is called a pair of shifting distance functions.

Remark 3.2. [7] If (ϕ, φ) is a pair of shifting distance functions, $\phi(t) \leq \varphi(t)$, then $t = 0$.

Example 3.3. The pair (ϕ, φ) defined by $\phi(t) = \arctan(1+2t)$, $\varphi(t) = \arctan(1+t)$ is a pair of shifting distance functions on $[0,+\infty)$.

The F control function $F : [0,+\infty)^3 \to [0,\infty)$ was introduced by Karapinar et al. [10]. We extend it to functions of four variables.

Definition 3.4. A function $F : [0,+\infty)^4 \to [0,\infty)$ is a control function if it satisfies the conditions:

(F_1) $\max\{a,b\} \leq F(a,b,c,d)$;
(F_2) $F(a,0,0,0) = a$;
(F_3) F is continuous.

We denote this class of functions F by $\mathbb{F}$.

Example 3.5. The following functions belong to $\mathbb{F}$.

1. $F(a,b,c,d) = a+b+c+d$,
2. $F(a,b,c,d) = \max\{a,b\} + \ln(c+d+1)$.

Theorem 3.6. *Let (X,S) be an S-complete S-metric space, and $T,g : X \to X$ functions such that $TX \subset gX$. Assume gX is closed and satisfies the condition:*

$$\begin{aligned} &\phi\left(F(S(Tx,Ty,Tz),\psi(Tx),\psi(Ty),\psi(Tz))\right) \\ &\le \varphi\left(F(S(gx,gy,gz),\psi(gx),\psi(gy),\psi(gz))\right), \end{aligned} \tag{3.1}$$

for all $x,y,z \in X$, where $F \in \mathbb{F}$, (ϕ,φ) is a pair of shifting distance function, and $\psi : X \to R_+$ is a lower semi-continuous function. Then T and g have a coincidence point. If (T,g) is weakly compatible, then T and g have a unique common fixed point.

Proof. Step 1. Let $x_0 \in X$. Since $TX \subset gX$, there exists $x_1 \in X$ such that $gx_1 = Tx_0$. There exists $x_2 \in X$ such that $gx_2 = Tx_1$. Continue this procedure, we get a sequence $\{x_n\}$ satisfying

$$gx_n = Tx_{n-1}, \quad n = 1,2,3,\cdots. \tag{3.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, then $Tx_{n_0} = gx_{n_0}$, which means that T and g have a coupled point and the proof is ended. In the following we assume $gx_{n_0} \neq gx_{n_0+1}$ for $n \in \mathbb{N}$. In (3.1) we let $(x,y,z) = (x_n,x_n,x_{n+1})$. From (3.2) we obtain

$$\begin{aligned} &\phi\left(F(S(gx_{n+1},gx_{n+1},gx_{n+2}),\psi(gx_{n+1}),\psi(gx_{n+1}),\psi(gx_{n+2}))\right) \\ \le&\varphi\left(F(S(gx_n,gx_n,gx_{n+1}),\psi(gx_n),\psi(gx_n),\psi(gx_{n+1}))\right). \end{aligned}$$

In light of (1) of Definition 3.1 we have

$$\{F(S(gx_n,gx_n,gx_{n+1}),\psi(gx_n),\psi(gx_n),\psi(gx_{n+1}))\}$$

is a decreasing sequence. So there exists $r \ge 0$ such that

$$\lim_{n\to\infty} F(S(gx_n,gx_n,gx_{n+1}),\psi(gx_n),\psi(gx_n),\psi(gx_{n+1})) = r.$$

It follows from (2) of Definition 3.1 that $r = 0$.
By virtue of (F_1) of Definition 3.4 we gain that

$$\lim_{n\to\infty} \max\{S(gx_n,gx_n,gx_{n+1}),\psi(gx_n)\} = 0. \tag{3.3}$$

Step 2. To prove $\{gx_n\}$ is an S-Cauchy sequence. If $\lim\limits_{n\to\infty} S(gx_n,gx_n,gx_{n+1}) \neq 0$, then there exists $\varepsilon > 0$, two subsequence $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$ with $m(k) > n(k) \ge k$ and $m(k)$ is the smallest positive integer which satisfy

$$S(gx_{n(k)},gx_{n(k)},gx_{m(k)}) \ge \varepsilon. \tag{3.4}$$

We have

$$S(gx_{n(k)},gx_{n(k)},gx_{m(k)-1}) < \varepsilon. \tag{3.5}$$

Since

$$\begin{aligned}\varepsilon \leq & S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\ \leq & S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)-1}) + 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) \\ \leq & \varepsilon + 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}),\end{aligned}$$

and (3.3) we gain

$$\lim_{n\to\infty} S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) = \varepsilon.$$

In light of

$$\begin{aligned}& S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\ & \leq S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)-1}) + 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) \\ & \leq S(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)-1}) + 2S(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) \\ & + 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}),\end{aligned}$$

and

$$\begin{aligned}& S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}) \\ & \leq S(gx_{m(k)-1}, gx_{m(k)-1}, gx_{n(k)}) + 2S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{n(k)}) \\ & \leq S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + 2S(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) \\ & + 2S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{n(k)}),\end{aligned}$$

we get

$$\lim_{n\to\infty} S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}) = \varepsilon.$$

In (3.1), let $(x, y, z) = (x_{n(k)-1}, x_{n(k)-1}, x_{m(k)-1})$, we derive that

$$\phi(A_k) \leq \varphi(B_k),$$

with

$$A_k = F(S(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}), \psi(gx_{n(k)}), \psi(gx_{n(k)}), \psi(gx_{m(k)})),$$

$$B_k = F(S(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)-1}), \psi(gx_{n(k)-1}), \psi(gx_{n(k)-1}), \psi(gx_{m(k)-1})),$$

From the property of F we get $\lim_{n\to\infty} A_k = \lim_{n\to\infty} B_k = F(\varepsilon, 0, 0, 0) = \varepsilon$.
On account of (2) of Definition 3.1 we get $\varepsilon = 0$. It is a contradiction. So $\{gx_n\}$ is an S-Cauchy sequence. From the S-completely of (X, S), we obtain that the sequence $\{gx_n\}$ is S-convergent. Since gX is a closed set there exists gx such that

$$\lim_{n\to\infty} gx_n = gx.$$

Step 3. To prove $gx = Tx$.
From the lower semi-continuity of ψ we get $\psi(gx) \leq \liminf_{n\to\infty} \psi(gx_n) = 0$.
On account of (3.1) and (1) of Definition 3.1 we get

$$\begin{aligned}& \lim_{n\to\infty} F(S(Tx, Tx, Tx_n), \psi(Tx), \psi(Tx), \psi(Tx_n)) \\ \leq & \lim_{n\to\infty} F(S(gx, gx, gx_n), \psi(gx), \psi(gx), \psi(gx_n)) = 0.\end{aligned}$$

In light of $\lim_{n\to\infty} S(Tx, Tx, Tx_n) = 0$, we have

$$S(Tx, Tx, gx) \leq 2S(Tx, Tx, Tx_n) + S(gx, gx, Tx_n) = 0,$$

i.e. $Tx = gx$; thus, T and g have a coupled point x.
Step 4. If T and g are weakly compatible, then $Tgx = gTx = ggx$, i.e. gx is also a coupled point of (T, g). Assume there exists another $y \in X$ such that $Ty = gy$. From (3.1) we have

$$\begin{aligned}&\phi(F(S(Tx, Tx, Ty), \psi(Tx), \psi(Tx), \psi(Ty)))\\ \leq\ &\varphi(F(S(gx, gx, gy), \psi(gx), \psi(gx), \psi(gy))).\end{aligned}$$

From Remark 3.2 we get $F(S(Tx, Tx, Ty), \psi(Tx), \psi(Tx), \psi(Ty)) = 0$, i.e.

$$gx = gy = Tx = Ty,$$

which implies that the coupled point is unique. We derive $Tgx = ggx = gx$. T and g have a unique common fixed point gx. □

In light of Theorem 3.6, if we take $g = I$, the identity mapping on X, we deduce the following corollary.

Corollary 3.7. *Let (X, S) be an S-complete S-metric space, and $T : X \to X$ be a function satisfying the condition:*

$$\phi\left(F(S(Tx, Ty, Tz), \psi(Tx), \psi(Ty), \psi(Tz))\right) \leq \varphi\left(F(S(x, y, z), \psi(x), \psi(y), \psi(z))\right), \tag{3.6}$$

for all $x, y, z \in X$, where $F \in \mathbb{F}$, (ϕ, φ) is a pair of shifting distance functions, and $\psi : X \to R_+$ is a lower semi-continuous function. Then T has a unique fixed point.

From Theorem 3.6, if the function $F(a, b, c, d) = a + b + c + d$ and $\psi(t) \equiv 0$, we derive the following corollary.

Corollary 3.8. *Let (X, S) be an S-complete S-metric space, and $T, g : X \to X$ be two functions such that $TX \subset gX$. Assume gX is closed and satisfies the condition:*

$$\phi\left(S(Tx, Ty, Tz)\right) \leq \varphi\left(S(gx, gy, gz)\right) \tag{3.7}$$

for all $x, y, z \in X$, where $F \in \mathbb{F}$, (ϕ, φ) is a pair of shifting distance functions, and $\psi : X \to R_+$ is a lower semi-continuous function. Then T and g have a coincidence point. If (T, g) is weakly compatible, then T and g have a unique common fixed point.

Corollary 3.9. *Let (X, S) be an S-complete S-metric space, and $T, g : X \to X$ are two functions such that $TX \subset gX$. Assume gX is a closed set, and for all $x, y \in X$,*

$$\begin{aligned}&F(S(Tx, Ty, Tz), \psi(Tx), \psi(Ty), \psi(Tz))\\ \leq\ &\alpha\left(F(S(gx, gy, gz), \psi(gx), \psi(gy), \psi(gz))\right) F(S(gx, gy, gz), \psi(gx), \psi(gy), \psi(gz))\end{aligned} \tag{3.8}$$

where $F \in \mathbb{F}$, the function $\alpha : R_+ \to [0, 1)$ satisfies the condition:

$$\lim_{n\to\infty} \alpha(t_n) = 1 \ \Rightarrow\ \lim_{n\to\infty} t_n = 0,$$

and $\psi : X \to R_+$ is a lower semi-continuous function. Then T and g have a coupled point. Moreover, if T and g are weakly compatible, then T and g have a unique common fixed point.

Proof. It is obvious that $(I, \alpha I)$ is a pair of shifting distance functions. From Theorem 3.6 we can derive the conclusion. □

Example 3.10. Let $X = [0, 2]$, $S(x, y, z) = |x-z|+|y-z|$; then (X, S) is an S-metric space. Define the mapping $T, g : X \to X$ by $Tx = \frac{x}{5}$, $gx = \frac{5x}{6}$.

Let $\psi(x) = x$, $F(a, b, c, d) = a + b + c + d$, $\phi(x) = \ln(5x)$, $\varphi(x) = \ln(2x)$ on $[0, 2]$.

It is easy to know that (ϕ, φ) is a pair of shifting distance functions. For every $x, y \in X$ we have

$$\begin{aligned}
&\phi(F(S(Tx, Ty, Tz), \psi(Tx), \psi(Ty), \psi(Tz))) \\
=&\ln\left(5\left(\left|\frac{x}{3}-\frac{z}{3}\right|+\left|\frac{y}{3}-\frac{z}{3}\right|+\frac{x}{3}+\frac{y}{3}+\frac{z}{3}\right)\right) \\
\leq&\ln\left(2\left(\left|\frac{2x}{3}-\frac{2z}{3}\right|+\left|\frac{2y}{3}-\frac{2z}{3}\right|+\frac{2x}{3}+\frac{2y}{3}+\frac{2z}{3}\right)\right) \\
=&\varphi(F(S(gx, gy, gz), \psi(gx), \psi(gy), \psi(gz))).
\end{aligned}$$

The conditions of Theorem 3.6 are satisfied and the mappings T and g have a unique common fixed point 0.

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