Fixed Point Theory, 23(2022), No. 2, 429-446
DOI: 10.24193/fpt-ro.2022.2.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

HYBRID INERTIAL ALGORITHM FOR FIXED POINT AND EQUILIBRIUM PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we proposed a hybrid inertial algorithm for approximating fixed points of noncommutative generic 2-generalized Bregman nonspreading mappings with equilibrium in reflexive Banach space. Also, we proved that the sequence generated by such algorithm converges strongly to the common fixed points of such mappings and solved some equilibrium problems in the space. The result established improved and generalized some recently announced results in the literature. A numerical example is given at end of the paper to ascertain some least level of improvement.
Key Words and Phrases: 2-generalized hybrid mapping, normally 2-generalized hybrid mapping, 2-generalized nonspreading mapping, generic 2-generalized Bregman nonspreading mapping, equilbrium problems.
2020 Mathematics Subject Classification: 47H09, 47H10, 47J25.

1. Introduction

Let H a real Hilbert space and C be a nonempty subset of H. A point $x \in C$ is called a fixed point of a map $T : C \to H$ if $Tx = x$. Denote the set of fixed points of T by $F(T)$ i.e. $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to H$ is called 2-generalized hybrid [20] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2$$
$$\leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2, \ \forall\ x, y \in C.$$

Takahashi [27] obtained weak and strong convergence theorems for noncommutative 2-generalized hybrid mappings in Hilbert spaces.

As an extension of 2-generalized hybrid mapping, a normally 2-generalized hybrid mapping was introduced in Hilbert spaces by Kondo and Takahahasi [19].

A mapping $T : C \to C$ is called normally 2-generalized hybrid [19] if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that (a) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$; (b) $\sum_{i=1}^{3} \alpha_i > 0$ and

$$\begin{aligned}(c)\ &\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + \alpha_3\|x - Ty\|^2 \\ + \quad &\beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + \beta_3\|x - y\|^2 \leq 0,\ \forall\ x, y \in C.\end{aligned}$$

In 2018, Hojo et al. [17] proved weak and strong convergence theorems for commutative normally 2-generalized hybrid mappings in Hilbert spaces. They established that the sequence $\{x_n\} \subset C$ defined by

$$\begin{cases} x_1 = x \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l, x_n \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}.$$

converges strongly to $z_0 = P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Recently, Takahashi et al. [28] proved strong convergence theorem by hybrid method for two noncommutative normally 2-generalized hybrid mappings in Hilbert spaces. They proved that the sequence $\{x_n\} \subset C$ defined by

$$\begin{cases} x_1 = x \in C \\ y_n = a_n x_n + b_n(\gamma_n S + (1-\gamma_n)T)x_n + c_n(\delta_n S^2 + (1-\delta_n)T^2)x_n \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_1 \quad \forall n \in \mathbb{N} \end{cases}. \tag{1.1}$$

converges strongly to $z_0 = P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is the metric projection of H on $F(S) \cap F(T)$.

Let $f : E \to (-\infty, +\infty]$ be a convex function. We denote by $\mathrm{dom}f$ the domain of f; that is $\mathrm{dom}f = \{x \in E : f(x) < \infty\}$. For any $x \in \mathrm{int}(\mathrm{dom}(f))$ and $y \in E$, the derivative of f at x in the direction y is defined by

$$f'(x, y) := \lim_{t\to 0} \frac{f(x + ty) - f(x)}{t}. \tag{1.2}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, the gradient of f at x is the linear functional $\nabla f(x) : E \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f'(x, y)$, for any $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at every $x \in \mathrm{int}(\mathrm{dom}(f))$. The function f is said to be Fréchet differentiable at x if the limit in (1.2) is attained uniformly in y, $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset $C \subset \mathrm{int}(\mathrm{dom}(f))$ if the limit (1.2) is attained uniformly for $x \in E$ and $\|y\| = 1$. It is well known that if a continuous convex function f is Gâteaux differentiable (resp. Fréchet differentiable) in $\mathrm{int}(\mathrm{dom}(f))$, then ∇f is norm-to-weak* continuous (resp. continuous) in $\mathrm{int}(\mathrm{dom}(f))$ (see also [7]).

Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ a strictly convex and Gâteaux differentiable function. The function $D_f : \mathrm{dom}f \times\ \mathrm{int}(\mathrm{dom}(f)) \to [0, +\infty)$, defined by

$$D_f(x,y) := f(x) - f(y) - \langle \nabla f(y), x-y \rangle, \tag{1.3}$$

is called the Bregman distance with respect to f (see [14]).

Remark 1.1. If E is smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$ for all $x \in E$ where $J : E \to E^*$ is the normalized duality mapping. Hence $D_f(x,y) = \phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,\ \forall\ x,y \in E$. Also if E is Hilbert space, then $D_f(x,y) = \|x-y\|^2,\ \forall x,y \in E$.

Observe that from (1.3), we have for any $x \in dom f$ and $y,z \in int(dom(f))$.

$$D_f(x,z) = D_f(x,y) + D_f(y,z) + \langle x-y, \nabla f(y) - \nabla f(z) \rangle. \tag{1.4}$$

which is called the three point identity.
As an extension and generalization of the normally 2-generalized hybrid mapping, Ali and Haruna [3] introduced a generic 2-generalized Bregman nonspreading mapping in a real reflexive Banach space. A mapping $T : C \to C$ is called generic 2-generalized Bregman nonspreading mapping if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2,\ \delta_1, \delta_2 \in \mathbb{R}$ such that (i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$, (ii) $\sum_{i=1}^{3} \alpha_i > 0$ and

$$\begin{aligned}
(iii)\ & \alpha_1 D_f(T^2x, Ty) + \alpha_2 D_f(Tx, Ty) + \alpha_3 D_f(x, Ty) + \beta_1 D_f(T^2x, y) \\
+ \quad & \beta_2 D_f(Tx, y) + \beta_3 D_f(x, y) \\
\leq \quad & \gamma_1 \big(D_f(Ty, T^2x) - D_f(Ty, x)\big) + \gamma_2 \big(D_f(Ty, Tx) - D_f(Ty, x)\big) \\
+ \quad & \delta_1 \big(D_f(y, T^2x) - D_f(y, x)\big) + \delta_2 \big(D_f(y, Tx) - D_f(y, x)\big),
\end{aligned} \tag{1.5}$$

for all $x, y \in C$. such mapping is called $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generic 2-generalized Bregman nonspreading mapping. See, for example, [[2],[4],[21]], the other mappings which the generic 2-generalized Bregman nonspreading mapping contained as special cases in the Banach spaces.

Remark 1.2. If $E = H$ is a real Hibert space, then $D_f(x,y) = \|x-y\|^2$ and consequently the generic 2-generalized Bregman nonspreading mapping reduces to $(\alpha_1', \alpha_2', \alpha_3', \beta_1', \beta_2', \beta_3')$ normally 2-generalized hybrid in the sense of [19] where

$$\alpha_1' = \alpha_1 - \gamma_1,\ \alpha_2' = \alpha_2 - \gamma_2,\ \alpha_3' = \alpha_3 + \gamma_1 + \gamma_2$$

and

$$\beta_1' = \beta_1 - \delta_1,\ \beta_2' = \beta_2 - \delta_2,\ \beta_3' = \beta_3 + \delta_1 + \delta_2.$$

With regards to the generic 2-generalized nonspreading mappings, the following results were proved, see [5] for details.

Lemma 1.1. *Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty subset of $int(dom(f))$ and $T : C \to C$ be a generic 2-generalized Bregman nonspreading mapping. If $x_n \rightharpoonup p$, $(x_n - Tx_n) \to 0$ and $(x_n - T^2x_n) \to 0$ as $n \to \infty$, then $p \in F(T)$.*

Lemma 1.2. *Let C be a nonempty subset of $int(dom(f))$ and $T : C \to C$ be a generic 2-generalized Bregman nonspreading mapping. If $F(T) \neq \emptyset$, then T is quasi Bregman nonexpansive.*

Motivated and inspired by the above results, we prove that the sequence defined by the proposed algorithm converges strongly to the common fixed point of generic 2-generalized Bregman nonspreading mappings which in turns, solved some equilblirium problems in a real reflexive Banach space. Our result improved and generalized the results of Takahashi et al.[28]. In fact, a numerical example shows that a sequence generated by hybrid inertial algorithm which is corollary to our main result converges faster than that of Takahashi et al.[28].

2. Preliminaries

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the the dual space of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. The Fenchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

Observe that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*),\ \forall x \in E,\ x^* \in E^*.$$

It is well known that if $f : E \to (-\infty, +\infty]$ is a proper, convex and lower semi-continuous, then $f^* : E^* \to (-\infty, +\infty]$ is proper, convex and weak* lower semi-continuous function; see for example [26].

A sublevel of f is the set of the form $\text{lev}_{\leq}^{f} r := \{x \in E : f(x) \leq r\}$ for $r \in \mathbb{R}$.

A function f on E is coercive [16] if every sublevel of f is bounded, equivalently

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Let $B_r := \{x \in E : \|x\| \leq r\}$ for all $r > 0$ and $S_E := \{x \in E : \|x\| = 1\}$. A function f on E is said to be

(i) strongly coercive [30] if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

(ii) locally bounded if $f(B_r)$ is bounded for all $r > 0$.

(iii) locally uniformly smooth ([30]) if $\forall\ r > 0$, the $\lim_{t \to 0} \frac{\sigma_r(t)}{t} = 0$, where $\sigma_r : [0, +\infty) \to [0, +\infty]$ is the function defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} (\alpha f(x + (1-\alpha)ty) + (1-\alpha) f(x - \alpha ty) - f(x)) \times (\alpha(1-\alpha))^{-1}$$

for all $t \geq 0$.

(iv) locally uniformly convex (or uniformly convex on bounded subsets of E ([30])) if $\forall\ r, t > 0$ the $\rho_r(t) > 0$, where $\rho_r : [0, +\infty) \to [0, +\infty]$ is the gauge of uniform convexity of f, defined by

$$\rho_r(t) = \inf_{x, y \in B_r, \|x-y\| = t, \alpha \in (0,1)} (\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y)) \times (\alpha(1-\alpha))^{-1}$$

for all $t \geq 0$.

The following result is proved in [30].

Lemma 2.1. [30]. Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent

(1) f is bounded on bounded sets and uniformly smooth on bounded sets;
(2) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded sets.
(3) $dom f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded sets.

Let $x \in \text{int}(\text{dom}(f))$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y),\ \forall y \in E\}.$$

Definition 2.1. (see [9]) The function f is said to be:

(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom$f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.1. Let E be a reflexive Banach space. Then we have:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [9] Theorem 5.4);
(ii) $(\partial f)^{-1} = \partial f^*$;
(iii) f is Legendre if and only if f^* is Legendre (see [9],Corrolary 5.5);
(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran$\nabla f$ = dom∇f^* = int(dom(f^*)) and ran ∇f^* = dom∇f = int(dom(f)), (see [9], Theorem 5.10).

Various examples of Legendre functions were given in [8, 9] . One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f coincides with the generalized duality mapping of E, i.e, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces.

Definition 2.2. [12, 18] Let E be a Banach space. The function $f : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(i) f is continuous, strictly convex and Gâteaux differentiable;
(ii) the set $\{y \in E : D_f(x, y) < r\}$ is bounded for all $x \in E$ and $r > 0$.

The following result can be found in [1] [see also [13], [18]]

Lemma 2.2. *Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V_f be a function $V_f : E \times E^* \to [0, +\infty)$ associated with f defined by*

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*)\ \forall x \in E,\ x^* \in E^*. \tag{2.1}$$

Then the following assertions hold:

(i) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))\ \forall x \in E, x^* \in E^*$.
(ii) $V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*)\ \forall x \in E,\ x^* \in E^*$.

Also from equation (2.1), it is obvious that $D_f(x,y) = V_f(x, \nabla f(y))$ and V_f is convex in the second variable. Therefore for $t \in (0,1)$ and $x, y \in E$, we have

$$D_f(z, \nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))) \leq tD_f(z,x) + (1-t)D_f(z,y). \tag{2.2}$$

A Bregman projection [11] of $x \in int(dom(f))$ onto the nonempty, closed and convex set $C \subset dom f$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

The following is well-known concerning Bregman projections

Lemma 2.3 ([13])**.** *Let C be nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*
(a) $z = P_C^f x$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$.
(b) $D_f(y, P_C^f x) + D_f(P_C^f x, x) \leq D_f(y,x)$ $\forall x \in E, y \in C$.

Lemma 2.4. [23] *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Let $r > 0$ be a constant,*

$$B_r := \{z \in E : \|z\| \leq r\},\ B_r^* := \{z^* \in E^* : \|z^*\| \leq r\}$$

let ρ_r and ρ_r^* be the gauges of uniform convexity of g and g^* respectively. Then,

(i) *for any $x, y \in B_r$ and $\alpha \in (0,1)$,*

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(\|x-y\|)$$

(ii) *for any $x, y \in B_r$, $\rho_r(\|x-y\|) \leq D_g(x,y)$*

(iii) *If in addition g is bounded on bounded subsets and uniformly convex on bounded subsets of E, then for any $x \in E$,$y^*, z^* \in B_r^*$ and $\alpha \in (0,1)$,*

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_g(x,y^*) + (1-\alpha)V_g(x,z^*) - \alpha(1-\alpha)\rho_r^*(\|y^* - z^*\|);$$

(iv) *If in addition g is bounded on bounded subsets, uniformly convex and uniformly smooth on bounded subsets of E, then for any $x \in E$,$y^*, z^* \in B_r^*$,*

$$\rho_r^*(\|x^* - y^*\|) \leq D_g(x^*, y^*).$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in int(dom(f))$ is the function

$$v_f(x,.) : int(dom(f)) \times [0, +\infty] \to [0, +\infty]$$

defined by

$$v_f(x,t) = \inf\{D_f(y,x) : y \in dom f, \|y - x\| = t\}.$$

The function f is totally convex at x if $v_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at every point $x \in int(dom(f))$ and is said to be totally convex on bounded sets if $v_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $V_f : int(dom(f)) \times [0, +\infty] \to [0, +\infty]$ defined by

$$V_f(B,t) = \inf\{v_f(x,t) : x \in B \cap dom f\}.$$

Lemma 2.5. [25] *If $x \in int(dom(f))$, then the following statements are equivalent:*
(i) The function f is totally convex at x;
(ii) for any sequence $\{y_n\} \subset dom f$,

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0$$

Lemma 2.6. [29] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $int(dom f^*)$. Let $x \in int(dom(f))$. If $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$ is bounded, then so is the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Lemma 2.7. [22] *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following are equivalent.*

(1) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.8. [24] *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $int(dom f)$ and $T : C \to C$ be a quasi -Bregman nonexpansive mapping. Then $F(T)$ is closed and convex.*

The equilibrium problem with respect to a bifunction $g : C \times C \to \mathbb{R}$ is to find a point $x \in C$ such that $g(x, y) \geq 0$ for all $y \in C$. Denote the set of solutions of the equilibrium problem by $EP(g)$, i.e.

$$EP(g) = \{x \in C : g(x, y) \geq 0 \ \forall y \in C\}.$$

Numerous problems can be reduced to finding solution of the equilibrium problem among which can be found in physics, optimization and economics. To solve equilibrium problems, some of the methods been proposed include that of Blum and Oettli [10] and Combettes and Hirstoaga [15].
To solve equilibrium problem, the bifunction $g : C \times C \to \mathbb{R}$ is assumed to satisfy the following conditions as can be seen in [10]:

(A1) $g(x, x) = 0 \quad \forall x \in C$.
(A2) g is monotone that is, $g(x, y) + g(y, x) \leq 0 \quad \forall x, y \in C$.
(A3) $\limsup_{t\to\infty} g(x + t(z - x), y) \leq g(x, y), \quad \forall x, y, z \in C$.
(A4) The function $y \to g(x, y)$ is convex and lower semi continuous.

The resolvent of the bifunction g [15] is the operator $T_r : E \to 2^C$ defined by

$$T_r x = \{x \in C : g(x, y) + \frac{1}{r}\langle \nabla f x - \nabla f z, y - x\rangle \geq 0, \ \forall y \in C\}.$$

Lemma 2.9. [24] *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions $(A1) - (A4)$, then the following hold:*

(i) *T_r is single-valued;*
(ii) *T_r is a Bregman firmly nonexpansive operator;*
(iii) *$F(T_r) = EP(g)$;*
(iv) *$EP(g)$ is closed and convex;*
(v) *For all $x \in E$ and $p \in F(T_r)$ one has $D_f(p, T_r x) + D_f(T_r x, x) \leq D_f(p, x)$.*

3. Main results

In this section, E is consider to be a real reflexive Banach space. We proposed a hybrid inertial algorithm for noncommutative generic 2-generalized Bregman non-spreading mappings with equilibrium in Banach spaces. We then prove that the sequence generated by such algorithm converges strongly to the common element of the set of fixed points of such mappings and the set of solutions of the equilblirium problem in the space.

Theorem 3.1. *Let $f : E \to \mathbb{R}$ be strongly coercive, Legendre, uniformly Fréchet differentiable and totally convex function which is bounded on bounded subsets of E. Let C be a nonempty, closed and convex subset of $int(domf)$ and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $S, T : C \to C$ generic 2-generalized Bregman nonspreading mappings such that $\mathcal{F} = F(S) \cap F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0, x_1 \in C \\ u_n = x_n + l_n(x_n - x_{n-1}) \\ y_n = \nabla f^*(\alpha_n \nabla f u_n + \beta_n \nabla f v_n + \gamma_n \nabla f w_n) \\ z_n = T_{r_n} y_n \\ C_n = \{p \in C : D_f(p, z_n) \leq D_f(p, u_n)\}, \\ Q_n = \{p \in C : \langle \nabla f x_1 - \nabla f x_n, x_n - p\rangle \geq 0\}, \\ x_{n+1} = P^f_{C_n \cap Q_n}(x_1) \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where

$$v_n = \nabla f^*(\delta_n \nabla f S u_n + (1 - \delta_n)\nabla f T u_n),$$
$$w_n = \nabla f^*(\lambda_n \nabla f S^2 u_n + (1 - \lambda_n)\nabla f T^2 u_n)$$

with the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\} \subset [a, b] \subset (0, 1)$ and

$$\alpha_n + \beta_n + \gamma_n = 1.$$

Then $\{x_n\}$ converges strongly to $z = P^f_{\mathcal{F}}(u)$, where $P^f_{\mathcal{F}}(u)$ is the Bregman projection of E onto $\mathcal{F}$.

Proof. We first guarantee that the sequence $\{x_n\}$ is well defined. From the definition of C_n, we see that $D_f(p, z_n) \leq D_f(p, u_n)$ if and only if

$$D_f(u_n, z_n) + \langle \nabla u_n - \nabla z_n, p - u_n\rangle \leq 0.$$

Thus, it is an evident that both C_n, Q_n and $C_n \cap Q_n$ are closed and convex. Also, since the mappings S and T are generic 2-generalized Bregman nonspreading with nonempty fixed point sets then by Lemma 1.2, they are quasi nonexpansive. Hence by Lemma 2.8, both S and T are closed and convex.
We let $z \in \mathcal{F} = F(S) \cap F(T) \cap EP(g) \neq \emptyset$ so that

$$\begin{aligned} D_f(z, v_n) &= D_f(z, \nabla f^*(\delta_n \nabla f S u_n + (1 - \delta_n)\nabla f T u_n)) \\ &\leq \delta_n D_f(z, S u_n) + (1 - \delta_n) D_f(z, T u_n) \\ &\leq \delta_n D_f(z, u_n) + (1 - \delta_n) D_f(z, u_n) \\ &= D_f(z, u_n). \end{aligned}$$

Similarly,

$$\begin{aligned} D_f(z, w_n) &= D_f(z, \nabla f^*(\lambda_n \nabla f S^2 u_n + (1-\lambda_n)\nabla f T^2 u_n)) \\ &\leq \lambda_n D_f(z, S^2 u_n) + (1-\lambda_n) D_f(z, T^2 u_n) \\ &\leq \lambda_n D_f(z, u_n) + (1-\lambda_n) D_f(z, u_n) \\ &= D_f(z, u_n). \end{aligned}$$

Also,

$$\begin{aligned} D_f(z, y_n) &= D_f(z, \alpha_n \nabla f u_n + \beta_n \nabla f v_n + \gamma_n \nabla f w_n)) \\ &\leq \alpha_n D_f(z, u_n) + \beta_n D_f(z, v_n) + \gamma D_f(z, w_n) \\ &\leq \alpha_n D_f(z, u_n) + \beta D_f(z, u_n) + \delta D_f(z, u_n) \\ &= D_f(z, u_n). \end{aligned}$$

Using (b) of Lemma 2.3, we obtain

$$\begin{aligned} D_f(z, z_n) &= D_f(z, T_{r_n} y_n)) \\ &\leq D_f(z, y_n) - D_f(T_{r_n} y_n, y_n) \\ &\leq D_f(z, u_n). \end{aligned}$$

This implies, $z \in C_n$. Thus $\mathcal{F} \subset C_n$. Now, we show that $\mathcal{F} \subset C_n \cap Q_n$ $\forall n \in \mathbb{N}$ and we do this by induction. For $n = 1$, we see that $\mathcal{F} \subset C_1 \cap Q_1$ since $\mathcal{F} \subset C_1$ and $Q_1 = C$. Suppose for some $k \geq 1$, $\mathcal{F} \subset C_k \cap Q_k$ then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P^f_{C_k \cap Q_k} x_1$. Thus, from the property of Bregman projection we have

$$\langle \nabla f x_{k+1} - \nabla f x_1, p - x_{k+1} \rangle \geq 0,$$

for all $p \in C_k \cap Q_k$ and since $\mathcal{F} \subset C_k \cap Q_k$, we have

$$\langle \nabla f x_{k+1} - \nabla f x_1, z - x_{k+1} \rangle \geq 0,$$

for all $z \in \mathcal{F}$. This implies $\mathcal{F} \subset C_{k+1} \cap Q_{k+1}$ and therefore $\mathcal{F} \subset C_n \cap Q_n$ $\forall n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is well defined.

Next is to show that the sequence $\{x_n\}$ is bounded. We know from the definition of Q_n that $x_n = P^f_{Q_n} x_1$. Using (b) of Lemma 2.3, we get

$$\begin{aligned} D_f(x_n, x_1) &= D_f(P^f_{Q_n} x_1, x_1) \\ &\leq D_f(z, x_1) - D_f(z, x_n) \\ &\leq D_f(z, x_1) \ \forall z \in \mathbb{F} \subset Q_n. \end{aligned}$$

This implies that the sequence $\{D_f(x_n, x_1)\}$ is bounded. Thus, by Lemma 2.6, the sequence $\{x_n\}$ is bounded too.

Also, since $x_n = P^f_{Q_n} x_1$ and $x_{n+1} \subset Q_n$ then using (1.4), we have that

$$\begin{aligned} 0 &\leq \langle \nabla f x_n - \nabla f x_1, x_{n+1} - x_n \rangle \\ &= D_f(x_{n+1}, x_1) - D_f(x_{n+1}, x_n) - D_f(x_n, x_1) \\ &\leq D_f(x_{n+1}, x_1) - D_f(x_n, x_1). \end{aligned} \tag{3.2}$$

This implies that $D_f(x_n, x_1) \le D_f(x_{n+1}, x_1)$. Thus, the sequence $\{D_f(x_n, x_1)\}$ is monotone increasing and since it is bounded then $\lim_{n\to\infty} D_f(x_n, x_1)$ exists.
From (3.2), we see that

$$D_f(x_{n+1}, x_n) \le D_f(x_{n+1}, x_1) - D_f(x_n, x_1).$$

Using the fact that $\lim_{n\to\infty} D_f(x_n, x_1)$ exists, we get that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0.$$

Thus, by Lemma 2.7

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Using (3.1) and (3.3), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.4}$$

This implies $\{u_n\}$ is bounded. Also, using (3.3) and (3.4) we get

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.5}$$

Thus, by Lemma 2.7, $\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0$.
Since $x_{n+1} \subset C_n$, it implies $D_f(x_{n+1}, z_n) \le D_f(x_{n+1}, u_n)$.
Thus, $\lim_{n\to\infty} D_f(x_{n+1}, z_n) = 0$. Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.6}$$

Using (3.5) and (3.6) we get

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.7}$$

Hence, we obtain that the

$$\lim_{n\to\infty} \|\nabla f u_n - \nabla f z_n\| = 0 \text{ and } \lim_{n\to\infty} D_f(z_n, u_n) = 0.$$

From the definition of z_n and (b) of Lemma 2.3, we obtain

$$\begin{aligned} D_f(z_n, y_n) &= D_f(T_{r_n} y_n, y_n) \\ &\le D_f(z, y_n) - D_f(z, z_n) \\ &\le D_f(z, u_n) - D_f(z, z_n) \\ &\le D_f(z_n, u_n) + \|\nabla f u_n - \nabla f z_n\| \|z - u_n\| \to 0 \ as \ n \to \infty. \end{aligned}$$

Thus, by Lemma 2.5

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.9}$$

On the other hand,

$$\begin{aligned} D_f(z,y_n) &= D_f(z,\nabla f^*[\alpha_n\nabla fu_n+\beta_n\nabla fv_n+\gamma_n\nabla fw_n]) \\ &= V_f(z,\alpha_n\nabla fu_n+\beta_n\nabla fv_n+\gamma_n\nabla fw_n) \\ &\le \alpha_n V_f(z,\nabla u_n)+\beta_n V_f(z,\nabla v_n)+\gamma_n V_f(z,\nabla w_n) \\ &- \alpha_n\beta_n\rho_r^*(\|\nabla fu_n-\nabla fv_n)\| \qquad (3.10) \\ &= \alpha_n D_f(z,u_n)+\beta_n D_f(z,v_n)+\gamma_n D_f(z,w_n) \\ &- \alpha_n\beta_n\rho_r^*(\|\nabla fu_n-\nabla fv_n)\| \qquad (3.11) \\ &\le \alpha_n D_f(z,u_n)+\beta_n D_f(z,u_n)+\gamma_n D_f(z,u_n) \\ &- \alpha_n\beta_n p_s^*\big(\|\nabla f(u_n)-\nabla f(v_n)\|\big) \\ &= D_f(z,u_n)-\alpha_n\beta_n p_r^*\big(\|\nabla f(u_n)-\nabla f(v_n)\|\big). \end{aligned}$$

Thus,

$$D_f(z,y_n)\le D_f(z,u_n)-\alpha_n\beta_n p_r^*\big(\|\nabla f(u_n)-\nabla f(v_n)\|\big). \qquad (3.12)$$

Similarly,

$$D_f(z,y_n)\le D_f(z,u_n)-\alpha_n\gamma_n p_r^*\big(\|\nabla f(u_n)-\nabla f(w_n)\|\big). \qquad (3.13)$$

These imply that

$$\alpha_n\beta_n p_r^*\big(\|\nabla f(u_n)-\nabla f(v_n)\|\big)\le D_f(z,u_n)-D_f(z,y_n) \qquad (3.14)$$

and

$$\alpha_n\gamma_n p_r^*\big(\|\nabla f(u_n)-\nabla f(w_n)\|\big)\le D_f(z,u_n)-D_f(z,y_n). \qquad (3.15)$$

Using the property of p_r^* and the fact that $\alpha_n,\beta_n,\gamma_n\in[a,b]\subset(0,1)$, taking limit as $n\to\infty$ of (3.14) and (3.15), we have

$$\lim_{n\to\infty}\|\nabla f(u_n)-\nabla f(v_n)\|=0,\quad \lim_{n\to\infty}\|\nabla f(u_n)-\nabla f(w_n)\|=0. \qquad (3.16)$$

Also,

$$\begin{aligned} D_f(z,v_n) &= D_f(z,\nabla f^*(\delta_n\nabla fSu_n+(1-\delta_n)\nabla Tu_n)) \\ &\le \delta_n D_f(z,Su_n)+(1-\delta_n)D_f(z,Tu_n) \\ &- \delta_n(1-\delta_n)\rho_r^*(\|\nabla fSu_n-\nabla Tu_n\|) \qquad (3.17) \\ &\le D_f(z,u_n)-\delta_n(1-\delta_n)\rho_r^*(\|\nabla fSu_n-\nabla Tu_n\|). \end{aligned}$$

This implies,

$$\delta_n(1-\delta_n)\rho_r^*(\|\nabla fSu_n-\nabla Tu_n\|)\le D_f(z,u_n)-D_f(z,v_n). \qquad (3.18)$$

Similarly,

$$\lambda_n(1-\lambda_n)\rho_r^*(\|\nabla fS^2u_n-\nabla T^2u_n\|)\le D_f(z,u_n)-D_f(z,w_n). \qquad (3.19)$$

Using the property of p_r^* and the fact that $\delta_n,\lambda_n\in[a,b]\subset(0,1)$, taking limit as $n\to\infty$ of (3.18) and (3.19), we get

$$\lim_{n\to\infty}\|\nabla f(Su_n)-\nabla f(Tu_n)\|=0,\quad \lim_{n\to\infty}\|\nabla f(S^2u_n)-\nabla f(T^2u_n)\|=0. \qquad (3.20)$$

This implies,

$$\lim_{n\to\infty}\|Su_n-Tu_n\|=0,\quad \lim_{n\to\infty}\|S^2u_n-T^2u_n\|=0. \qquad (3.21)$$

Using (3.16) and (3.20), we get

$$\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(Tu_n)\| = 0, \quad \lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(T^2 u_n)\| = 0. \tag{3.22}$$

Thus,

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0, \quad \lim_{n\to\infty} \|u_n - T^2 u_n\| = 0. \tag{3.23}$$

From (3.21) and (3.23), we can deduce that

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0, \quad \lim_{n\to\infty} \|u_n - S^2 u_n\| = 0. \tag{3.24}$$

By the boundedness of $\{u_n\}$ and reflexivity of E, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup u$. Thus, it follows from (3.23), (3.24) and Lemma 1.1 that $u \in F(S) \cap F(T)$. Also, using (3.7), we see that $z_{n_k} \rightharpoonup u$. Since $z_n = T_{r_n} y_n$ then from the definition of T_r, we get

$$g(z_n, y) + \frac{1}{r_n} \langle \nabla f z_n - \nabla f y_n, y - z_n \rangle \geq 0, \quad \forall y \in C.$$

Hence

$$g(z_{n_k}, y) + \frac{1}{r_{n_k}} \langle \nabla f z_{n_k} - \nabla f y_{n_k}, y - z_{n_k} \rangle \geq 0, \quad \forall y \in C.$$

Using $(A2)$, we get

$$\begin{aligned} g(y, z_{n_k}) &\leq -g(z_{n_k}, y) \\ &\leq \frac{1}{r_{n_k}} \langle \nabla f z_{n_k} - \nabla f y_{n_k}, y - z_{n_k} \rangle \\ &\leq \frac{1}{r_{n_k}} \|\nabla f z_{n_k} - \nabla f y_{n_k}\| \|y - z_{n_k}\| \quad \forall y \in C. \end{aligned}$$

Taking limit as $k \to \infty$ of the above inequality and with use of $(A4)$ and the fact that $z_{n_k} \rightharpoonup u$, we get $g(y, u) \leq 0$. Define $y_t = ty + (1-t)u$ for $0 < t < 1$ and $y \in C$. Since $u, y \in C$ then $y_t \in C$ which yield that $g(y_t, u) \leq 0$.
Using $(A1)$ we see that

$$\begin{aligned} 0 = g(y_t, y_t) &\leq tg(y_t, y) + (1-t)g(y_t, u) \\ &\leq tg(y_t, y). \end{aligned}$$

Thus, $g(y_t, y) \geq 0$. Now letting $t \to 0$ and using $(A3)$ we see that $g(u, y) \geq 0$ for any $y \in C$. This implies $u \in EP(g)$. Hence $u \in \mathcal{F}$.
Also, from (3.4), we have $x_{n_k} \rightharpoonup u$. Now put $v = P^f_{\mathcal{F}} x_1$. Since $x_{n+1} = P^f_{C_n \cap Q_n} x_1$ and $v \in C_n \cap Q_n$, we get $D_f(x_{n+1}, x_1) \leq D_f(v, x_1)$.
Since $D_f(., x)$ is lower semi continuous and convex and thus weakly lower semi continuous on int(domf) then from the fact that $x_{n_k} \rightharpoonup u$, wee see that

$$D_f(u, x_1) \leq \liminf_{k\to\infty} D_f(x_{n_k}, x_1) \leq D_f(v, x_1).$$

From the definition of v, we can conclude that $u = v$ and the sequence $x_n \rightharpoonup v$. We finally show that $x_n \to v$. Now using the three point identity

$$\begin{aligned}
\limsup_{n\to\infty} D_f(x_n, v) &= \limsup_{n\to\infty}[D_f(x_n, x_1) + D_f(x_1, v) + \langle \nabla f x_1 - \nabla f v, x_n - x_1\rangle] \\
&\leq \limsup_{n\to\infty}[D_f(v, x_1) + D_f(x_1, v) + \langle \nabla f x_1 - \nabla f v, x_n - x_1\rangle] \\
&= \limsup_{n\to\infty}[\langle \nabla f v - \nabla f x_1, v - x_1\rangle - \langle \nabla f v - \nabla f x_1, x_n - x_1\rangle] \\
&= \limsup_{n\to\infty}\langle \nabla f v - \nabla f x_1, v - x_n\rangle = 0.
\end{aligned}$$

Thus, we obtain $\lim_{n\to\infty} D_f(x_n, v) = 0$. Hence by Lemma 2.6 we get $x_n \to v$ as $n \to \infty$. This completes the proof. □

As a consequence, in view of Remark 1.2, the following results are obtained by applying Theorem 3.1.

Corollary 3.1.1. *Let C be a nonempty, closed convex subset of a real Hilbert space and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Let $S, T : C \to C$ be normally 2-generalized hybrid mapping with $f(x) = \|x\|^2$ such that*

$$\mathcal{F} = F(S) \cap F(T) \cap EP(g) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence defined by

$$\begin{cases}
x_0, x_1 \in C \\
u_n = x_n + l_n(x_n - x_{n-1}) \\
y_n = \alpha_n u_n + \beta_n v_n + \gamma_n w_n \\
z_n = T_{r_n} y_n \\
C_n = \{p \in C : \|z_n - p\|^2 \leq \|u_n - p\|^2\}, \\
Q_n = \{p \in C : \langle x_1 - x_n, x_n - p\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}(x_1) \quad n \in \mathbb{N},
\end{cases}.$$

where

$$v_n = \delta_n S u_n + (1 - \delta_n) T u_n,$$
$$w_n = \lambda_n S^2 u_n + (1 - \lambda_n) T^2 u_n$$

with the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\} \subset [a, b] \subset (0, 1)$ and

$$\alpha_n + \beta_n + \gamma_n = 1.$$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(u)$, where $P_{\mathcal{F}}(u)$ is the metric projection of E onto $\mathcal{F}$.

Proof. By remark 1.2, the generic 2-generalized Bregman nonspreading mapping reduces to normally 2-generalized hybrid mapping in Hilbert space i.e. there exists $\alpha_1', \alpha_2', \alpha_3', \beta_1', \beta_2', \beta_3' \in \mathbb{R}$ such that

$$\begin{aligned}
\alpha_1'\|T^2x - Ty\|^2 &+ \alpha_2'\|Tx - Ty\|^2 + \alpha_3'\|x - Ty\|^2 \\
&+ \beta_1'\|T^2x - y\|^2 + \beta_2'\|Tx - y\|^2 + \beta_3'\|x - y\|^2 \leq 0 \ \forall\ x, y \in C,
\end{aligned}$$

where

$$\alpha_1' = \alpha_1 - \gamma_1,\ \alpha_2' = \alpha_2 - \gamma_2,\ \alpha_3' = \alpha_3 + \gamma_1 + \gamma_2$$

and

$$\beta_1' = \beta_1 - \delta_1,\ \beta_2' = \beta_2 - \delta_2,\ \beta_3' = \beta_3 + \delta_1 + \delta_2$$

satisfying

$$\sum_{i=1}^{3}(\alpha_i' + \beta_i') = \sum_{i=1}^{3}(\alpha_i + \beta_i) \geq 0$$

and

$$\sum_{i=1}^{3}\alpha_i' = \sum_{i=1}^{3}\alpha_i > 0.$$

Thus, by Theorem 3.1 we see that the sequence $\{x_n\}$ converges strongly to $z = \mathrm{P}_{\mathcal{F}}(u)$. This completes the proof. □

Corollary 3.1.2. *Let C be a nonempty closed convex subset of a real Hilbert space and Let $S, T : C \to C$ be normally 2-generalized hybrid mapping with $f(x) = \|x\|^2$ such that $\mathcal{F} = F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0, x_1 \in C \\ u_n = x_n + l_n(x_n - x_{n-1}) \\ y_n = \alpha_n u_n + \beta_n v_n + \gamma_n w_n \\ C_n = \{p \in C : \|y_n - p\|^2 \leq \|u_n - p\|^2\}, \\ Q_n = \{p \in C : \langle x_1 - x_n, x_n - p\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1) \quad n \in \mathbb{N}, \end{cases} . \tag{3.25}$$

where

$$v_n = \delta_n S u_n + (1 - \delta_n) T u_n,$$
$$w_n = \lambda_n S^2 u_n + (1 - \lambda_n) T^2 u_n$$

with the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\lambda_n\} \subset [a, b] \subset (0, 1)$ and

$$\alpha_n + \beta_n + \gamma_n = 1.$$

Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(u)$, where $P_{\mathcal{F}}(u)$ is the metric projection of E onto $\mathcal{F}$. This completes the proof.

A numerical example

Here, we present a numerical example to show that the convergence of a sequence generated by our inertial algorithm (3.25) in corollary 3.1.2 is faster than that of (1.1) which does not involve the intertial condition. Let $E = \mathbb{R}$ and $C = [0, 2]$. Let $S, T : C \to C$ be defined by

$$Sx = Tx = \begin{cases} 0, & x \in [0, 2) \\ 1, & x = 2. \end{cases}$$

Observe that for the choice of real numbers

$$\alpha_{=}\alpha_2 = \alpha_3 = \delta_1 = 1$$

and

$$\beta_1 = \beta_2 = \beta_3 = \gamma_1 = \gamma_2 = \delta_2 = -1,$$

we see that

(1) $\sum_{i=1}^{3} \alpha_i > 0$;

(2) $\sum_{i=1}^{3} (\alpha_i + \beta_i) \geq 0$; and

(3) $h(x, y) \leq 0$, for all $x, y \in C$, where

$$\begin{aligned} h(x,y) &= \alpha_1(T^2x - Ty)^2 + \alpha_2(Tx - Ty)^2 + \alpha_3(x - Ty)^2 \\ &+ \beta_1(T^2x - y)^2 + \beta_2(Tx - y)^2 + \beta_3(x - y)^2 \\ &- \gamma_1(Ty - T^2x)^2 + \gamma_1(Ty - x)^2 - \gamma_2(Ty - Tx)^2 + \gamma_2(Ty - x)^2 \\ &- \delta_1(y - T^2x)^2 + \delta_1(y - x)^2 - \delta_2(y - Tx)^2 + \delta_2(y - x)^2. \end{aligned}$$

Therefore, S and T are generic 2-generalized Bregman nonspreading mappings, see [3] for details.

It is clear that S and T are such that $0 \in F(S) = F(T)$. Thus, $0 \in F(S) \cap F(T)$. Taking

$$\{a_n\} = \{b_n\} = \{c_n\} = \{\alpha_n\} = \{\beta_n\} = \{\lambda_n\} = \{\gamma_n\} = \{\delta_n\} = \left\{\frac{n}{2n+1}\right\} \subset (0,1)$$

and $\{l_n\} = \left\{\frac{n}{2n+30}\right\} \subset (0,1)$, it follows from (1.1) and (3.25) that a sequence $\{x_n\}$ generated by the following algorithms

$$\begin{cases} x_1 \in [0,2]; \\ y_n = \begin{cases} \frac{n}{2n+1} x_n, \ x_n \in [0,2) \\ \frac{n}{2n+1}(x_n + 1), \ x_n = 2; \end{cases} \\ C_n = \left\{z \in [0,2] : z \leq \frac{x_n + y_n}{2}\right\}; \\ Q_n = \{z \in [0,2] : z \geq x_n\}; \\ C_n \cap Q_n = \left\{z \in [0,2] : x_n \leq z \leq \frac{x_n + y_n}{2}\right\}; \\ x_{n+1} = P_{C_n \cap Q_n}(x_1) = \frac{x_n + y_n}{2}, \ n \geq 1; \end{cases} \tag{3.26}$$

and respectively,

$$\begin{cases} x_0, x_1 \in [0,2]; \\ u_n = x_n + \frac{n}{2n+30}(x_n - x_{n-1}); \\ y_n = \begin{cases} \frac{n}{2n+1} u_n, \ u_n \in [0,2) \\ \frac{n}{2n+1}(u_n + 1), \ u_n = 2; \end{cases} \\ C_n = \left\{p \in [0,2] : p \leq \frac{u_n + y_n}{2}\right\}; \\ Q_n = \{p \in [0,2] : p \geq x_n\}; \\ C_n \cap Q_n = \left\{p \in [0,2] : x_n \leq p \leq \frac{u_n + y_n}{2}\right\}; \\ x_{n+1} = P_{C_n \cap Q_n}(x_1) = \frac{u_n + y_n}{2}, \ n \geq 1; \end{cases} \tag{3.27}$$

converge strongly to $0 \in F(S) \cap F(T)$. Clearly from Figure 1 below our inertial algorithm which is a corollary to our main result converges faster than that of Takahashi et al. [28] which does not involve the inertial condition.

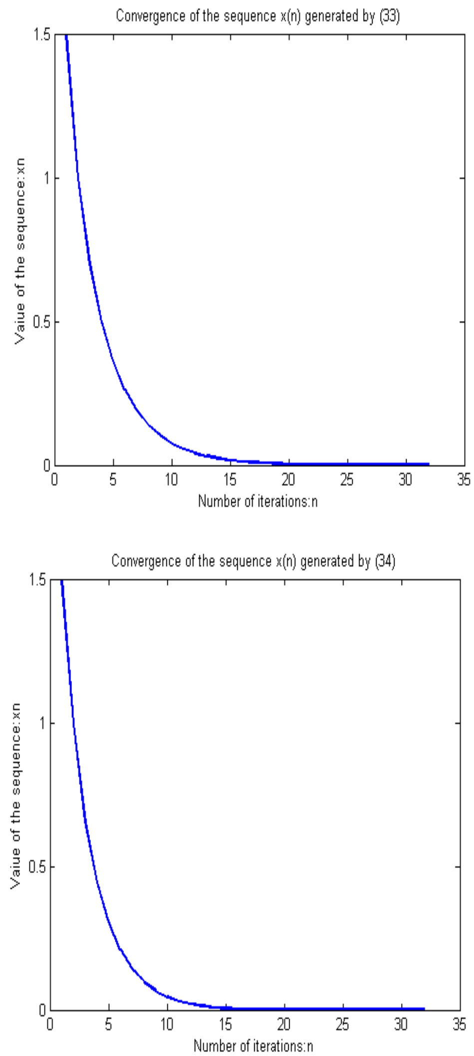

FIGURE 1. The graph of sequence $\{x_n\}$ generated by (3.26) (respectively, (3.27)) versus number of iterations $n := 1, 2, \cdots, 30$, with initial choices of $x_1 = 1.5000.$ and $x_2 = 1.0000$.

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Received: July 29, 2020; Accepted: January 29, 2022.