Fixed Point Theory, 23(2022), No. 1, 385-390
DOI: 10.24193/fpt-ro.2022.1.24
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CANTOR'S INTERSECTION THEOREM IN THE SETTING OF $\mathcal{F}$-METRIC SPACES

SUMIT SOM*, LAKSHMI KANTA DEY** AND WUTIPHOL SINTUNAVARAT***

*Department of Mathematics,
National Institute of Technology Durgapur, India
E-mail: somkakdwip@gmail.com

**Department of Mathematics,
National Institute of Technology Durgapur, India
E-mail: lakshmikdey@yahoo.co.in

***Department of Mathematics and Statistics,
Faculty of Science and Technology,
Thammasat University Rangsit Center,
Pathum Thani 12120, Thailand
E-mail: wutiphol@mathstat.sci.tu.ac.th
(Corresponding author)

Abstract. This paper deals with an open problem posed by Jleli and Samet in [1, M. Jleli and B. Samet, On a new generalization of metric spaces, J. Fixed Point Theory Appl, 20(3) 2018]. In [1, Remark 5.1], they asked whether the Cantor's intersection theorem can be extended to $\mathcal{F}$-metric spaces or not. In this manuscript, we give an affirmative answer to this open question. Additionally, keeping in mind the fact that totally boundedness is not a topological property, in the setting of $\mathcal{F}$-metric spaces are equivalent to that of usual metric spaces.
Key Words and Phrases: $\mathcal{F}$-metric space, metrizability, Cantor's intersection theorem.
2020 Mathematics Subject Classification: 54A20, 54E35, 54E50, 47H10.

1. Introduction and preliminaries

Recently, Jleli and Samet [1] proposed a new generalization of the classical metric space concept. By means of a certain class of functions, the authors defined the notion of an $\mathcal{F}$-metric space. Firstly, we will recall the definition of such kind of spaces. Consider $\mathcal{F}$ be any class of functions $f : (0, \infty) \to \mathbb{R}$ which satisfy the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t \Rightarrow f(s) \leq f(t)$;
($\mathcal{F}_2$) for every sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0, +\infty)$, we have

$$\lim_{n\to\infty} t_n = 0 \Longleftrightarrow \lim_{n\to+\infty} f(t_n) = -\infty.$$

Now, we like to give the definition of an $\mathcal{F}$-metric space and some needed definition related to this space.

Definition 1.1 ([1])**.** Let X be a non-empty set and $D : X \times X \to [0, \infty)$ be a given mapping. Suppose that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ satisfying the following conditions for all $(x, y) \in X \times X$:

(D1) $D(x, y) = 0 \Longleftrightarrow x = y$;
(D2) $D(x, y) = D(y, x)$;
(D3) for each $N \in \mathbb{N}$ with $N \geq 2$ and for every $\{u_i\}_{i=1}^N \subseteq X$ with $(u_1, u_N) = (x, y)$, we have

$$D(x, y) > 0 \Longrightarrow f(D(x, y)) \leq f\left(\sum_{i=1}^{N-1} D(u_i, u_{i+1})\right) + \alpha.$$

Then D is said to be an $\mathcal{F}$-metric on X and the pair (X, D) is said to be an $\mathcal{F}$-metric space.

Definition 1.2 ([1])**.** Let (X, D) be an $\mathcal{F}$-metric space. A subset $\mathcal{O}$ of X is said to be $\mathcal{F}$-open if for every $x \in \mathcal{O}$, there is some $r > 0$ such that $B_D(x, r) \subseteq \mathcal{O}$, where

$$B_D(x, r) = \{y \in X : D(x, y) < r\}.$$

We say that a subset $\mathcal{C}$ of X is $\mathcal{F}$-closed if $X \setminus C$ is $\mathcal{F}$-open.

We denote by $\tau_{\mathcal{F}}$ the family of all $\mathcal{F}$-open subsets of an $\mathcal{F}$-metric space (X, D). It is easy to see that $\tau_{\mathcal{F}}$ is a topology on X.

Definition 1.3 ([1])**.** Let (X, D) be an $\mathcal{F}$-metric space and $\{x_n\}$ be a sequence in X.

(1) We say that $\{x_n\}$ is $\mathcal{F}$-convergent to $x \in X$ if for every $\mathcal{F}$-open subset $\mathcal{O}_x$ of X containing x, there exists some $N \in \mathbb{N}$ such that $x_n \in \mathcal{O}_x$ for all $n \geq N$.
(2) We say that $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence if $\lim\limits_{n,m \to \infty} D(x_n, x_m) = 0$.
(3) We say that X is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to some point in X.

Remark 1.4. In an $\mathcal{F}$-metric space (X, D), a sequence $\{x_n\} \subseteq X$ is $\mathcal{F}$-convergent to $x \in X$ if and only if $\lim\limits_{n \to \infty} D(x_n, x) = 0$. Moreover, the limit of an $\mathcal{F}$-convergent sequence is unique.

Definition 1.5 ([1])**.** A subset A of an $\mathcal{F}$-metric space (X, D) is said to be $\mathcal{F}$-compact if A is compact with respect to the topology $\tau_{\mathcal{F}}$ on X.

Definition 1.6 ([1])**.** A subset A of an $\mathcal{F}$-metric space (X, D) is said to be $\mathcal{F}$-totally bounded if for every $r > 0$, there exists a finite sequence $\{x_i\}_{i=1}^n \subseteq A$ such that

$$A \subseteq \bigcup_{i=1,2,\ldots,n} B_D(x_i, r).$$

By means of the article [3], the authors proved that an $\mathcal{F}$-metric space (X, D) is metrizable under a suitable metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x,y) = \inf\left\{ \sum_{i=1}^{N-1} D(u_i, u_{i+1}) :\ N \in \mathbb{N} \text{ with } N \geq 2, \right.$$
$$\left. \{u_i\}_{i=1}^{N} \subseteq X \text{ with } (u_1, u_N) = (x, y) \right\}. \tag{1.1}$$

They also showed that the notions of a Cauchy sequence, the completeness, the Banach contraction principle are equivalent with that of metric spaces.

In this paper, based on the metric defined by (1.1), we give an affirmative answer to the open question posed by Jleli and Samet in [1].

2. Main results

In this section, we will prove the Cantor's intersection theorem for $\mathcal{F}$-metric spaces. Before proving this theorem, we will give a lemma which will be needed for proving the theorem. From now on, D will denote the $\mathcal{F}$-metric on a non-empty set X, d will denote the metric defined by (1.1). Also, $\tau_{\mathcal{F}}$ and τ_d denotes the topologies generated by the metrics D and d, respectively. Before stating the lemma, we first want to introduce the notion of $\mathcal{F}$-boundedness in the setting of $\mathcal{F}$-metric spaces.

Definition 2.1. Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then $A \subseteq X$ is said to be $\mathcal{F}$-bounded if there exists $M > 0$ such that $D(x, y) \leq M$ for all $x, y \in A$.

Lemma 2.2. *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and let $A \subseteq X$ be $\mathcal{F}$-bounded. Then A is bounded with respect to the metric d and $diam_d(A) \leq diam_D(A)$, where*

$$diam_d(A) = \sup\{d(x, y) : x, y \in A\}$$

and

$$diam_D(A) = \sup\{D(x, y) : x, y \in A\}.$$

Proof. Let $A \subseteq X$ be $\mathcal{F}$-bounded. Then there exists $M > 0$ such that $D(x, y) \leq M$ for all $x, y \in A$. By the definition of the metric d in (1.1), we have,

$$d(x, y) \leq D(x, y) \text{ for all } x, y \in X \implies d(x, y) \leq M \text{ for all } x, y \in A.$$

This shows that A is bounded with respect to the metric d. Proof of the second part follows similarly, so omitted. □

Theorem 2.3 (**Cantor's Intersection Theorem**). *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then X is $\mathcal{F}$-complete if and only if for every decreasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of non-empty, $\mathcal{F}$-closed subsets of X with $diam_D(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{i=1}^{\infty} F_i$ contains exactly one point.*

Proof. First of all, we will suppose that X is $\mathcal{F}$-complete. Then X is complete with respect to the metric d [3, Theorem 2.2 (iii)]. We now suppose that $\{F_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty, $\mathcal{F}$-closed subsets of X with $diam_D(F_n) \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, we have

$$X \setminus F_n \in \tau_{\mathcal{F}} \quad \Longrightarrow \quad X \setminus F_n \in \tau_d.$$

So F_n is closed with respect to the metric d for all $n \in \mathbb{N}$. Also by Lemma 2.2, we obtain $diam_d(F_n) \leq diam_D(F_n) \to 0$ as $n \to \infty$. By using Cantor's intersection theorem for standard metric spaces, we can say that, the intersection $\bigcap_{i=1}^{\infty} F_i$ contains exactly one point.

For the reverse part, we will suppose that $\{F_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty, $\mathcal{F}$-closed subsets of X with $diam_D(F_n) \to 0$ as $n \to \infty$ and the intersection $\bigcap_{i=1}^{\infty} F_i$ contains exactly one point. So by similar arguments, we can say that $\{F_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty, closed subsets of X with respect to the metric d with $diam_d(F_n) \to 0$ as $n \to \infty$ and the intersection $\bigcap_{i=1}^{\infty} F_i$ contains exactly one point. Again by using Cantor's intersection theorem for standard metric spaces, we can say that X is complete with respect to the metric d. So X is $\mathcal{F}$-complete by [3, Theorem 2.2 (iii)]. □

Now, we will prove that the notion of compactness in the setting of $\mathcal{F}$-metric spaces is equivalent to that of standard metric spaces.

Theorem 2.4. *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then $A \subseteq X$ is $\mathcal{F}$-compact if and only if A is compact with respect to the metric d.*

Proof. ($\Longrightarrow$) Suppose that $A \subseteq X$ is $\mathcal{F}$-compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of A with respect to the metric d. So, $U_\alpha \in \tau_d \ \forall \ \alpha \in \Lambda \Rightarrow U_\alpha \in \tau_{\mathcal{F}} \ \forall \ \alpha \in \Lambda$. Since A is $\mathcal{F}$-compact so there exists a finite set $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} U_\alpha$. But as $\tau_d = \tau_{\mathcal{F}}$, so $U_\alpha \in \tau_d \ \forall \ \alpha \in \Lambda_0$. This shows that A is compact with respect to the metric d.
For the converse part, the arguments are similar, so omitted. □

Theorem 2.5. *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then X is $\mathcal{F}$-compact if and only if for every collection $\{F_\alpha\}_{\alpha\in\Lambda}$ of $\mathcal{F}$-closed subsets of X, having the finite intersection property, the intersection $\bigcap_{\alpha\in\Lambda} F_\alpha$ of all elements of Λ is non-empty.*

Proof. Suppose that X is $\mathcal{F}$-compact. So by Theorem 2.4, we can say that X is compact with respect to the metric d. Now suppose $\{F_\alpha\}_{\alpha\in\Lambda}$ be a collection of $\mathcal{F}$-closed subsets of X having the finite intersection property. So $\{F_\alpha\}_{\alpha\in\Lambda}$ will be a collection of closed subsets of X with respect to the metric d having the finite intersection property. Now we can use [2, Theorem 26.9] to conclude the result. □

In [1], Jleli and Samet defined the concept of $\mathcal{F}$-totally boundedness and it is to be noted that totally boundedness is not a topological property. But our next theorem ensures that the concept of $\mathcal{F}$-totally boundedness is equivalent to that of standard metric spaces.

In the upcoming theorem, we will use the notation $B_d(x, r)$, where x is in a metric space (X, d) and $r > 0$, which is defined by $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

Theorem 2.6. *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then $A \subseteq X$ is $\mathcal{F}$-totally bounded if and only if A is totally bounded with respect to to the metric d.*

Proof. ($\Longrightarrow$) Suppose that $A \subseteq X$ is $\mathcal{F}$-totally bounded and let $\varepsilon > 0$. So there exists a finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^{n} B_D(a_i, \varepsilon)$. Now by the definition of the metric d, we have $B_D(a_i, \varepsilon) \subseteq B_d(a_i, \varepsilon)$ for all $i \in \{1, 2, \ldots, n\}$. Consequently $A \subseteq \bigcup_{i=1}^{n} B_d(a_i, \varepsilon)$. This shows that A is totally bounded with respect to to the metric d.

($\Longleftarrow$) Let A be totally bounded with respect to to the metric d. Let $\varepsilon > 0$. It can be easily seen from the definition of $\mathcal{F}$-metric and the metric d that, for any $t > 0, x, y \in X, y \neq x$,

$$f(D(x, y)) \leq f(d(x, y) + t) + \alpha. \tag{2.1}$$

Now, by $\mathcal{F}_2$ condition, for $(f(\varepsilon) - \alpha)$, there exists a $\delta > 0$ such that if $0 < t < \delta$ then $f(t) < f(\varepsilon) - \alpha$. Since A is totally bounded with respect to the metric d, so for $\frac{\delta}{2} > 0$, there exists a finite subset $\{b_1, b_2, \ldots, b_t\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^{t} B_d(b_i, \frac{\delta}{2})$. Now we will show that $B_d(b_i, \frac{\delta}{2}) \subseteq B_D(b_i, \varepsilon)\ \forall\ i \in \{1, 2, \ldots, t\}$. Let $y \in B_d(b_i, \frac{\delta}{2})$ and if $|B_d(b_i, \frac{\delta}{2})| = 1$, then clearly $y \in B_D(b_i, \varepsilon)$. On the other hand, if $|B_d(b_i, \frac{\delta}{2})| > 1$ and $y \in B_d(b_i, \frac{\delta}{2}), y \neq b_i$, then $d(y, b_i) < \frac{\delta}{2}$. Now, using (2.1) we have,

$$\begin{aligned} f(D(y, b_i)) \leq f\left(d(y, b_i) + \frac{\delta}{2}\right) + \alpha \quad &\Longrightarrow \quad f(D(y, b_i)) < f(\varepsilon) \\ &\Longrightarrow \quad D(y, b_i) < \varepsilon. \end{aligned}$$

So, $B_d(b_i, \frac{\delta}{2}) \subseteq B_D(b_i, \varepsilon)$ for all $i \in \{1, 2, \ldots, t\}$. This shows that $A \subseteq \bigcup_{i=1}^{t} B_D(b_i, \varepsilon)$. Hence, A is $\mathcal{F}$-totally bounded. $\square$

In [1, Proposition 4.9 (ii)], Jleli and Samet showed that if $A \subseteq X$ is $\mathcal{F}$-compact, then A is $\mathcal{F}$-totally bounded but did not say anything about the converse. In the next theorem we will consider the converse part.

Theorem 2.7. *Let (X, D) be an $\mathcal{F}$-metric space with $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Then the following are equivalent:*

(i) *X is $\mathcal{F}$-complete and $\mathcal{F}$-totally bounded;*
(ii) *X is $\mathcal{F}$-compact;*
(iii) *X is compact with respect to the metric d;*
(iv) *X is complete and totally bounded with respect to the metric d.*

Proof. (i) $\Rightarrow$ (ii) Suppose that X is $\mathcal{F}$-complete and $\mathcal{F}$-totally bounded. Then by [3, Theorem 2.2 (iii)], we can conclude that X is complete with respect to the metric d and by Theorem 2.6, X is totally bounded with respect to the metric d. This implies that X is compact with respect to the metric d and by Theorem 2.4, we can conclude that X is $\mathcal{F}$-compact.

(ii) $\Rightarrow$(i) Suppose that X is $\mathcal{F}$-compact. So by Theorem 2.4, X is compact with respect to the metric d. This implies that X is complete with respect to the metric d. By [3, Theorem 2.2 (iii)], we can conclude that X is $\mathcal{F}$-complete. The other part has already proved in [1].

(ii) $\Rightarrow$ (iii) It follows from Theorem 2.4.
(iii) $\Rightarrow$ (iv) It follows from the theory of standard metric spaces.
(iv) $\Rightarrow$ (i) It follows from [3, Theorem 2.2 (iii)] and Theorem 2.6. □

Acknowledgement. The Research is funded by the Council of Scientific and Industrial Research (CSIR), Government of India under the Grant Number: $25(0285)/18/EMR-II$. This project is funded by National Research Council of Thailand (NRCT) N41A640092. Finally, we would like to thanks all valuable comments from the reviewers.

References

[1] M. Jleli, B. Samet, *On a new generalization of metric spaces*, J. Fixed Point Theory Appl., **20**(2018), 128.
[2] J.R. Munkres, *Topology*, Pearson, 1974.
[3] S. Som, A. Bera, L.K. Dey, *Some remarks on the metrizability of $\mathcal{F}$-metric spaces*, J. Fixed Point Theory Appl., *22*(2020), 17.

Received: July 25, 2019; Accepted: March 27, 2021.