Fixed Point Theory, 23(2022), No. 2, 447-472
DOI: 10.24193/fpt-ro.2022.2.02
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

IMPLICIT AND EXPLICIT VISCOSITY METHODS FOR HIERARCHICAL VARIATIONAL INEQUALITIES ON HADAMARD MANIFOLDS

QAMRUL HASAN ANSARI*, FEEROZ BABU** AND MOHD. ZEESHAN***

*,***Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail: qhansari@gmail.com and amu.zeeshan143@gmail.com

**Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India
E-mail: firoz77b@gmail.com

Abstract. In this paper, we consider a hierarchical variational inequality problem defined over the set of zeros of a set-valued monotone vector field in the setting of Hadamard manifolds. We also consider bilevel variational inequality problems and bilevel optimization problems as special cases of our variational inequality problem. We develop implicit and explicit viscosity methods for solving our problem for weakly contraction mappings. An inexact version of the explicit viscosity method is also studied. At the end, we provide two examples and computational experiments to illustrate implicit and explicit viscosity methods.

Key Words and Phrases: Hierarchical variational inequality problems; weakly contraction mappings, Hadamard manifolds, monotone vector fields, nonexpansive mappings, bilevel variational inequality problems, bilevel optimization problems.

2020 Mathematics Subject Classification: 49J53, 49J40, 47H04, 47J22, 58C30, 58E35, 47J25, 47J20.

1. Introduction

A variational inequality problem defined over the set of fixed points of a mapping is called a hierarchical variational inequality problem, also known as hierarchical fixed point problem. Several real-life problems, namely, signal recovery [11], beamforming [29] and power control [16] problems can be written in the form of a hierarchical variational inequality problem. For details and applications of hierarchical variational inequality problems, we refer to [7, 11, 16, 29, 34, 35] and the references therein. During the last decade, several people have considered hierarchical variational inequality problems defined over the set of zeros of a mapping in the setting of Hilbert / Banach spaces, see, for example, [26, 34, 27] and the references therein.

Since the nonconvex optimization problems can be treated as convex ones in the setting of manifolds, and because of many real life problems can be written as optimization problems in the setting of manifolds, many people have developed the theory of optimization, theory of variational inequalities, fixed point theory, etc., in the setting of manifolds, see, for example, [2, 3, 4, 5, 15, 6, 20, 21, 12, 14, 18, 24, 28, 13, 19, 17, 22, 32, 31] and the references therein. Very recently, we [3] developed viscosity approximation method for ϕ-contraction mappings for solving hierarchical variational inequality problems in the setting of Hadamard manifolds. It is worth to mention that the viscosity approximation method was first introduced by Moudafi [23] for solving variational inequality problems in the setting of Hilbert spaces. It is further extended and studied in [23, 26, 27, 34, 35] in the setting of Hilbert / Banach spaces.

The main motivation of this paper is to consider the hierarchical variational inequality problem defined over the set of zeros of a set-valued monotone vector field in the setting of Hadamard manifolds, and to develop implicit and explicit viscosity type approximation methods under weakly contraction mappings for solving such problems. It is well-known that the concept of a weakly contraction mapping is weaker than the concept of ϕ-contraction mapping (see [1, 3]).

The present paper is organized as follows: In the next section, we present some basic terminologies and tools from Riemannian / Hadamard manifolds which will be used throughout the paper. In Section 3, we first recall resolvent and Yosida approximation of a set-valued vector field. Then we formulate a variational inequality problem defined over the set of zeros of a set-valued monotone vector field. We also consider the bilevel variational inequality problem and the bilevel optimization problem as special cases of our variational inequality problem. In Section 4, we propose implicit type viscosity methods for solving our problem and discuss their convergence results. As a particular case, we derive the implicit viscosity methods and their convergence results for bilevel variational inequality problems and bilevel optimization problems. Section 5 deals with explicit viscosity method and its convergence result. An inexact version of the explicit viscosity method is also discussed. As particular cases, we derive the explicit viscosity method and its convergence result for bilevel variational inequality problems and bilevel optimization problems. In the last section, some examples are presented to check the numerical authenticity of implicit and explicit viscosity methods.

2. Preliminaries

2.1. **Basic concepts from manifold.** Let $\mathbb{M}$ be a finite dimensional differentiable manifold, $T_x\mathbb{M}$ be a tangent space of $\mathbb{M}$ at $x \in \mathbb{M}$ and $T\mathbb{M} = \bigcup_{x\in\mathbb{M}} T_x\mathbb{M}$ be the tangent bundle of $\mathbb{M}$. Let $x \in \mathbb{M}$ and $\langle \cdot, \cdot \rangle_x : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R}$ be a scalar product on $T_x\mathbb{M}$. The smooth mapping $\langle \cdot, \cdot \rangle : x \longmapsto \langle \cdot, \cdot \rangle_x$ is known as a Riemannian metric on $\mathbb{M}$. The corresponding norm to the inner product $\langle \cdot, \cdot \rangle_x$ on $T_x\mathbb{M}$ is denoted by $\| \cdot \|_x$. We omit the subscript x if no confusion occurs. A differentiable manifold $\mathbb{M}$ with a Riemannian metric $\langle \cdot, \cdot \rangle$ is called a Riemannian manifold.

Let $x, y \in \mathbb{M}$ and $\gamma : [a, b] \to \mathbb{M}$ be a piecewise smooth curve joining x to y. The length of the curve γ is defined by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt,$$

where $\dot{\gamma}(t)$ is a tangent vector at $\gamma(t)$ in the tangent space $T_{\gamma(t)}\mathbb{M}$. The minimal length of all such curves joining x to y is known as Riemannian distance and is denoted by $d(x, y)$.

A single-valued vector field on $\mathbb{M}$ is a C^∞ mapping $A : \mathbb{M} \to T\mathbb{M}$ such that for every $x \in \mathbb{M}$, a tangent vector $A(x) \in T_x\mathbb{M}$ is assigned. A C^∞ vector field A along γ is said to be parallel if $\nabla_{\dot{\gamma}(t)} A = \mathbf{0}$, where ∇ is Levi-Civita connection and $\mathbf{0}$ denotes the zero tangent vector. If $\dot{\gamma}(t)$ is parallel along γ, i.e., $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = \mathbf{0}$, then γ is called a geodesic and in this case $\|\dot{\gamma}(t)\|$ is constant. Moreover, γ is said to be normalized geodesic if $\|\dot{\gamma}(t)\| = 1$. A geodesic joining x to y in the Riemannian manifold $\mathbb{M}$ is said to be a minimal geodesic if its length is equal to $d(x, y)$.

A Riemannian manifold $\mathbb{M}$ is said to be complete if for any $x \in \mathbb{M}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By Hopf-Rinow Theorem [28], if $\mathbb{M}$ is a complete Riemannian manifold, then any pair of points in $\mathbb{M}$ can be joined by a minimal geodesic. Moreover, $(\mathbb{M}, d)$ is a complete metric space. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x u = \gamma(1; x), \quad \forall u \in T_x\mathbb{M},$$

where $\gamma(\cdot; x)$ is the geodesic starting from x with velocity u, i.e., $\gamma(0; x) = x$ and $\dot{\gamma}(0; x) = u$. It is known that $\exp_x tu = \gamma(t; x)$ for any real number t, and $\exp_x \mathbf{0} = \gamma(0; x) = x$. Note that the exponential map $\exp_x$ is differentiable on $T_x\mathbb{M}$ for any $x \in \mathbb{M}$. It is well-known that the derivative $\mathrm{D}\exp_x(\mathbf{0})$ of $\exp_x(\mathbf{0})$ is equal to the identity vector of $T_x\mathbb{M}$. Therefore, by the inverse mapping theorem, there exists an inverse exponential map $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. Moreover, for any $x, y \in \mathbb{M}$, we have $d(x, y) = \|\exp_x^{-1} y\|$. For further details, we refer to [28].

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

The rest of the paper, unless otherwise specified, we assume that $\mathbb{M}$ is a finite dimensional Hadamard manifold.

We recall some known properties of the exponential map in the setting of Hadamard manifolds.

Lemma 2.1. [17] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Hadamard manifold $\mathbb{M}$ such that $x_n \to x_0 \in \mathbb{M}$. Then the following assertions hold:*

(a) *For any $y \in \mathbb{M}$, we have*

$$\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y \text{ and } \exp_y^{-1} x_n \to \exp_y^{-1} x_0.$$

(b) *If $u_n \in T_{x_n}\mathbb{M}$ and $u_n \to u_0$, then $u_0 \in T_{x_0}\mathbb{M}$.*

(c) *Given $u_n, v_n \in T_{x_n}\mathbb{M}$ and $u_0, v_0 \in T_{x_0}\mathbb{M}$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n\rangle \to \langle u_0, v_0\rangle$.*

Proposition 2.2. [28] *Let $\mathbb{M}$ be a Hadamard manifold. Then for all $x \in \mathbb{M}$, the exponential map $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism, and for any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic $\gamma : [0, 1] \to \mathbb{M}$ joining $x = \gamma(0)$ to $y = \gamma(1)$ which is in fact a minimal geodesic defined by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y, \quad \forall t \in [0, 1].$$

A subset C of a Riemannian manifold $\mathbb{M}$ is said to be geodesic convex if for any two points x and y in C, any geodesic joining x to y is contained in C, i.e., for all

$a, b \in \mathbb{R}$ and for any geodesic $\gamma : [a, b] \to \mathbb{M}$ such that $x = \gamma(a)$ and $y = \gamma(b)$, we have $\gamma(at + (1-t)b) \in C$ for all $t \in [0, 1]$.
A function $f : \mathbb{M} \to \mathbb{R}$ is said to be geodesic convex if for any geodesic $\gamma : [a, b] \to \mathbb{M}$, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(at + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall t \in [0, 1] \text{ and } \forall a, b \in \mathbb{R}.$$

Proposition 2.3. [28] *The Riemannian distance $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is a geodesic convex function with respect to the product Riemannian metric, i.e., given any pair of geodesics $\gamma_1 : [0, 1] \to \mathbb{M}$ and $\gamma_2 : [0, 1] \to \mathbb{M}$, the following inequality holds for all $t \in [0, 1]$:*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $x \in \mathbb{M}$, the function $d(\cdot, x) : \mathbb{M} \to \mathbb{R}$ is a geodesic convex function.

We now mention some geometric properties from the finite dimensional Hadamard manifold $\mathbb{M}$ which are similar to the settings of Euclidean space $\mathbb{R}^n$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a Hadamard manifold $\mathbb{M}$ is a set consisting of three points x_1, x_2 and x_3, and three minimal geodesics γ_i joining x_i to x_{i+1}, where $i = 1, 2, 3 \pmod 3$.

Proposition 2.4. [28] *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$. For each $i = 1, 2, 3 \pmod 3$, let $\gamma_i : [0, l_i] \to \mathbb{M}$ be the geodesic joining x_i to x_{i+1}, $l_i = L(\gamma_i)$ and α_i be the angle between tangent vectors $\dot{\gamma}_i(0)$ and $-\dot{\gamma}_{i-1}(l_{i-1})$. Then,*

(a) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;
(b) $l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$.

As in [17], Proposition 2.4 (b) can be written in terms of Riemannian distance and exponential map as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle \leq d^2(x_{i-1}, x_i), \qquad (2.1)$$

since

$$\left\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}.$$

For further details, we refer to [14].

Lemma 2.5. [18] *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$. Then, there exists $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\| \quad \textit{and} \quad d(x, z) = \|x' - z'\|.$$

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta\ (x, y, z)$, which is unique up to isometry of $\mathbb{M}$. The points x', y', z' are called comparison points to the points x, y, z, respectively.

Lemma 2.6. [18] *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$ and $\Delta(x', y', z')$ be its comparison triangle.*

(a) *Let $\theta_1, \theta_2, \theta_3$ (respectively, $\theta'_1, \theta'_2, \theta'_3$) be the angles of $\Delta(x, y, z)$ (respectively, $\Delta(x', y', z')$) at the vertices x, y, z (respectively, x', y', z'). Then,*

$$\theta'_1 \geq \theta_1, \quad \theta'_2 \geq \theta_2 \quad \text{and} \quad \theta'_3 \geq \theta_3.$$

(b) *Let w be a point on the geodesic joining x to y and w' be its comparison point in the interval $[x', y']$. If $d(w, x) = \|w' - x'\|$ and $d(w, y) = \|w' - y'\|$, then $d(w, z) \leq \|w' - z'\|$.*

A parallel transport on the tangent bundle $T\mathbb{M}$ along γ with respect to ∇ is a linear map $\mathcal{P}_{\gamma,\gamma(b),\gamma(a)} : T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ defined by

$$\mathcal{P}_{\gamma,\gamma(b),\gamma(a)}(v) = A(\gamma(b)), \quad \forall a, b \in \mathbb{R} \text{ and } \forall v \in T_{\gamma(a)}\mathbb{M},$$

where A is the unique vector field such that $\nabla_{\dot{\gamma}(t)} A = \mathbf{0}$ for all t and $A(\gamma(a)) = v$. When γ is a minimal geodesic joining x to y, we write $\mathcal{P}_{y,x}$ instead of $\mathcal{P}_{\gamma,y,x}$. For further details, we refer to [13, 28].

2.2. **Basic concepts from nonlinear analysis.** Let $\mathbb{M}$ be a Hadamard manifold and C be a nonempty closed geodesic convex subset of $\mathbb{M}$. The projection of a point $x \in \mathbb{M}$ onto C is defined by

$$P_C(x) = \{z \in C : d(x, z) \leq d(x, y), \forall y \in C\}.$$

Proposition 2.7. [32] *Let C be a nonempty closed geodesic convex subset of a Hadamard manifold $\mathbb{M}$. Then for any $x \in \mathbb{M}$, $P_C(x)$ is a singleton set. Also, for any point $x \in \mathbb{M}$, the following statements are equivalent:*

(a) *$y = P_C(x)$;*
(b) *$\left\langle \exp_y^{-1} x, \exp_y^{-1} z \right\rangle \leq 0$ for all $z \in C$.*

Definition 2.8. [19] A mapping $T : C \subseteq \mathbb{M} \to \mathbb{M}$ is said to be firmly nonexpansive if for any $x, y \in C$, the function $\varphi : [0, 1] \to [0, +\infty)$ defined by

$$\varphi(t) := d\left(\exp_x t \exp_x^{-1} T(x), \exp_y t \exp_y^{-1} T(y)\right), \quad \forall t \in [0, 1],$$

is nonincreasing.

Li et al. [19] proved that every firmly nonexpansive map is nonexpansive, that is,

$$d(T(x), T(y)) \leq d(x, y), \quad \forall x, y \in C.$$

Alber and Guerre-Delabriere [1] defined the concept of a weakly contraction mapping and studied the existence of a unique fixed point for such mappings in the setting of Hilbert spaces. Rhoades [25] extended the result of Alber and Guerre-Delabriere [1] for complete metric spaces.

Definition 2.9. A mapping $f : \mathbb{M} \to \mathbb{M}$ is said to be weakly contraction if

$$d(f(x), f(y)) \leq d(x, y) - \psi(d(x, y)), \quad \forall x, y \in \mathbb{M},$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$ and $\lim\limits_{t\to\infty} \psi(t) = +\infty$. The function ψ is called a comparison function.

Remark 2.10. (a) If $\psi(t) = kt$ for all $t \geq 0$, where $k \in (0,1)$, then every weakly contraction mapping is contraction with constant $1-k$.

(b) Note that weakly contraction mappings are nonexpansive.

(c) For all $t \geq 0$, $\psi(t) = \frac{t^2}{1+t}$ is a comparison function.

Theorem 2.11. [25] *Let* $(\mathbb{M}, d)$ *be a complete metric space and* $f : \mathbb{M} \to \mathbb{M}$ *be a weakly contraction mapping. Then,* f *has a unique fixed point.*

Very recently, we [3] considered the following ϕ-contraction mapping in the setting of Hadamard manifolds. It was originally introduced by Boyd and Wong [9] in the setting of metric spaces.

Definition 2.12. A mapping $f : \mathbb{M} \to \mathbb{M}$ is said to be ϕ-contraction if

$$d(f(x), f(y)) \leq \phi(d(x,y)), \quad \forall x, y \in \mathbb{M},$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is an upper semicontinuous function such that $\phi(t) < t$ for all $t > 0$.

Remark 2.13. Every weakly contraction mapping is ϕ-contraction.
Indeed, let $f : \mathbb{M} \to \mathbb{M}$ be a weakly contraction mapping with the comparison function $\psi : [0, +\infty) \to [0, +\infty)$ which is continuous and nondecreasing such that $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$ and $\lim\limits_{t\to\infty} \psi(t) = +\infty$. Then, the function $\phi : [0, +\infty) \to [0, +\infty)$ defined by $\phi(t) = t - \psi(t)$ is an upper semicontinuous function and $\phi(t) < t$ for all $t > 0$. Also, $d(f(x), f(y)) \leq d(x,y) - \psi(d(x,y)) = \phi(d(x,y))$ for all $x, y \in \mathbb{M}$. Therefore, f is ϕ-contraction.

The following lemma will be used to prove the main result of this paper.

Lemma 2.14. [1] *Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *and* $\{\beta_n\}_{n\in\mathbb{N}}$ *be two sequences of positive real numbers such that* $\lim\limits_{n\to\infty} \frac{\beta_n}{\mu_n} = 0$ *and* $\sum\limits_{n=1}^{\infty} \mu_n = +\infty$. *Let* $\{w_n\}_{n\in\mathbb{N}}$ *be a sequence of positive real numbers satisfying the following recursive inequality:*

$$w_{n+1} \leq w_n - \mu_n \psi(w_n) + \beta_n, \quad \forall n \in \mathbb{N},$$

where $\psi : [0, +\infty) \to [0, +\infty)$ *is a continuous and nondecreasing function such that* $\psi(0) = 0$ *and* $\psi(t) > 0$ *for all* $t > 0$. *Then,* $\lim\limits_{n\to\infty} w_n = 0$.

3. Formulation of the Problems

Throughout the paper, unless otherwise specified, we assume that $\mathbb{M}$ is a Hadamard manifold. Let $\Omega(\mathbb{M})$ denote the set of all single-valued vector fields $B : \mathbb{M} \to T\mathbb{M}$ such that $B(x) \in T_x\mathbb{M}$ for each $x \in \mathbb{M}$. Let $\mathcal{X}(\mathbb{M})$ be the set of all set-valued vector fields $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ such that $A(x) \subseteq T_x\mathbb{M}$ for each $x \in \mathbb{M}$. The domain $D(A)$ of A is defined by $D(A) = \{x \in \mathbb{M} : A(x) \neq \emptyset\}$.

Definition 3.1. [21] A single-valued vector field $B \in \Omega(\mathbb{M})$ is said to be monotone if

$$\left\langle B(x), \exp_x^{-1} y \right\rangle \leq \left\langle B(y), -\exp_y^{-1} x \right\rangle, \quad \forall x, y \in \mathbb{M}.$$

Definition 3.2. [20] Let $T : C \subset \mathbb{M} \rightrightarrows \mathbb{M}$ be a set-valued mapping. The complementary vector field $A \in \mathcal{X}(\mathbb{M})$ of T is defined by

$$A(x) = -\exp_x^{-1} T(x), \quad \forall x \in C,$$

where $\exp_x^{-1} T(x) = \{\exp_x^{-1} y \in T_x\mathbb{M} : y \in T(x)\}$ for all $x \in \mathbb{M}$.

Theorem 3.3. [20] *If $T : C \subset \mathbb{M} \to \mathbb{M}$ is a nonexpansive mapping, then its complementary vector field is monotone.*

Now, we recall the definition of resolvent associated with a set-valued vector field.

Definition 3.4. [19] For a given $\lambda > 0$, the resolvent of a set-valued vector field $A \in \mathcal{X}(\mathbb{M})$ of order λ is a set-valued map $J_\lambda^A : \mathbb{M} \rightrightarrows D(A)$ defined by

$$J_\lambda^A(x) := \{z \in \mathbb{M} : x \in \exp_z \lambda A(z)\}, \quad \forall x \in \mathbb{M},$$

and the Yosida approximation of A of order λ is defined by

$$A_\lambda(x) = -\frac{1}{\lambda} \exp_x^{-1} J_\lambda^A(x), \quad \forall x \in \mathbb{M}.$$

Note that the Yosida approximation A_λ of A is the complementary vector field of the corresponding resolvent.

Remark 3.5. [19] For $\lambda > 0$, the range of resolvent J_λ^A is contained in the domain of A and

$$\mathrm{Fix}(J_\lambda^A) = A^{-1}(\mathbf{0}).$$

Definition 3.6. [12] A set-valued vector field $A \in \mathcal{X}(\mathbb{M})$ is said to be monotone if for any $x, y \in D(A)$,

$$\left\langle u, \exp_x^{-1} y \right\rangle \leq \left\langle v, -\exp_y^{-1} x \right\rangle, \quad \forall u \in A(x) \text{ and } \forall v \in A(y).$$

Lemma 3.7. [19] *A set-valued vector field $A \in \mathcal{X}(\mathbb{M})$ is monotone if and only if J_λ^A is single-valued and firmly nonexpansive.*

Remark 3.8. Note that the set of fixed points of a nonexpansive mapping is closed and geodesic convex (see [2, 19]). If $A \in \mathcal{X}(\mathbb{M})$ is monotone, then by Lemma 3.7, J_λ^A is single-valued and firmly nonexpansive, and hence nonexpansive. Therefore, $\mathrm{Fix}(J_\lambda^A) = A^{-1}(\mathbf{0})$ is closed and geodesic convex.

Lemma 3.9. [2] *Let $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ be a monotone set-valued vector field. Then, for $\mu > 0$ and $\eta > 0$, we have*

$$d\left(J_\mu^A(x), J_\eta^A(x)\right) \leq \frac{|\mu - \eta|}{\mu} d\left(x, J_\mu^A(x)\right), \quad \forall x \in \mathbb{M}. \tag{3.1}$$

Lemma 3.10. [19] *Let $A \in \mathcal{X}(\mathbb{M})$ be monotone, $\lambda > 0$ and $x \in D(A_\lambda)$. Then,*

$$A_\lambda(x) \in \mathcal{P}_{x, J_\lambda^A(x)} A\left(J_\lambda^A(x)\right) \quad \text{and} \quad \|A_\lambda(x)\| \leq |||A(x)|||,$$

where $|||A(x)||| = \inf\{\|u\| : u \in A(x)\}$. Moreover, $\|A_\lambda(x)\| = \|A(J_\lambda^A(x))\|$.

Let $f : \mathbb{M} \to \mathbb{M}$ be a weakly contraction mapping and $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ be a set-valued monotone vector field such that $A^{-1}(\mathbf{0}) \neq \emptyset$. The hierarchical variational inequality problem (in short, HVIP) is to find $\bar{x} \in A^{-1}(\mathbf{0})$ such that

$$\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} x \right\rangle \leq 0, \quad \forall x \in A^{-1}(\mathbf{0}). \tag{3.2}$$

We denote by $\mathbb{S}$ the solution set of HVIP (3.2). Clearly, by Remark 3.8, the assumption $A^{-1}(\mathbf{0}) \neq \emptyset$ and Proposition 2.7, HVIP (3.2) can equivalently be written as the following fixed point problem:

$$\text{Find } \bar{x} \in \mathbb{M} \quad \text{such that} \quad \bar{x} = P_{A^{-1}(\mathbf{0})} f(\bar{x}), \tag{3.3}$$

where $P_{A^{-1}(\mathbf{0})}$ is the metric projection on the nonempty closed geodesic convex set $A^{-1}(\mathbf{0})$ in the Hadamard manifold $\mathbb{M}$.

Special Cases

Bilevel variational inequality problems. A variational inequality problem defined over the set of solutions of another variational inequality problem is known as bilevel variational inequality problem. For further details, applications and a nice survey, we refer [33].

Let C be a nonempty closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$, and $B : C \to T\mathbb{M}$ be a single-valued monotone vector field. The variational inequality problem in the setting of Hadamard manifolds was first considered and studied by Németh [21] which is defined as follows:

$$\text{Find } \bar{y} \in C \quad \text{such that} \quad \left\langle B(\bar{y}), \exp_{\bar{y}}^{-1} z \right\rangle \geq 0, \quad \forall z \in C. \tag{3.4}$$

The solution set of the variational inequality problem (3.4) is denoted by $\mathrm{VIP}(B; C)$. The normal cone to C at $y \in C$ is defined by

$$N_C(y) := \left\{ u \in T_y\mathbb{M} : \left\langle u, \exp_y^{-1} z \right\rangle \leq 0, \, \forall z \in C \right\}.$$

Note that the problem (3.4) is equivalent to the following inclusion problem:

$$\text{Find } \bar{y} \in C \quad \text{such that} \quad \mathbf{0} \in B(\bar{y}) + N_C(\bar{y}), \tag{3.5}$$

where $B + N_C$ is a set-valued monotone vector field (see [17]). Let $R_\lambda^B := J_\lambda^{B+N_C}$, that is,

$$\begin{aligned} R_\lambda^B(y) &= J_\lambda^{B+N_C}(y) \\ &= \{ w : y \in \exp_w \lambda(B + N_C)(w) \} \\ &= \left\{ w : \frac{1}{\lambda} \exp_w^{-1} y - B(w) \in N_C(w) \right\} \\ &= \left\{ w : \left\langle \lambda B(w) - \exp_w^{-1} y, \exp_w^{-1} z \right\rangle \geq 0, \, \forall z \in C \right\}, \quad \forall y \in \mathbb{M}. \end{aligned}$$

Since $B + N_C$ is a set-valued monotone vector field, by Lemma 3.7, R_λ^B is single-valued and firmly nonexpnasive, and hence, nonexpansive. Clearly, $\mathrm{Fix}(R_\lambda^B) = \mathrm{VIP}(B; C)$. If we consider $A = B + N_C$, then $\mathrm{VIP}(B; C) = \mathrm{Fix}(R_\lambda^B) = A^{-1}(\mathbf{0})$, and therefore, the HVIP (3.2) reduces to the following bilevel variational inequality problem:

$$\text{Find } \bar{x} \in \mathrm{VIP}(B; C) \quad \text{such that} \quad \left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \leq 0, \quad \forall y \in \mathrm{VIP}(B; C), \tag{3.6}$$

where $f : \mathbb{M} \to \mathbb{M}$ is a weakly contraction mapping.

Bilevel optimization problems. An optimization problem defined over the set of solutions of another optimization problem is known as bilevel optimization problem. For further detail and applications, we refer [30, 10].

Let $F : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and geodesic convex function with the domain $\mathrm{D}(F) := \{x \in \mathbb{M} : F(x) \neq +\infty\}$. Let C be a nonempty closed and geodesic convex subset of a Hadamard manifold $\mathbb{M}$. The constraint minimization problem is defined as

$$\min_{x \in C} F(x). \tag{3.7}$$

The solution set of the constraint minimization problem is denoted by $\operatorname*{argmin}_{x \in C} F(x)$, that is,

$$\operatorname*{argmin}_{x \in C} F(x) := \{\bar{x} \in \mathbb{M} : F(\bar{x}) \leq F(y), \forall y \in C\}.$$

The directional derivative of a proper geodesic convex function $F : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ at x in direction $u \in T_x\mathbb{M}$ is

$$F'(x; u) := \lim_{t \to 0^+} \frac{F(\exp_x tu) - F(x)}{t}.$$

The resolvent of F, defined by

$$\mathrm{Prox}_{\lambda, F}(x) := \operatorname*{argmin}_{y \in C} \left\{ F(y) + \frac{1}{2\lambda} d^2(x, y) \right\}, \tag{3.8}$$

where $\lambda > 0$, is single-valued and firmly nonexpansive (see [14]). The set of fixed points of resolvent of F is denoted by $\mathrm{Fix}(\mathrm{Prox}_{\lambda,F})$. By ([8, Lemma 3.2]),

$$\operatorname*{argmin}_{x \in C} F(x) \; = \; \mathrm{Fix}(\mathrm{Prox}_{\lambda,F}). \tag{3.9}$$

The gradient ∇G of a geodesic convex differentiable function $G : \mathbb{M} \to \mathbb{R}$ at $x \in \mathbb{M}$ [13] is defined by $\langle \nabla G(x), u \rangle := G'(x; u)$ for all $u \in T_x\mathbb{M}$.

Proposition 3.11. [24] *Let $\mathbb{M}$ be a Riemannian manifold and $G : \mathbb{M} \to \mathbb{R}$ be a differentiable function. Then, G is geodesic convex if and only if ∇G is a monotone vector field.*

Since every weakly contraction mapping is nonexpansive, therefore by Theorem 3.3, the complimentary vector field of a weakly contraction mapping is monotone. Also, by Proposition 3.11, the gradient of a geodesic convex function is monotone. Therefore, without loss of generality, we can consider a weakly contraction mapping $f : \mathbb{M} \to \mathbb{M}$ and a geodesic convex differentiable function $G : \mathbb{M} \to \mathbb{R}$ such that $\nabla G = -\exp_{\cdot}^{-1} f$. For $\lambda > 0$, if we consider $J_\lambda^A = \mathrm{Prox}_{\lambda,F}$ and adopt the technique of [3, Proposition 3.10], then HVIP (3.2) reduces to the following bilevel minimization problem (in short, BMP):

$$\min_{x \in \operatorname{argmin}\ F} G(x), \tag{3.10}$$

where argmin F denotes $\operatorname*{argmin}_{x \in C} F(x)$. It is considered and studied by Cabot [10] and Solodov [30] in the setting of Euclidean spaces.

4. Implicit Methods

To propose the implicit methods for solving HVIP (3.2), we first prove that the mappings $J_\lambda^A f$ and $J_\lambda^A(\exp_. t \exp_.^{-1} f)$ have a unique fixed point where f is a weakly contraction mapping and $\lambda > 0$, $t \in [0,1]$.

Proposition 4.1. *Let $f : \mathbb{M} \to \mathbb{M}$ be a weakly contraction mapping and $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ be a set-valued monotone vector field. Then the following assertions hold:*

(a) *For each $\lambda > 0$, $J_\lambda^A f$ has a unique fixed point.*
(b) *For each $\lambda > 0$ and $t \in [0,1]$, $J_\lambda^A(\exp_. t \exp_.^{-1} f)$ has a unique fixed point.*

Proof. (a) Since J_λ^A is nonexpansive and f is weakly contraction, we have that $J_\lambda^A f$ is weakly contraction, that is,

$$d(J_\lambda^A(f(x)), J_\lambda^A(f(y))) \leq d(f(x), f(y)) \leq d(x,y) - \psi(d(x,y)), \quad \forall x, y \in \mathbb{M}.$$

So, by Theorem 2.11, $J_\lambda^A f$ has a unique fixed point.
(b) Since J_λ^A is nonexpansive for all $\lambda > 0$ and f is weakly contraction, we have

$$\begin{aligned} d\big(J_\lambda^A\left(\exp_x t \exp_x^{-1} f(x)\right), J_\lambda^A\left(\exp_y t \exp_y^{-1} f(y)\right)\big) & \\ \leq d(\exp_x t \exp_x^{-1} f(x), \exp_y t \exp_y^{-1} f(y)) & \\ \leq (1-t)d(x,y) + td(f(x), f(y)) & \\ \leq (1-t)d(x,y) + t(d(x,y) - \psi(d(x,y))) & \\ = d(x,y) - t\psi(d(x,y)), \quad \forall x, y \in \mathbb{M}. & \end{aligned}$$

Hence, $J_\lambda^A(\exp_. t \exp_.^{-1} f)$ is weakly contraction with the comparison function $\varphi = t\psi$ which is continuous and nondecreasing such that $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$ and $\lim_{t\to\infty} \varphi(t) = +\infty$. By Theorem 2.11, there exists a unique fixed point of $J_\lambda^A(\exp_. t \exp_.^{-1} f)$. □

In view of the above proposition, we suggest the following implicit methods to find the solutions of HVIP (3.2):

$$y_\lambda := J_\lambda^A(f(y_\lambda)), \quad \forall \lambda > 0. \tag{4.1}$$

and

$$x_t := J_\lambda^A(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t)), \quad \forall \lambda > 0 \text{ and } \forall t \in [0,1]. \tag{4.2}$$

The implicit methods (4.1) and (4.2) improve and extend several known methods in the following ways:

(a) The implicit method (4.1) extends the one considered by Wong et al. [34] for the problem (3.2) from Banach space settings to Hadamard manifolds.
(b) The implicit method (4.2) extends and improves the one considered in [27] for a contraction mapping f in the setting of Banach spaces.
(c) If $f(x) = u$ is a fixed point in $\mathbb{M}$, then the implicit method (4.2) reduces to the one considered in [5].
(d) In [6, 3], we studied the HVIP (3.2) for nonexpansive mappings, weakly contraction mappings and ϕ-contraction mappings in the setting of Hadamard manifolds and also considered some particular cases. If we consider f to be a

ϕ - contraction mapping, then the map $J_\lambda^A(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t))$ may not be a ϕ-contraction. Therefore, in view of [9, Theorem 1], J_λ^A $(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t))$ may not have a fixed point, and hence, the methods in [6, 3] may not be applicable. Thus, the implicit method (4.2) is applicable for more general class than the one considered in [6, 3].

Theorem 4.2. *Let $f : \mathbb{M} \to \mathbb{M}$ be a weakly contraction mapping with the comparison function ψ, and $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ be a set-valued monotone vector field such that $A^{-1}(\mathbf{0}) \neq \emptyset$. Then the following assertions hold:*

(a) *The path $(y_\lambda)_{\lambda>0}$ generated by (4.1) converges to a solution of HVIP (3.2) as $\lambda \to \infty$.*
(b) *The path $(x_t)_{t\in[0,1]}$ generated by (4.2) converges to a solution of HVIP (3.2) as $t \to 0$.*

Proof. (a) Since $A^{-1}(\mathbf{0}) \neq \emptyset$, we can assume that $y \in A^{-1}(\mathbf{0})$ such that $y_\lambda \neq y$, otherwise y_λ will converge to y. Then for $\lambda > 0$, we have

$$\begin{aligned} d(y_\lambda, y) &= d\left(J_\lambda^A(f(y_\lambda)), J_\lambda^A(y)\right) \\ &\leq d(f(y_\lambda), y) \\ &\leq d(f(y_\lambda), f(y)) + d(f(y), y) \\ &\leq d(y_\lambda, y) - \psi(d(y_\lambda, y)) + d(f(y), y), \end{aligned}$$

that is,

$$\psi(d(y_\lambda, y)) \leq d(f(y), y). \tag{4.3}$$

Now we prove that $(d(y_\lambda, y))_{\lambda>0}$ is bounded. Suppose contrary that $(d(y_\lambda, y))_{\lambda>0}$ is not bounded. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0, +\infty)$ with $\lambda_k \to \infty$ as $k \to \infty$ such that

$$d(y_{\lambda_k}, y) > k, \quad \forall k \in \mathbb{N}.$$

Since ψ is a comparison function, so it is nondecreasing and $\lim\limits_{t\to\infty} \psi(t) = +\infty$; from the above inequality and (4.3), we have

$$\psi(k) < \psi(d(y_{\lambda_k}, y)) \leq d(f(y), y), \quad \forall k \in \mathbb{N},$$

that is,

$$\lim_{k\to\infty} \psi(k) < d(f(y), y),$$

a contradiction. Hence, $(d(y_\lambda, y))_{\lambda>0}$ is bounded.
For $r > 0$, let A_r be the complimentary vector field of A. Then, by Yosida approximation and Lemma 3.10, we have

$$d\left(y_\lambda, J_r^A(y_\lambda)\right) = \left\|\exp_{y_\lambda}^{-1} J_r^A(y_\lambda)\right\| = r\|A_r(y_\lambda)\| \leq r|||A(y_\lambda)||| = r\left\|A(J_\lambda^A(f(y_\lambda)))\right\|.$$

By Lemma 3.10, $A_\lambda(x) \in \mathcal{P}_{x,J_\lambda(x)} A(J_\lambda(x))$ for all $\lambda > 0$, and therefore,

$$\|A_\lambda(f(y_\lambda))\| = \|A(J_\lambda^A(f(y_\lambda)))\|.$$

Then the above inequality becomes

$$\begin{aligned} d\left(y_\lambda, J_r^A(y_\lambda)\right) &\le r\,\|A_\lambda(f(y_\lambda))\| = \frac{r}{\lambda}\left\|\exp^{-1}_{f(y_\lambda)} J_\lambda^A(f(y_\lambda))\right\| \\ &= \frac{r}{\lambda}\left\|\exp^{-1}_{f(y_\lambda)} y_\lambda\right\| \le \frac{r}{\lambda}K_1, \end{aligned} \tag{4.4}$$

for some $K_1 > 0$. Since $(d(y_\lambda, y))_{\lambda>0}$ is bounded for every $y \in A^{-1}(\mathbf{0})$, and so is $(y_\lambda)_{\lambda>0}$. Therefore, we may assume that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $(0,+\infty)$ such that $\lim\limits_{n\to\infty} \lambda_n = +\infty$ and the subsequence (y_{λ_n}) of $(y_\lambda)_{\lambda>0}$ converges to $\bar{y}$. For $r > 0$, from (4.4), we have

$$d\left(y_{\lambda_n}, J_r^A(y_{\lambda_n})\right) \le \frac{r}{\lambda_n}K_1.$$

By taking limit as $n \to \infty$, we get

$$d\left(\bar{y}, J_r^A(\bar{y})\right) = 0, \quad \forall r > 0.$$

This implies that $\bar{y} \in \mathrm{Fix}(J_r^A)$. By Remark 3.5, we have $\bar{y} \in A^{-1}(\mathbf{0})$.
Since $y_\lambda = J_\lambda^A f(y_\lambda)$, it follows that

$$\exp^{-1}_{y_\lambda} f(y_\lambda) \in A(y_\lambda).$$

By monotonicity of A and for any $y \in A^{-1}(\mathbf{0})$, we have

$$\left\langle \exp^{-1}_{y_\lambda} f(y_\lambda), \exp^{-1}_{y_\lambda} y \right\rangle \le 0,$$

and so,

$$\left\langle \exp^{-1}_{y_{\lambda_n}} f(y_{\lambda_n}), \exp^{-1}_{y_{\lambda_n}} y \right\rangle \le 0, \quad \forall y \in A^{-1}(\mathbf{0}). \tag{4.5}$$

Since $y_{\lambda_n} \to \bar{y}$ as $\lambda_n \to \infty$ as $n \to \infty$, it follows that

$$\left\langle \exp^{-1}_{\bar{y}} f(\bar{y}), \exp^{-1}_{\bar{y}} y \right\rangle \le 0, \quad \forall y \in A^{-1}(\mathbf{0}). \tag{4.6}$$

In order to show that the path $(y_\lambda)_{\lambda>0}$ converges to an element of $A^{-1}(\mathbf{0})$, we assume that $(\lambda_{n'})$ is another sequence in $(0,+\infty)$ such that $y_{\lambda_{n'}} \to \tilde{y}$ as $\lambda_{n'} \to \infty$. By (4.4), we obtain $\tilde{y} \in \mathrm{Fix}(J_\lambda^A)$ for all $\lambda > 0$. From (4.6), we have

$$\left\langle \exp^{-1}_{\tilde{y}} f(\tilde{y}), \exp^{-1}_{\tilde{y}} y \right\rangle \le 0, \quad \forall y \in A^{-1}(\mathbf{0}).$$

In particular, for $\bar{y} \in A^{-1}(\mathbf{0})$, we have

$$\left\langle \exp^{-1}_{\tilde{y}} f(\tilde{y}), \exp^{-1}_{\tilde{y}} \bar{y} \right\rangle \le 0. \tag{4.7}$$

Since f is weakly contraction, so is nonexpansive, and therefore the complementary vector field $-\exp^{-1}_{\cdot} f$ of f is monotone. Thus, we have

$$\left\langle -\exp^{-1}_{\tilde{y}} f(\tilde{y}), \exp^{-1}_{\tilde{y}} \bar{y} \right\rangle + \left\langle -\exp^{-1}_{\bar{y}} f(\bar{y}), \exp^{-1}_{\bar{y}} \tilde{y} \right\rangle \le 0.$$

By combining above inequality with the inequality (4.7), we obtain

$$\langle \exp^{-1}_{\bar{y}} f(\bar{y}), \exp^{-1}_{\bar{y}} \tilde{y} \rangle \ge 0.$$

Since the inequality (4.6) holds for all $y \in A^{-1}(\mathbf{0})$, so it holds for a particular $\tilde{y} \in A^{-1}(\mathbf{0})$, that is,

$$\left\langle \exp^{-1}_{\bar{y}} f(\bar{y}), \exp^{-1}_{\bar{y}} \tilde{y} \right\rangle \le 0.$$

By combining above two inequalities, we obtain

$$\left\langle \exp_{\bar{y}}^{-1} f(\bar{y}), \exp_{\bar{y}}^{-1} \tilde{y} \right\rangle = 0. \tag{4.8}$$

Now, we have two cases.

CASE 1. If $\exp_{\bar{y}}^{-1} f(\bar{y}) \neq \mathbf{0}$, then from (4.8), we have $\exp_{\bar{y}}^{-1} \tilde{y} = \mathbf{0}$. This implies that $d(\bar{y}, \tilde{y}) = \| \exp_{\bar{y}}^{-1} \tilde{y}\| = 0$, that is, $\bar{y} = \tilde{y}$. Hence, $(y_\lambda)_{\lambda>0}$ converges to a unique element of $A^{-1}(\mathbf{0})$.

CASE 2. If $\exp_{\bar{y}}^{-1} f(\bar{y}) = \mathbf{0}$, then $d(f(\bar{y}), \bar{y}) = 0$, that is, $\bar{y} = f(\bar{y})$. Putting $\bar{y} = f(\bar{y})$ in (4.7), we get

$$\left\langle \exp_{\tilde{y}}^{-1} f(\tilde{y}), \exp_{\tilde{y}}^{-1} f(\bar{y}) \right\rangle \leq 0. \tag{4.9}$$

Consider a geodesic triangle $\Delta(f(\tilde{y}), \tilde{y}, f(\bar{y}))$. Then by inequality (2.1), we have

$$d^2(f(\tilde{y}), \tilde{y}) + d^2(\tilde{y}, f(\bar{y})) - 2\left\langle \exp_{\tilde{y}}^{-1} f(\tilde{y}), \exp_{\tilde{y}}^{-1} f(\bar{y}) \right\rangle \leq d^2(f(\tilde{y}), f(\bar{y})). \tag{4.10}$$

By combining (4.9) and (4.10), we get

$$d^2(f(\tilde{y}), \tilde{y}) + d^2(\tilde{y}, f(\bar{y})) \leq d^2(f(\tilde{y}), f(\bar{y})), \tag{4.11}$$

that is,

$$d(\tilde{y}, f(\bar{y})) \leq d(f(\tilde{y}), f(\bar{y})).$$

Since f is weakly contraction, we have

$$d(\tilde{y}, f(\bar{y})) \leq d(f(\tilde{y}), f(\bar{y})) \leq d(\tilde{y}, \bar{y}) - \psi(d(\tilde{y}, \bar{y})) \leq d(\tilde{y}, \bar{y}).$$

This implies that $d(\tilde{y}, f(\bar{y})) \leq d(\tilde{y}, \bar{y})$. Since $A^{-1}(\mathbf{0})$ is nonempty closed and geodesic convex, by the definition of metric projection, we get $f(\bar{y}) = P_{A^{-1}(\mathbf{0})}\tilde{y}$, and so, $\bar{y} = P_{A^{-1}(\mathbf{0})}\tilde{y}$. Since $\bar{y}, \tilde{y} \in A^{-1}(\mathbf{0})$, we have $\tilde{y} = \bar{y}$. Hence, $(y_\lambda)_{\lambda>0}$ converges to a unique element of $A^{-1}(\mathbf{0})$.

(b) Let $x \in A^{-1}(\mathbf{0})$, $\lambda > 0$ and $t \in [0,1]$, we have

$$\begin{aligned} d(x_t, x) &= d\left(J_\lambda^A(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t)), J_\lambda^A(x)\right) \\ &\leq d\left(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t), x\right) \\ &\leq (1-t)d(x_t, x) + td(f(x_t), x) \\ &\leq (1-t)d(x_t, x) + t(d(f(x_t), f(x)) + d(f(x), x)), \end{aligned}$$

that is,

$$\begin{aligned} d(x_t, x) &\leq d(x_t, x) - \psi(d(x_t, x)) + d(f(x), x) \\ \psi(d(x_t, x)) &\leq d(f(x), x). \end{aligned}$$

As in (a), $(d(x_t, x))_{t\in[0,1]}$ is bounded, and so is $(x_t)_{t\in[0,1]}$. Let $z_t = \exp_{x_t} t \exp_{x_t}^{-1} f(x_t)$. For $\lambda > 0$, we have

$$\begin{aligned} d\left(x_t, J_\lambda^A(x_t)\right) &= d\left(J_\lambda^A(z_t), J_\lambda^A(x_t)\right) \leq d(z_t, x_t) \\ &\leq ((1-t)d(x_t, x_t) + td(f(x_t), x_t)) = td(f(x_t), x_t). \end{aligned} \tag{4.12}$$

Since $(x_t)_{t\in[0,1]}$ is bounded, we may assume that the sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,1]$ be such that $\lim\limits_{n\to\infty} t_n = 0$ and the subsequence (x_{t_n}) of $(x_t)_{t\in[0,1]}$ converges to $\bar{x}$. For $\lambda > 0$, from the above inequality, we have

$$d\left(x_{t_n}, J_\lambda^A(x_{t_n})\right) \leq t_n d(f(x_{t_n}), x_{t_n}). \tag{4.13}$$

By taking limit as $n \to \infty$, we get

$$d\left(\bar{x}, J_\lambda^A(\bar{x})\right) = 0.$$

This implies that $\bar{x} \in A^{-1}(\mathbf{0})$. Since $x_t = J_\lambda^A(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t))$, we get

$$\frac{1}{\lambda} \exp_{x_t} t \exp_{x_t}^{-1} f(x_t) \in A(x_t),$$

that is,

$$\frac{t}{\lambda} \exp_{x_t}^{-1} f(x_t) \in A(x_t).$$

By monotonicity of A and for any $x \in A^{-1}(\mathbf{0})$, we have

$$\left\langle \exp_{x_t}^{-1} f(x_t), \exp_{x_t}^{-1} x \right\rangle \leq 0, \quad \forall x \in A^{-1}(\mathbf{0}),$$

and so,

$$\left\langle \exp_{x_{t_n}}^{-1} f(x_{t_n}), \exp_{x_{t_n}}^{-1} x \right\rangle \leq 0, \quad \forall x \in A^{-1}(\mathbf{0}). \tag{4.14}$$

Note that $x_{t_n} \to \bar{x}$ as $t_n \to 0$ and every weakly contraction mapping is continuous, we have

$$\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} x \right\rangle \leq 0, \quad \forall x \in A^{-1}(\mathbf{0}). \tag{4.15}$$

In order to show that the path $(x_t)_{t\in[0,1]}$ converges to an element of $A^{-1}(\mathbf{0})$, we assume that $(t_{n'})$ is another sequence in $[0,1]$ such that $x_{t_{n'}} \to \tilde{x}$ as $t_{n'} \to 0$. By (4.13), we obtain $\tilde{x} \in \mathrm{Fix}(J_\lambda^A)$ for all $\lambda > 0$. From (4.15), we have

$$\left\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} x \right\rangle \leq 0, \quad \forall x \in A^{-1}(\mathbf{0}).$$

In particular, for $\bar{x} \in A^{-1}(\mathbf{0})$, we get

$$\left\langle \exp_{\tilde{x}}^{-1} f(\tilde{x}), \exp_{\tilde{x}}^{-1} \bar{x} \right\rangle \leq 0. \tag{4.16}$$

By the same argument as in part (a), we can show that $(x_t)_{t\in[0,1]}$ converges to a unique point of $A^{-1}(\mathbf{0})$. $\square$

By considering $A = B + N_C$ in Theorems 4.2, we have the following result for the problem (3.6).

Corollary 4.3. *Let $f : \mathbb{M} \to \mathbb{M}$ be a weakly contraction mapping with the comparison function ψ, and $B : \mathbb{M} \to T\mathbb{M}$ be a single-valued monotone vector field such that the solution set of BVIP (3.6) is nonempty. Then, the following assertions hold:*

(a) *The path $(y_\lambda)_{\lambda>0}$ generated by*

$$y_\lambda := R_\lambda^B(f(y_\lambda)), \quad \forall \lambda > 0, \tag{4.17}$$

converges to a solution of BVIP (3.6) as $\lambda \to \infty$.

(b) *The path* $(x_t)_{t\in[0,1]}$ *generated by*

$$x_t := R^B_\lambda(\exp_{x_t} t \exp^{-1}_{x_t} f(x_t)), \quad \forall \lambda > 0 \text{ and } \forall t \in [0,1], \tag{4.18}$$

converges to a solution of BVIP (3.6) *as* $t \to 0$.

By taking $J^A_\lambda = \mathrm{Prox}_{\lambda,F}$ and $-\exp^{-1}_{\cdot} f = \nabla G$ in Theorems 4.2, we obtain the following result for the problem (3.10).

Corollary 4.4. *Let* $F : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous, geodesic convex function and* $G : \mathbb{M} \to \mathbb{R}$ *be a geodesic convex differentiable function and the solution set of BMP* (3.10) *is nonempty. Then, the following assertions hold:*

(a) *The path* $(y_\lambda)_{\lambda>0}$ *generated by*

$$y_\lambda := \mathrm{Prox}_{\lambda,F}(\exp_{y_\lambda}(-\nabla G(y_\lambda))), \quad \forall \lambda > 0, \tag{4.19}$$

converges to a solution of BMP (3.10) *as* $\lambda \to \infty$.

(b) *The path* $(x_t)_{t\in[0,1]}$ *generated by*

$$x_t := \mathrm{Prox}_{\lambda,F}(\exp_{x_t}(-t\,\nabla G(x_t))), \quad \forall \lambda > 0 \text{ and } \forall t \in [0,1], \tag{4.20}$$

converges to a solution of BMP (3.10) *as* $t \to 0$.

5. Explicit Methods

Algorithm 5.1. Choose an arbitrary element $x_1 \in \mathbb{M}$ and define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_{n+1} := J^A_{\lambda_n}\big(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n)\big), \quad \forall n \in \mathbb{N}, \tag{5.1}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $0 < \lambda \le \lambda_n < +\infty$ are the sequences of nonnegative real numbers such that the following conditions hold:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = +\infty$;

(iii) $\lim\limits_{n\to\infty} \frac{1}{\alpha_n}|\alpha_n - \alpha_{n-1}| = 0$;

(iv) $\lim\limits_{n\to\infty} \frac{1}{\alpha_n}|\lambda_n - \lambda_{n-1}| = 0$.

When $\mathbb{M} = \mathbb{H}$ is a Banach space, Algorithm 5.1 is discussed in [27].

Theorem 5.2. *Let* $f : \mathbb{M} \to \mathbb{M}$ *be a weakly contraction mapping with the comparison function* ψ *and* $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ *be a set-valued monotone vector field such that* $A^{-1}(\mathbf{0}) \neq \emptyset$. *Then, the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by Algorithm* 5.1 *converges to a solution of HVIP* (3.2).

Proof. Let $z_n = \exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n)$, and $\gamma_n : [0,1] \to \mathbb{M}$ be the geodesic such that $\gamma_n(0) = x_n$ and $\gamma_n(1) = f(x_n)$ for each $n \in \mathbb{N}$. Clearly, $z_n = \gamma_n(\alpha_n)$. Let $\bar{x} \in A^{-1}(\mathbf{0})$, that is, $\bar{x} = J^A_{\lambda_n}(\bar{x})$. Then by Proposition 2.3, we have

$$\begin{aligned}
d(z_n, \bar{x}) &= d(\gamma_n(\alpha_n), \bar{x}) \\
&\leq (1-\alpha_n) d(\gamma_n(0), \bar{x}) + \alpha_n d(\gamma_n(1), \bar{x}) \\
&= (1-\alpha_n) d(x_n, \bar{x}) + \alpha_n d(f(x_n), \bar{x}) \\
&\leq (1-\alpha_n) d(x_n, \bar{x}) + \alpha_n d(f(x_n), f(\bar{x})) + \alpha_n d(f(\bar{x}), \bar{x}) \\
&\leq (1-\alpha_n) d(x_n, \bar{x}) + \alpha_n d(x_n, \bar{x}) - \alpha_n \psi(d(x_n, \bar{x})) + \alpha_n d(f(\bar{x}), \bar{x}) \\
&= d(x_n, \bar{x}) - \alpha_n \psi(d(x_n, \bar{x})) + \alpha_n d(f(\bar{x}), \bar{x}). \qquad (5.2)
\end{aligned}$$

From (5.1), we obtain

$$\begin{aligned}
d(x_{n+1}, \bar{x}) &= d(J^A_{\lambda_n}(z_n), J^A_{\lambda_n}(\bar{x})) \leq d(z_n, \bar{x}) \\
&\leq d(x_n, \bar{x}) - \alpha_n \psi(d(x_n, \bar{x})) + \alpha_n d(f(\bar{x}), \bar{x}). \qquad (5.3)
\end{aligned}$$

Set $\mu := \inf\{\psi(d(x_n, \bar{x}))/d(x_n, \bar{x}) : x_n \neq \bar{x}, n \in \mathbb{N}\}$ for all $\bar{x} \in A^{-1}(\mathbf{0})$. Then, from the above inequality, we have

$$\begin{aligned}
d(x_{n+1}, \bar{x}) &\leq (1-\mu\alpha_n) d(x_n, \bar{x}) + \alpha_n d(f(\bar{x}), \bar{x}) \\
&\leq \max\left\{ d(x_n, \bar{x}), \frac{1}{\mu} d(f(\bar{x}), \bar{x}) \right\} \\
&\;\;\vdots \\
&\leq \max\left\{ d(x_1, \bar{x}), \frac{1}{\mu} d(f(\bar{x}), \bar{x}) \right\}. \qquad (5.4)
\end{aligned}$$

Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Hence, we may assume a constant $K > 0$ such that $d(x_n, \bar{x}) \leq K$. Since $\alpha_n \to 0$ as $n \to \infty$, $\{\alpha_n\}_{n\in\mathbb{N}}$ is bounded. Therefore, there exists a constant $\eta > 0$ such that $\alpha_n \leq \eta$ for all $n \in \mathbb{N}$. Thus, from (5.2), we have

$$\begin{aligned}
d(z_n, \bar{x}) &\leq d(x_n, \bar{x}) - \alpha_n \psi(d(x_n, \bar{x})) + \alpha_n d(f(\bar{x}), \bar{x}) \\
&\leq d(x_n, \bar{x}) + \alpha_n d(f(\bar{x}), \bar{x}) \\
&\leq K + \eta d(f(\bar{x}), \bar{x}),
\end{aligned}$$

and therefore, $\{z_n\}_{n\in\mathbb{N}}$ is bounded. Since f is weakly contraction, we have

$$\begin{aligned}
d(f(x_n), \bar{x}) &\leq d(f(x_n), f(\bar{x})) + d(f(\bar{x}), \bar{x}) \\
&\leq d(x_n, \bar{x}) - \psi(d(x_n, \bar{x})) + d(f(\bar{x}), \bar{x}) \\
&< d(x_n, \bar{x}) + d(f(\bar{x}), \bar{x}) \\
&\leq K + d(f(\bar{x}), \bar{x}),
\end{aligned}$$

which implies that $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded.

From the convexity of Riemannian distance, we obtain

$$\begin{aligned} d(z_n, z_{n-1}) &= d(\gamma_n(\alpha_n), \gamma_{n-1}(\alpha_{n-1})) \\ &\leq d(\gamma_n(\alpha_n), \gamma_{n-1}(\alpha_n)) + d(\gamma_{n-1}(\alpha_n), \gamma_{n-1}(\alpha_{n-1})) \\ &\leq (1-\alpha_n) d(\gamma_n(0), \gamma_{n-1}(0)) + \alpha_n d(\gamma_n(1), \gamma_{n-1}(1)) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(x_{n-1}, f(x_{n-1})) \\ &= (1-\alpha_n) d(x_n, x_{n-1}) + \alpha_n d(f(x_n), f(x_{n-1})) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(x_{n-1}, f(x_{n-1})) \\ &\leq (1-\alpha_n) d(x_n, x_{n-1}) + \alpha_n d(x_n, x_{n-1}) - \alpha_n \psi(d(x_n, x_{n-1})) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(x_{n-1}, f(x_{n-1})) \\ &\leq d(x_n, x_{n-1}) - \alpha_n \psi(d(x_n, x_{n-1})) + |\alpha_n - \alpha_{n-1}| K_2, \end{aligned} \tag{5.5}$$

where $K_2 = \sup_{n\in\mathbb{N}}\{d(x_{n-1}, f(x_{n-1}))\}$ is a constant. By Lemma 3.9, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(J^A_{\lambda_n}(z_n), J^A_{\lambda_{n-1}}(z_{n-1})) \\ &\leq d(J^A_{\lambda_n}(z_n), J^A_{\lambda_n}(z_{n-1})) + d(J^A_{\lambda_n}(z_{n-1}), J^A_{\lambda_{n-1}}(z_{n-1})) \\ &\leq d(z_n, z_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_{n-1}} d(z_{n-1}, J^A_{\lambda_{n-1}}(z_{n-1})) \\ &\leq d(x_n, x_{n-1}) - \alpha_n \psi(d(x_n, x_{n-1})) + |\alpha_n - \alpha_{n-1}| K_2 + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} K_3, \end{aligned}$$

where $K_3 = \sup_{n\in\mathbb{N}}\left\{d(z_{n-1}, J^A_{\lambda_{n-1}}(z_{n-1}))\right\}$ is a constant. From (ii), (iii) and Lemma 2.14, we conclude that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5.6}$$

Now, we prove that

$$\limsup_{n\to\infty} \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n \right\rangle \leq 0$$

where $\bar{x} = P_{A^{-1}(\mathbf{0})} f(\bar{x})$. Since the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{f(x_n)\}_{n\in\mathbb{N}}$ are bounded, so is $\left\{\left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n \right\rangle\right\}_{n\in\mathbb{N}}$. Hence, its upper limit exists. We may assume a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n \right\rangle = \lim_{j\to\infty} \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_{n_j} \right\rangle,$$

and $x_{n_j} \to \tilde{z}$. By the convexity of Riemanninan distance, we have

$$\begin{aligned} d(z_n, x_n) &\leq d(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n), x_n) \\ &\leq (1-\alpha_n) d(x_n, x_n) + \alpha_n d(f(x_n), x_n) \\ &= \alpha_n d(f(x_n), x_n). \end{aligned}$$

Since $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded, and so is $\{d(f(x_n), x_n)\}_{n\in\mathbb{N}}$. Since $\alpha_n \to 0$, taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(z_n, x_n) = 0.$$

On the other hand, we have

$$\begin{aligned} d\left(J^A_{\lambda_n} x_n, x_n\right) &\leq d\left(J^A_{\lambda_n} x_n, J^A_{\lambda_n} z_n\right) + d\left(J^A_{\lambda_n} z_n, x_n\right) \\ &\leq d(z_n, x_n) + d(x_{n+1}, x_n), \end{aligned}$$

and therefore, $\lim_{n\to\infty} d\big(J^A_{\lambda_n} x_n, x_n\big) = 0$. For any $\mu > 0$, let A_μ be the complimentary vector field of A. Then, by Yosida approximation and Lemma 3.10, we have

$$\begin{aligned} d\left(J^A_{\lambda_n}(x_n), J^A_\mu(J^A_{\lambda_n}(x_n))\right) &= \left\| \exp^{-1}_{J^A_{\lambda_n}(x_n)} J^A_\mu(J^A_{\lambda_n}(x_n)) \right\| \\ &= \mu \|A_\mu(J^A_{\lambda_n}(x_n))\| \leq \mu |||A(J^A_{\lambda_n}(x_n))|||. \end{aligned}$$

By Lemma 3.10, $A_\lambda(x) \in \mathcal{P}_{x,J_\lambda(x)} A(J_\lambda(x))$ for all $\lambda > 0$, and therefore,

$$\|A_{\lambda_n}(J^A_{\lambda_n}(x_n))\| = \|A(J^A_{\lambda_n}(x_n))\|.$$

Then the above inequality becomes

$$\begin{aligned} d\left(J^A_{\lambda_n}(x_n), J^A_\mu(J^A_{\lambda_n}(x_n))\right) &\leq \mu \left\|A_{\lambda_n}(J^A_{\lambda_n}(x_n))\right\| = \frac{\mu}{\lambda_n} \left\|\exp^{-1}_{x_n} J^A_{\lambda_n}(x_n)\right\| \\ &\leq \frac{\mu}{\lambda} d\left(x_n, J^A_{\lambda_n}(x_n)\right) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{5.7}$$

and so,

$$\begin{aligned} d\left(x_n, J^A_\mu x_n\right) &\leq d\left(x_n, J^A_{\lambda_n}(x_n)\right) + d\left(J^A_{\lambda_n}(x_n), J^A_\mu(J^A_{\lambda_n}(x_n))\right) \\ &\quad + d\left(J^A_\mu(J^A_{\lambda_n}(x_n)), J^A_\mu(x_n)\right) \\ &\leq 2d\left(x_n, J^A_{\lambda_n}(x_n)\right) + d\left(J^A_{\lambda_n}(x_n), J^A_\mu(J^A_{\lambda_n}(x_n))\right). \end{aligned}$$

It follows that $\lim_{n\to\infty} d\left(x_n, J^A_\mu x_n\right) = 0$. This implies that $\bar{x} \in A^{-1}(\mathbf{0})$. Since

$$\left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} y \right\rangle \leq 0$$

for any $y \in A^{-1}(\mathbf{0})$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n \right\rangle &= \lim_{j\to\infty} \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_{n_j} \right\rangle \\ &= \left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} \tilde{z} \right\rangle \leq 0. \end{aligned}$$

It follows that there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ in $(0, +\infty)$ with $\lim_{n\to\infty} c_n = 0$ such that

$$\left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n \right\rangle \leq c_n, \quad \forall n \in \mathbb{N}.$$

We next prove that $\lim_{n\to\infty} d(x_n, \bar{x}) = 0$. Fix $n \geq 0$ and set $s_n = f(x_n)$. Consider the geodesic triangles $\Delta(s_n, x_n, \bar{x})$, $\Delta(f(\bar{x}), x_n, \bar{x})$ and $\Delta(f(\bar{x}), x_n, s_n)$. Then by Lemma 2.5, there exist comparison triangles $\Delta(s'_n, x'_n, \bar{x}')$, $\Delta(f(\bar{x})', x'_n, \bar{x}')$ and $\Delta(f(\bar{x})', x'_n, s'_n)$s uch that

$$d(s_n, x_n) = \|s'_n - x'_n\|, \quad d(x_n, \bar{x}) = \|x'_n - \bar{x}'\| \quad \text{and} \quad d(s_n, \bar{x}) = \|s'_n - \bar{x}'\|,$$

$$d(f(\bar{x}), \bar{x}) = \|f(\bar{x})' - \bar{x}'\|, \quad d(x_n, \bar{x}) = \|x'_n - \bar{x}'\| \quad \text{and} \quad d(s_n, f(\bar{x})) = \|s'_n - f(\bar{x})'\|.$$

Let α and α' denote the angles at $\bar{x}$ and $\bar{x}'$ in the triangles $\Delta(f(\bar{x}), x_n, \bar{x})$ and $\Delta(f(\bar{x})', x'_n, \bar{x}')$, respectively. Therefore, $\alpha \leq \alpha'$ by Lemma 2.6 (a), and so,

$\cos\alpha' \leq \cos\alpha$. Let $z'_n := \alpha_n s'_n + (1-\alpha_n)x'_n$ be the comparison point of z_n. Then, by Lemma 2.6 (b), we have

$$\begin{aligned}
&d^2(x_{n+1},\bar{x}) = d^2(J^A_{\lambda_n} z_n, J^A_{\lambda_n}\bar{x})\\
&\leq d^2(z_n,\bar{x}) \leq \|z'_n - \bar{x}'\|^2\\
&= \|\alpha_n s'_n + (1-\alpha_n)x'_n - \bar{x}'\|^2\\
&= \|\alpha_n(s'_n - \bar{x}') + (1-\alpha_n)(x'_n - \bar{x}')\|^2\\
&= \alpha_n^2\|s'_n - \bar{x}'\|^2 + (1-\alpha_n)^2\|x'_n - \bar{x}'\|^2 + 2\alpha_n(1-\alpha_n)\left\langle s'_n - \bar{x}', x'_n - \bar{x}'\right\rangle\\
&= \alpha_n^2\|s'_n - \bar{x}'\|^2 + (1-\alpha_n)^2\|x'_n - \bar{x}'\|^2 + 2\alpha_n(1-\alpha_n)\big(\left\langle s'_n - f(\bar{x})', x'_n - \bar{x}'\right\rangle\\
&\qquad + \left\langle f(\bar{x})' - \bar{x}', x'_n - \bar{x}'\right\rangle\big)\\
&\leq \alpha_n^2\|s'_n - \bar{x}'\|^2 + (1-\alpha_n)^2\|x'_n - \bar{x}'\|^2 + 2\alpha_n(1-\alpha_n)\big(\|s'_n - f(\bar{x})'\|\|x'_n - \bar{x}'\|\\
&\qquad + \|f(\bar{x})' - \bar{x}'\|\|x'_n - \bar{x}'\|\cos\alpha'\big)\\
&\leq \alpha_n^2 d^2(f(x_n),\bar{x}) + (1-\alpha_n)^2 d^2(x_n,\bar{x}) + 2\alpha_n(1-\alpha_n)\big(d(f(x_n),f(\bar{x}))d(x_n,\bar{x})\\
&\qquad + d(f(\bar{x}),\bar{x})d(x_n,\bar{x})\cos\alpha\big)\\
&\leq \alpha_n^2 d^2(f(x_n),\bar{x}) + (1-\alpha_n)^2 d^2(x_n,\bar{x}) + 2\alpha_n(1-\alpha_n)\big(d(x_n,\bar{x})\\
&\qquad - \psi(d(x_n,\bar{x}))d(x_n,\bar{x}) + 2\alpha_n(1-\alpha_n)d(f(\bar{x}),\bar{x})d(x_n,\bar{x})\cos\alpha,
\end{aligned}$$

that is,

$$\begin{aligned}
d^2(x_{n+1},\bar{x}) &\leq \alpha_n^2 d^2(f(x_n),\bar{x}) + (1-\alpha_n^2)d^2(x_n,\bar{x}) - 2\alpha_n(1-\alpha_n)\psi(d(x_n,\bar{x}))d(x_n,\bar{x})\\
&\quad + 2\alpha_n d(f(\bar{x}),\bar{x})d(x_n,\bar{x})\cos\alpha\\
&= (1-\alpha_n^2)d^2(x_n,\bar{x}) + \alpha_n^2 d^2(f(x_n),\bar{x}) - 2\alpha_n(1-\alpha_n)\psi(d(x_n,\bar{x}))d(x_n,\bar{x})\\
&\quad + 2\alpha_n\left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n\right\rangle\\
&\leq d^2(x_n,\bar{x}) + \alpha_n^2 d^2(f(x_n),\bar{x}) - 2\alpha_n(1-\alpha_n)\psi(d(x_n,\bar{x}))d(x_n,\bar{x})\\
&\quad + 2\alpha_n\left\langle \exp^{-1}_{\bar{x}} f(\bar{x}), \exp^{-1}_{\bar{x}} x_n\right\rangle.
\end{aligned}$$

Since $\{d(f(x_n),\bar{x})\}_{n\in\mathbb{N}}$ is bounded, we may assume that $d(f(x_n),\bar{x}) \leq K_4$ for some $K_4 > 0$. Let $u_n = d^2(x_n,\bar{x})$ for any $n\in\mathbb{N}$. Then, we have

$$u_{n+1} \leq u_n - \alpha_n\varphi(u_n) + \beta_n, \quad \forall n\in\mathbb{N},$$

where $\beta_n = \alpha_n^2 K_4^2 + 2\alpha_n c_n + 2\alpha_n^2\psi(\sqrt{u_n})\sqrt{u_n}$ and $\varphi(t) = 2\sqrt{t}\psi(\sqrt{t})$.
Since $u_n = d^2(x_n,\bar{x}) \leq K^2$ and ψ is nondecreasing, we have $\psi(\sqrt{u_n}) \leq \psi(K)$. Then $\beta_n \leq \alpha_n^2 K_4^2 + 2\alpha_n c_n + 2\alpha_n^2\psi(K) \quad K$. By condition (i), it follows $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = 0$. This together with (ii) and Lemma 2.14, we have $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{x}$. □

By considering $A = B + N_C$ in Theorems 5.2, we have the following result for BVIP (3.6).

Corollary 5.3. *Let $f:\mathbb{M}\to\mathbb{M}$ be a weakly contraction mapping with the comparison function ψ and $B:\mathbb{M}\to T\mathbb{M}$ be a single-valued monotone vector field such that the solution set of BVIP* (3.6) *is nonempty.*
Let $\{\alpha_n\}_{n\in\mathbb{N}}\subset[0,1]$ and $0<\lambda\leq\lambda_n<+\infty$ be sequences of nonnegative real numbers

satisfying the same conditions as in Algorithm 5.1. Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$x_{n+1} := R^B_{\lambda_n}\big(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n)\big), \quad \forall n \in \mathbb{N}, \tag{5.8}$$

converges to a solution of BVIP (3.6).

By taking $J^A_\lambda = \mathrm{Prox}_{\lambda,F}$ and $-\exp^{-1}_{\cdot} f = \nabla G$ in Theorems 5.2, we obtain the following result for BMP (3.10).

Corollary 5.4. *Let* $F : \mathbb{M} \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous, geodesic convex function and* $G : \mathbb{M} \to \mathbb{R}$ *be a geodesic convex differentiable function such that the solution set of BMP* (3.10) *is nonempty. Let* $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $0 < \lambda \leq \lambda_n < +\infty$ *be sequences of nonnegative real numbers satisfying the same conditions as in Algorithm* 5.1*. Then, the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$x_{n+1} := \mathrm{Prox}_{\lambda_n,F}\big(\exp_{x_n}(-\alpha_n \nabla G(x_n))\big), \quad \forall n \in \mathbb{N}, \tag{5.9}$$

converges to a solution of BMP (3.10).

We now present the following inexact version of Algorithm 5.1.

Algorithm 5.5. Choose $w_1 \in \mathbb{M}$ and generate a sequence $\{w_{n+1}\}_{n\in\mathbb{N}}$ as

$$\begin{aligned} u_n &:= \exp_{w_n} \alpha_n \exp^{-1}_{w_n} f(w_n), \\ w_{n+1} &:= J^A_{\lambda_n}(\bar{u}_n), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{5.10}$$

where $\exp^{-1}_{u_n} \bar{u}_n = e_n$ is a sequence of errors such that

$$\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0, \tag{5.11}$$

and $\{\alpha_n\}_{n\in\mathbb{N}}$ is the same as in Algorithm 5.1.

Clearly, $\|e_n\| = d(u_n, \bar{u}_n)$ for all $n \in \mathbb{N}$. If $\mathbb{M} = \mathbb{H}$ is a Banach space, then Algorithm 5.5 reduces to the algorithm studied in [27] where f is considered to be a contraction mapping. Moreover, if $f(x) = u$ is a fixed point in $\mathbb{M}$ for all $x \in \mathbb{M}$, then Algorithm 5.5 reduces to the algorithm studied in [5].

We now establish the following convergence result for the sequence generated by Algorithm 5.5 to a solution of HVIP (3.2).

Theorem 5.6. *Let* $f : \mathbb{M} \to \mathbb{M}$ *be a weakly contraction mapping with the comparison function* ψ *and* $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ *be a set-valued monotone vector field such that* $A^{-1}(\mathbf{0}) \neq \emptyset$*. Let* $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $0 < \lambda \leq \lambda_n < +\infty$ *be sequences of real numbers satisfying the conditions* (i)-(iv) *of Algorithm* 5.1*, and let* $\{e_n\}_{n\in\mathbb{N}}$ *be a sequence of errors satisfying the condition* (5.11)*. Then, the sequence* $\{w_n\}_{n\in\mathbb{N}}$ *defined by Algorithm* 5.5 *converges to a solution of HVIP* (3.2).

Proof. For $w_1 = x_1 \in \mathbb{M}$, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{M}$ generated by (5.1). Then from Theorem 5.2, $\{x_n\}_{n\in\mathbb{N}}$ converges to a solution, say $\bar{x}$, of HVIP (3.2), that is,

$\bar{x} = P_{A^{-1}(\mathbf{0})} f(\bar{x})$. It follows from (5.1) and Algorithm 5.5 that

$$\begin{aligned} d(x_{n+1}, w_{n+1}) &= d\left(J^A_{\lambda_n}\left(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n)\right), J^A_{\lambda_n}(\bar{u}_n)\right) \\ &\le d(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n), \bar{u}_n) \\ &\le d(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n), u_n) + d(u_n, \bar{u}_n) \\ &= d(\exp_{x_n} \alpha_n \exp^{-1}_{x_n} f(x_n), \exp_{w_n} \alpha_n \exp^{-1}_{w_n} f(w_n)) + \|e_n\| \\ &\le (1-\alpha_n) d(x_n, w_n) + \alpha_n d(f(x_n), f(w_n)) + \|e_n\| \\ &\le (1-\alpha_n) d(x_n, w_n) + \alpha_n \left(d(x_n, w_n) - \psi(d(x_n, w_n))\right) + \|e_n\| \\ &= d(x_n, w_n) - \alpha_n \psi(d(x_n, w_n)) + \|e_n\|. \end{aligned}$$

By Lemma 2.14, we obtain $\lim\limits_{n\to\infty} d(x_n, w_n) = 0$. Therefore, $\{w_n\}_{n\in\mathbb{N}}$ converges to a solution of HVIP (3.2). □

6. Computational Numerical Experiments

We illustrate the implicit methods (4.1) and (4.2) and the convergence Theorem 4.2 by the following example.

Example 6.1. Let $\mathbb{M} := \mathbb{R}^{++} = \{x \in \mathbb{R} : x > 0\}$ be a Hadamard manifold with Riemannian metric is defined by

$$\langle u, v\rangle = h(x)uv, \quad \forall u, v \in T_x\mathbb{M},$$

where $h : \mathbb{M} \to (0, +\infty)$ is given by $h(x) = 1/x^4$. It is easy to see that the tangent plane $T_x\mathbb{M}$ at $x \in \mathbb{M}$ is equal to $\mathbb{R}$ and $T\mathbb{M} = \mathbb{R}$. The geodesic $\gamma : [0,1] \to \mathbb{M}$ joining $x = \gamma(0)$ and $y = \gamma(1)$ in $\mathbb{M}$ is defined as

$$\gamma(t) = \exp_x t \exp^{-1}_x y = \frac{xy}{tx + (1-t)y}, \quad \forall t \in [0,1]. \tag{6.1}$$

The inverse exponential map is given by $\exp^{-1}_x y = \dot{\gamma}(0) = \frac{x}{y}(y - x)$ for all $x, y \in \mathbb{M}$. The Riemannian distance is given by

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|, \quad \forall x, y \in \mathbb{M}.$$

For further details, see [24]. We consider a set-valued vector field $A : \mathbb{M} \rightrightarrows T\mathbb{M}$ defined by

$$A(x) := \begin{cases} x(x-1), & \text{if } 0 < x < 1, \\ \mathbb{R}^+, & \text{if } x = 1, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{6.2}$$

where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Note that A is a set-valued monotone vector field. Clearly, $A^{-1}(\mathbf{0}) = \{1\} \neq \emptyset$ and the resolvent of A is given by

$$J^A_\lambda(x) = \frac{x + \lambda x}{1 + \lambda x}, \quad \forall x \in \mathbb{M}.$$

Indeed, let $J_\lambda^A(x) = z$ for any $x \in \mathbb{M}$. Then by the definition of resolvent, we have $\exp_z^{-1} x \in \lambda A(z)$, that is,

$$\frac{z}{\lambda x}(x-z) \in \begin{cases} z(z-1), & \text{if } 0 < z < 1, \\ \mathbb{R}^+, & \text{if } z = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

This implies that

$$J_\lambda^A(x) = \begin{cases} \frac{x+\lambda x}{1+\lambda x}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases} \quad \text{that is} \quad J_\lambda^A(x) = \frac{x+\lambda x}{1+\lambda x}, \quad \forall x > 0.$$

Now, we define a mapping $f : \mathbb{M} \to \mathbb{M}$ by $f(x) = 1 + x$ for all $x \in \mathbb{M}$. Then f is a weakly contraction mapping with the comparison function ψ given by $\psi(t) = \frac{t^2}{1+t}$ for all $t \geq 0$.

Indeed, for any $x, y > 0$, we have

$$\left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \left|\frac{\frac{1}{x}}{1+\frac{1}{x}} - \frac{\frac{1}{y}}{1+\frac{1}{y}}\right| = \frac{\left|\frac{1}{x} - \frac{1}{y}\right|}{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)}.$$

Since $|a-b| < a+b$ for all $a, b > 0$, we have $|a-b| < a+b+ab$, and so,

$$1 + |a-b| < 1 + a + b + ab = (1+a)(1+b)$$

for all $a, b > 0$. Therefore,

$$\left|\frac{1}{1+x} - \frac{1}{1+y}\right| = \frac{\left|\frac{1}{x} - \frac{1}{y}\right|}{\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)} < \frac{\left|\frac{1}{x} - \frac{1}{y}\right|}{1 + \left|\frac{1}{x} - \frac{1}{y}\right|},$$

and so,

$$\begin{aligned} d(f(x), f(y)) &< \frac{d(x,y)}{1+d(x,y)} = d(x,y) - \frac{d^2(x,y)}{1+d(x,y)} \\ &= d(x,y) - \psi(d(x,y)), \ \forall x, y \in \mathbb{M}, \end{aligned}$$

where $\psi(t) = \frac{t^2}{1+t}$ for all $t \geq 0$. Note that f is not a contraction map.

Let $\bar{x} = 1 \in A^{-1}(\mathbf{0})$. Then $f(\bar{x}) = 2$, $\exp_{\bar{x}}^{-1} f(\bar{x}) = \frac{1}{2}$, $\exp_{\bar{x}}^{-1} y = \frac{1}{y}(y-1)$ for any $y \in A^{-1}(\mathbf{0})$ and $h(\bar{x}) = 1$. With these settings, we have

$$\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle = \frac{1}{2} \cdot \frac{1}{y}(y-1) = 0, \quad \forall y \in A^{-1}(\mathbf{0}),$$

i.e., $\mathbb{S} = \{1\}$. Under the above settings, for $\lambda > 0$, (4.1) becomes

$$y_\lambda = J_\lambda^A(f(y_\lambda)) = \frac{(1+\lambda)(1+y_\lambda)}{1+\lambda(1+y_\lambda)} \quad \Rightarrow \quad y_\lambda = \sqrt{\frac{1+\lambda}{\lambda}}.$$

Taking limit as $\lambda \to \infty$, we obtain $\lim\limits_{\lambda \to \infty} y_\lambda = 1$ a solution of the problem (3.2).

Moreover, for $t \in [0,1]$ and $\lambda > 0$, (4.2) can be written as

$$x_t = J_\lambda^A(\exp_{x_t} t \exp_{x_t}^{-1} f(x_t)) = \frac{x_t(1+x_t)(1+\lambda)}{1-t+x_t+\lambda x_t(1+x_t)} \quad \Rightarrow \quad x_t = \sqrt{\frac{t+\lambda}{\lambda}}.$$

By taking limit as $t \to 0$, we have that $\lim_{t\to 0} x_t = 1$ a solution of the problem (3.2).

In the following example, we show that the explicit Algorithm 5.1 converges to a solution of the problem (3.2).

Example 6.2. Under the setting of Example 6.1, Algorithm 5.1 reduces to

$$x_{n+1} = J_{\lambda_n}^A(\exp_{x_n} \alpha_n \exp_{x_n}^{-1} f(x_n)) = \frac{x_n(1+x_n)(1+\lambda_n)}{1-\alpha_n+x_n+\lambda_n x_n(1+x_n)}, \quad \forall n \in \mathbb{N}.$$

We choose different initial points $x_1 = 0.01$, $x_1 = 0.5$ and $x_1 = 2$ and different parameters $\lambda_n = \frac{n}{\beta(n+\beta)}$ and $\alpha_n = \frac{1}{n+\beta}$ for all $n \in \mathbb{N}$, where $\beta = 1, 2, \ldots$, which satisfy assumptions (I) - (IV) of Algorithm 5.1. Then the convergence of the sequence generated by Algorithm 5.1 converges to 1 a solution of the problem (3.2), is shown in the following figures and table.

Error	$\lambda_n = \frac{n}{n+1}$ and $\alpha_n = \frac{1}{n+1}$			$\lambda_n = \frac{n}{2(n+2)}$ and $\alpha_n = \frac{1}{n+2}$		
No. iter.	$x_1 = 0.01$	$x_1 = 0.5$	$x_1 = 2$	$x_1 = 0.01$	$x_1 = 0.5$	$x_1 = 2$
1	0.04104	0.18293	0.24806	0.01125	0.04906	0.19393
2	0.08003	0.07640	0.13373	0.01708	0.06746	0.16770
3	0.01356	0.02090	0.07530	0.02549	0.08771	0.12489
8	$\cdots$	$\cdots$	0.00979	$\cdots$	$\cdots$	$\cdots$
11	0.00407	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
13	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0.00869
14	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0.00871	$\cdots$
17	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0.00022	$\cdots$
19	$\cdots$	$\cdots$	$\cdots$	0.00538	$\cdots$	$\cdots$
22	$\cdots$	$\cdots$	$\cdots$	0.00017	$\cdots$	$\cdots$
23	0.00097	$\cdots$	0.00097	$\cdots$	$\cdots$	$\cdots$
24	$\cdots$	0.00097	$\cdots$	$\cdots$	$\cdots$	$\cdots$
33	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	0.00099

Table 1. Computative error of Algorithm 5.1 by Example 6.2 for the choices of different parameters $\lambda_n = \frac{n}{\beta(n+\beta)}$ and $\alpha_n = \frac{1}{n+\beta}$ for $\beta = 1, 2$, different initial points $x_1 = 0.01$, $x_1 = 0.5$ and $x_1 = 2$ and the tolerance of error $= |x_{n+1} - x_n| < 10^{-4}$.

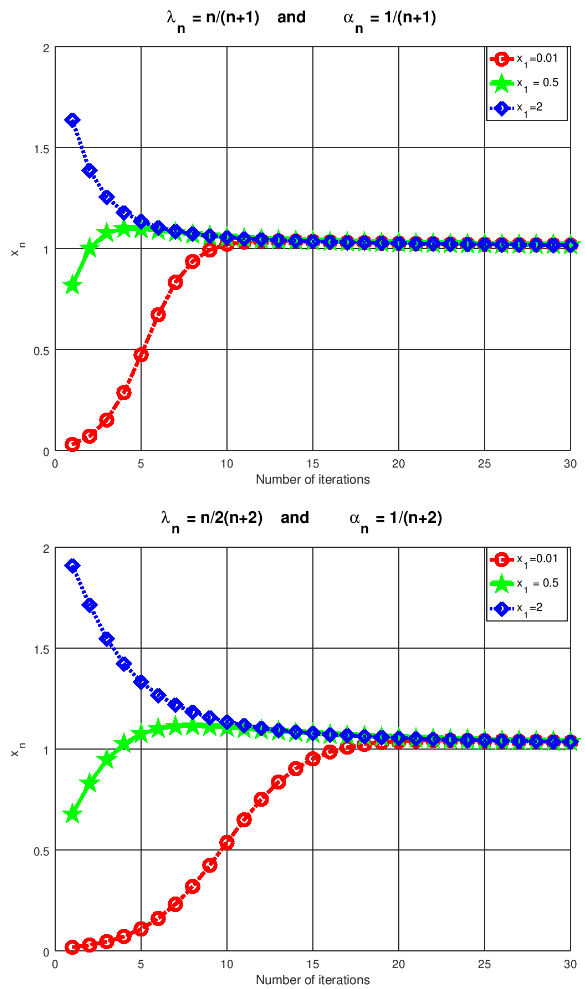

FIGURE 1. Numerical convergence of Algorithm 5.1 by computational method in Example 6.2 for the choices of different parameters $\lambda_n = \frac{n}{\beta(n+\beta)}$ and $\alpha_n = \frac{1}{n+\beta}$ for $\beta = 1, 2$ and different initial points $x_1 = 0.01$, $x_1 = 0.5$ and $x_1 = 2$.

Acknowledgements. In this research, first and last author were supported by a research grant of DST-SERB No. EMR/2016/005124.

References

[1] Y.I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, Oper. Theory Adv. Appl., **98**(1997), 7-22.

[2] S. Al-Homidan, Q.H. Ansari, F. Babu, *Halpern and Mann type algorithms for fixed points and inclusion problems on Hadamard manifolds*, Numer. Funct. Anal. Optim., **40**(2019), no. 6, 621-653.

[3] S. Al-Homidan, Q.H. Ansari, F. Babu, J.C. Yao, *Viscosity method with a ϕ-contraction mapping for hierarchical variational inequalities on Hadamard manifolds*, Fixed Point Theory, **21**(2020), no. 2, 561-584.

[4] Q.H. Ansari, F. Babu, X.-B. Li, *Variational inclusion problems in Hadamard manifolds*, J. Nonlinear Convex Anal., **19**(2018), no. 2, 219-237.

[5] Q.H. Ansari, F. Babu, J.C. Yao, *Regularization of proximal point algorithms in Hadamard manifolds*, J. Fixed Point Theory Appl., **21**(2019), Art. ID 25.

[6] Q.H. Ansari, F. Babu, M. Zeeshan, *Existence results and implicit viscosity methods for hierarchical problems on Hadamard manifolds*, J. Nonlinear Convex Anal., **21**(2020), no. 10, 2299-2324.

[7] Q.H. Ansari, L.-C. Ceng, H. Gupta, *Triple hierarchical variational inequalities. In Nonlinear Analysis: Approximation Theory, Optimization and Applications*, Edited by Qamrul Hasan Ansari, Birkhäuser, Springer, New Delhi, Heidelberg, New York, Dordrecht, London, (2014), 231-280.

[8] M. Bačák, *Computing medians and means in Hadamard spaces*, SIAM J. Optim., **24**(2014), 1542-1566.

[9] D.W. Boyd, J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 335-341.

[10] A. Cabot, *Proximal point algorithm controlled by a slowly vanishing term: applications to hierarchical minimization*, SIAM J. Optim., **15**(2005), 555-572.

[11] P.L. Combettes, *A block-iterative surrogate constraint splitting method for quadratic signal recovery*, IEEE Trans. Signal Process, **51**(2003), 1771-1782.

[12] J.X. da Cruz Neto, O.P. Ferreira, L.R. Lucambio Pérez, *Monotone point-to-set vector fields*, Balk. J. Geometry Appl., **5**(2000), no. 1, 69-79.

[13] M.P. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, Basel, Berlin, 1992.

[14] O.P. Ferreira, P.R. Oliveira, *Proximal point algorithms on Riemannian manifolds*, Optimization, **51**(2002), 257-270.

[15] S. Huang, *Approximations with weak contractions in Hadamard manifolds*, Linear Nonlinear Anal., **1**(2015), no. 2, 317-328.

[16] H. Iiduka, *Fixed point optimization algorithmand its application to power control in CDMA data networks*, Math. Prog., **133**(2012), 227-242.

[17] C. Li, G. López, V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. London Math. Soc., **79**(2009), 663-683.

[18] C. Li, G. López, V. Martín-Márquez, *Itetive algorithms for nonexpansive mappings on Hadamard manifolds*, Taiwanese J. Math., **14**(2010), no. 2, 541-559.

[19] C. Li, G. López, V. Martín-Márquez, *Resolvents of set-valued monotone vector fields in Hadamard manifolds*, Set-valued Anal., **19**(2011), 361-383.

[20] S.Z. Németh, *Monotonicity of the complementary vector field of a nonexpansive map*, Acta Math. Hungar., **84**(1999), no. 3, 189-197.

[21] S.Z. Németh, *Monotone vector fields*, Publ. Math. Debrecen, **54**(1999), 437-449.

[22] S.Z. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal., **52**(2003), 1491-1498.

[23] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241**(2000), 46-55.

[24] T. Rapcsák, *Smooth Nonlinear Optimization in $\mathbb{R}^n$*, Kluwer Academic Publishers, Dordrecht, 1997.

[25] B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., **47**(2001), 2683-2693.

[26] D.R. Sahu, Q.H. Ansari, J.C. Yao, *The Prox-Tikhonov forward method and application*, Taiwanese J. Math., **19**(2015), 481-503.
[27] D.R. Sahu, J.C. Yao, *The Prox-Tikhonov regularization method for the proximal point algorithm in Banach spaces*, J. Global Optim., **51**(2011), 641-655.
[28] T. Sakai, *Riemannian Geometry*, Amer. Math. Soc. Providence, RI, 1996.
[29] K. Slavakis, I. Yamada, *Robust wideband beamforming by the hybrid steepest descentmethod*, IEEE Trans. Signal Process, **55**(2007), 4511-4522.
[30] M. Solodov, *An explicit descent method for bilevel convex optimization*, J. Convex Anal., **14**(2007), 227-237,
[31] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994.
[32] R. Walter, *On the metric projections onto convex sets in Riemannian spaces*, Arch. Math., **XXV**(1974), 91-98.
[33] Z.P. Wan, J.W. Chen, *On bilevel variational inequalities*, J. Oper. Res. Soc. China, **1**(2013), 483-510.
[34] N.C. Wong, D.R. Sahu, J.C. Yao, *Solving variational inequalities involving nonexpansive type mappings*, Nonlinear Anal., **69**(2008), 4732-4753.
[35] H.K. Xu, *Viscosity method for hierarchical fixed point approach to variational inequalities*, Taiwanese J. Math., **14**(2010), no. 2, 463-478.

Received: November 17, 2020; Accepted: October 22, 2021.