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GENERALIZED QUASI-CONTRACTIONS ON WEAK ORTHOGONAL METRIC SPACES

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Abstract. In this sequel, we introduce and study the concept of the weak orthogonal metric spaces as a generalization of the orthogonal metric spaces. Besides, we define and study the generalized quasi-contractions on such spaces and illustrate several non-trivial examples to endorse our obtained results. Among other things, as corollaries we obtain the main results of some of the pioneering articles existing in the literature. Finally, we answer the open question posed by Gordji et al. [On orthogonal sets and Banach fixed point theorem, Fixed Point Theory, 18(2):569-578, 2017].

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1. Introduction and preliminaries

Let (X, d) be any metric space and let T be a self-mapping on X. Then the set $Fix(T) = \{x \in X : Tx = x\}$ is the fixed point set of T. An operator T is said to be a contraction on X if

$$d(Tx, Ty) \leq rd(x, y)$$

holds for all $x, y \in X$ and for some $r \in [0, 1)$. The well-known Banach contraction principle [3] states that if T is a contraction on a complete metric space X, then T has only one fixed point.

Over the years, the metric fixed point theory has enthralled many a number of mathematicians in finding new theories, solving many real-life phenomena and therefore, a considerable number of research articles were put in print where the generalized versions of the metric notion are investigated by making alterations to the basic metric axioms. Eventually, there are a handful of metric structures which have come into the light in the literature. For some recent significant books on fixed point theory, we refer to [1, 4, 8, 11, 13, 20, 21].

On the other hand, there are many important and interesting results in the fixed point theory in connection with the lattices and ordered sets (see [15, 14, 16, 23, 24] and the references therein). In this direction, very recently Petruşel and Rus [16] considered the following:

Let X be a non-empty set endowed with a metric d, an order relation $\leq$ and an operator $f : X \to X$, which satisfies two main assumptions:

(i) f is generalized monotone with respect to $\leq$;
(ii) f is a (generalized) contraction with respect to d on a certain subset Y of $X \times X$.

The authors apply this result to study some problems related to integral and differential equations, and moreover, several open questions are discussed.

Recently, Gordji et al. [9] came up with the novel and exciting notion of the orthogonal sets and subsequently orthogonal metric spaces, and also affirmed an extension of Banach [3] fixed point theorem in the newly proposed setting. Further, Baghani et al. [2] improved the main result of [9], and proved a result equivalent to the axiom of choice. As an application, they considered the existence and uniqueness of a solution for a Volterra-type integral equation in L_p spaces. Afterwards, Ramezani [17] studied generalized convex contractions on the orthogonal metric spaces, and further, Ramezani and Baghani [18] coined the notion of the strongly orthogonal sets and proved a genuine generalization of Banach fixed point theorem and Walter's [25] theorem.

Recently, Senapati et al. [22] improved and extended the idea of orthogonal metric spaces using w-distances and obtained some interesting results with an application in non-linear fractional differential equations. Now we recall some basic definitions from [9].

Definition 1.1. Let X be a non-empty set and let $\perp$ be a binary relation defined on $X \times X$. Then $(X, \perp)$ is said to be an orthogonal set (briefly, O-set) if there exists $x_0 \in X$ such that

$$(\forall y \in X, x_0 \perp y) \text{ or } (\forall y \in X, y \perp x_0).$$

The element x_0 is called an orthogonal element. An orthogonal set may have more than one orthogonal element.

Example 1.2. Let X be a normed linear space. We define $x \perp y$ if $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{C}$. Then for all $y \in X$, there exists $x = \theta \in X$ such that $||x + \lambda y|| \geq ||x||$ for all $\lambda \in \mathbb{C}$. This shows that $(X, \perp)$ is an orthogonal set.

Let us mention the following very interesting example from [9].

Example 1.3. Let X be the set of all people in the world. We define $x \perp y$ if x can give blood to y. If x_0 is a person such that his (her) blood type is O^-, then we have $x_0 \perp y$ for all $y \in X$. Hence $(X, \perp)$ is an O-set, and here, x_0 is not unique.

Definition 1.4. Let $(X, \perp)$ be an O-set. A sequence (x_n) in X is an orthogonal sequence (briefly, O-sequence) if for all $n \in \mathbb{N}$,

$$x_n \perp x_{n+1} \text{ or } x_{n+1} \perp x_n.$$

Definition 1.5. Let $(X, \perp, d)$ be an orthogonal metric space. A Cauchy sequence (x_n) in X is a Cauchy orthogonal sequence (briefly, Cauchy O-sequence) if it is an orthogonal sequence. Moreover, $(X, \perp, d)$ is said to be a complete orthogonal metric space (briefly, O-complete) if every Cauchy O-sequence converges in X. A mapping $T : X \to X$ is said to be orthogonally continuous (O-continuous) at $x \in X$ if for each O-sequence (x_n) converging to x implies that $Tx_n \to Tx$ as $n \to \infty$. Also, T is said to be $\perp$-continuous on X if T is $\perp$-continuous at each $x \in X$.

One can note that, every continuous mapping is $\perp$-continuous, but the converse is not true in general.

Definition 1.6. Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $T : X \to X$ is said to be an orthogonal Banach contraction (briefly, Banach $\perp$-contraction) if $0 \leq k < 1$ and

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ with $x \perp y$.

Definition 1.7. Let $(X, \perp)$ be an O-set. A mapping $f : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $f(x) \perp f(y)$. Also, $f : X \to X$ is said to be weakly $\perp$-preserving if $x \perp y$ implies $f(x) \perp f(y)$ or $f(y) \perp f(x)$.

The next result is the main theorem of [9] and can be considered as a real extension of Banach contraction principle.

Theorem 1.8. Let $(X, \perp, d)$ be an O-complete metric space. Let $f : X \to X$ be $\perp$-continuous, Banach $\perp$-contraction and $\perp$-preserving. Then f has a unique fixed point $x \in X$. Also, f is a Picard operator, that is, $\lim_{n\to\infty} f^n x = x$ for all $x \in X$.

For more notions, examples and properties related to the orthogonal sets and orthogonal metric spaces, the readers are referred to [9, 2, 22]. On the other hand, in 1971, Ćirić [6] (see also [8, 13, 19]) proved the following important and exciting generalization of the Banach contraction principle.

Theorem 1.9. Let (X, d) be a complete metric space and let f be a self-mapping on X. Then f is said to be a quasi-contraction on X if there exists $\lambda \in [0, 1)$ satisfying

$$d(fx, fy) \leq \lambda \max\{d(x, y), d(fx, x), d(fy, y), d(fx, y), d(fy, x)\},$$

for all $x, y \in X$. Then f has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $u = \lim_{n\to\infty} f^n x$.

In 2015, Kumam et al. [12] stated and proved a generalization of the aforementioned Ćirić fixed point theorem in metric spaces by using a new generalized quasi-contractive map.

Theorem 1.10. Let (X, d) be metric space and let T be a self-map on X. Then T is said to be a generalized quasi-contraction if

$$d(Tx, Ty) \leq kM(x, y)$$

for all $x, y \in X$, where $0 \leq k < 1$ and

$$\begin{aligned} M(x, y) \quad = \quad & \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y), \\ & d(T^2x, x), d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}. \end{aligned}$$

Then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $u = \lim_{n\to\infty} T^n x$.

In this paper, we introduce and study the concept of the weak orthogonal metric spaces as a generalization of the orthogonal metric spaces. Then we define and investigate the generalized quasi-contractions on such spaces. Among other things, as corollaries of our findings, we obtain a few known results from the literature. Additionally, we construct several constructive examples to endorse our findings and furthermore, we answer an open problem posed in [9].

2. Weak orthogonal metric spaces and related notions

In this section, we introduce the idea of a weak orthogonal set and subject to that, the idea of a weak orthogonal metric space. Further, we develop the essential fundamental concepts of continuity, Cauchy sequences and completeness on this background. Here we also note that, $\mathbb{N}_0$ stands for the set $\mathbb{N} \cup \{0\}$. We begin with the definition of weak orthogonal sets.

Definition 2.1. Let X be a non-empty set and let $\perp$ be a binary relation defined on $X \times X$. Then $(X, \perp)$ is said to be a weak orthogonal set (briefly, O_w-set) if there exists $x_0 \in X$ such that for all $y \in X$,

$$x_0 \perp y \text{ or } y \perp x_0.$$

The element x_0 is called a weak orthogonal element. Likewise an orthogonal set, a weak orthogonal set can have more than one weak orthogonal element. Again, two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$ or $y \perp x$.

Remark 2.2. From the definition, it is clear that every orthogonal set is a weak orthogonal set, but the converse may not be true. The following examples show that a weak orthogonal set is not an orthogonal set, generally.

Example 2.3. Let us set $X = \mathbb{R}$ and we define a binary relation $\perp$ on X by

$$x \perp y \text{ if } x \leq y.$$

It is very easy to check that $\perp$ is a weak orthogonal relation but not an orthogonal relation. For all $x \in X$ with $x \geq 0$, we have $0 \perp x$ and for all $x \leq 0$, we have $x \perp 0$. Hence, $(\mathbb{R}, \perp)$ is a weak orthogonal set. Note that, this set is not an orthogonal set since there is no element $x_0 \in X$ such that for all $x \in X$, $x_0 \perp x$ or for all $x \in X$, $x \perp x_0$ holds. Also note that, every element in X is a weak orthogonal element.

Example 2.4. We consider the linear space $M_{n \times n}(\mathbb{R})$ and

$$S = \{A \in M_{n \times n}(\mathbb{R}) : A \geq 0 \text{ or } A \leq 0\}.$$

Now we define a binary relation $\perp$ on S as $A \perp B$ if $A - B \geq 0$. Clearly for all positive semi-definite matrices $A \in S$, $A \perp 0$ and for all negative semi-definite matrices $A \in S$, $0 \perp A$. Therefore $(S, \perp)$ is a weak orthogonal set.

Example 2.5. Let H be an infinite dimensional Hilbert space and let

$$S = \{P, I + P : P \text{ be an orthogonal projection operator}\}.$$

Now we define a binary relation $\perp$ on S as $P_1 \perp P_2$ if $P_1 \geq P_2$. Therefore for all $P \in S$, we have either $P \perp I$ or $I \perp P$. Hence $(S, \perp)$ is a weak orthogonal set.

In the following discussion, we extend the notions of orthogonal sequences and Cauchy orthogonal sequences to weak orthogonal sequences and Cauchy weak orthogonal sequences, respectively.

Definition 2.6. Let $(X, \perp)$ be a weak orthogonal set (briefly, O_w-set). A sequence (x_n) in X is said to be a weak orthogonal sequence (briefly, O_w-sequence) if for all $n \in \mathbb{N}$,

$$x_n \perp x_{n+1} \text{ or } x_{n+1} \perp x_n.$$

Similarly, a Cauchy sequence (x_n) in X is said to be a Cauchy weak orthogonal sequence (briefly, Cauchy O_w-sequence) if for all $n \in \mathbb{N}$,

$$x_n \perp x_{n+1} \text{ or } x_{n+1} \perp x_n.$$

Remark 2.7. Every orthogonal sequence is a weak orthogonal sequence, but the converse is not true.

Example 2.8. Let us consider the weak orthogonal set in Example 2.3. We consider a sequence (x_n) in X by $x_n = (-1)^n \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly, for all $m \in \mathbb{N}$ with $n = 2m + 1$, $x_n \perp x_{n+1}$ and $n = 2m$, $x_{n+1} \perp x_n$. This shows that (x_n) is a weak orthogonal sequence but not an orthogonal sequence.

Now we introduce the notion of a weak orthogonal metric space.

Definition 2.9. Let $(X, \perp)$ be a weak orthogonal set and let d be any metric defined on X. Then $(X, \perp, d)$ is said to be a weak orthogonal metric space (O_w-metric space).

Definition 2.10. A weak orthogonal metric space $(X, \perp, d)$ is said to be a complete weak orthogonal metric space (briefly, O_w-complete) if every Cauchy O_w-sequence converges in X.

Definition 2.11. A self-map T on a weak orthogonal metric space $(X, \perp, d)$ is said to be weak orthogonality preserving (briefly, O_w-preserving) if $x \perp y \Rightarrow Tx \perp Ty$ or $Ty \perp Tx$ for all $x, y \in X$.

Here we draw the reader's attention to a basic difference between the Banach contraction in metric spaces and orthogonal Banach contraction in orthogonal metric spaces. It is very well-known that in metric spaces, every Banach contraction mapping is a continuous mapping. But in orthogonal metric spaces, Banach $\perp$-contraction condition does not give the guarantee of orthogonal continuity of a mapping. In this regard, we present the following simple example.

Example 2.12. We consider the orthogonal metric space $(X, \perp, d)$, where $X = \mathbb{R}$ and

$$x \perp y \text{ if } xy \in \mathbb{Q}.$$

Therefore, for all $x \in X$, there exists $0 \in \mathbb{R}$ such that $0 \perp x$ and hence, $(X, \perp, d)$ is an orthogonal set. We define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} 0, & x \in \mathbb{Q}^c; \\ \frac{x}{3}, & \text{otherwise.} \end{cases}$$

At first, we show that T is a Banach $\perp$-contraction. In order to show this, let us consider two non-zero numbers $x, y \in X$ with $x \perp y$. Then we must have either $x, y \in \mathbb{Q}$ or $x, y \in \mathbb{Q}^c$ which implies that

$$d(Tx, Ty) = \frac{x - y}{3} \leq \frac{1}{3} d(x, y)$$

or

$$d(Tx, Ty) = 0 \leq kd(x, y),$$

for all $k \in [0, 1)$. Let $x = 0$ and $y \in \mathbb{R}$ be a non-zero number. Then it is easy to check that $d(Tx, Ty) \leq kd(x, y)$ for some $k \in [0, 1)$. Therefore,

$$d(Tx, Ty) \leq kd(x, y)$$

for some $k \in [0, 1)$ and for all $x, y \in X$ with $x \perp y$. Therefore, T is a Banach $\perp$-contraction. Note that T is not a Banach contraction. For example, let $x = 1$ and $y = 1 + \frac{1}{\sqrt{11}}$. Then there exists no $k \in [0, 1)$ such that

$$d(Tx, Ty) = \frac{1}{3} \leq kd(x, y)$$

holds. Next, we claim that the mapping T is not O-continuous. To show this, we consider the sequence (x_n) in X where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ for each $n \in \mathbb{N}$. Clearly, (x_n) is an orthogonal sequence converging to e. It is easy to check that $Tx_n \to \frac{e}{3} \neq T(e) = 0$, which implies that T is not O-continuous.

Therefore to establish the fixed point, common fixed point and other related results in orthogonal metric spaces, we need to assume the condition of O-continuity of the mapping which is already defined in [9]. Now we are interested to extend the idea of O-continuity to orbitally O-continuity and then, orbitally weak O-continuity on the newly introduced setting. However, by the notation $O_T(x)$, we define orbit of a mapping T at $x \in X$, that is,

$$O_T(x) = \{T^n x : n = 0, 1, 2, \dots\}.$$

Definition 2.13. Let $(X, \perp, d)$ be an O-metric space and let T be a self-mapping on X. Then T is said to be orbitally O-continuous at $z \in X$ if for every O-sequence (y_n) in $O_T(x)$ for any $x \in X$,

$$y_n \to z \Rightarrow Ty_n \to Tz.$$

Definition 2.14. Let $(X, \perp, d)$ be an O-metric space and let T be a self-mapping on X. Then X is said to be T-orbitally O-complete if every Cauchy O-sequence (y_n) in $O_T(x)$ for any $x \in X$, converges in X.

The following example shows that an orbitally O-continuous mapping need not be O-continuous and subsequently, a T-orbitally O-complete space may not be O-complete.

Example 2.15. Let $X = (0, \infty)$ and let us define $x \perp y$ if $xy \leq x$ or y. Then for all $y \in X$, there exists $x = 1$, such that $xy \leq y$. So, $(X, \perp)$ is an O-set. We consider the usual metric d on X. Then $(X, \perp, d)$ is an O-metric space. Let $T : X \to X$ be defined as

$$T(x) = \begin{cases} 2, & x \in (0, 1); \\ 1, & x = 1; \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

Here, we claim that

(A) The function T is an orbitally O-continuous but not O-continuous.
(B) X is a T-orbitally O-complete metric space but not O-complete.

Proof. **(A)** We consider a sequence (x_n) in X such that $x_n = 1 - \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Clearly, this sequence is an O-sequence and converges to 1. For all $n \in \mathbb{N}$, $Tx_n = 2$ and $T1 = 1$, which implies that T is not an O-continuous mapping. It is easy to check that T is an orbitally O-continuous mapping.
(B) To prove this, we consider the following cases:
Case-I: Let us consider $x \in (0,1)$. Then

$$\begin{aligned} O_T(x) &= \{T^n x : n = 0,1,2,\dots\} \\ &= \left\{x, 2, \frac{1}{3}, 2, \frac{1}{3}, \dots\right\}. \end{aligned}$$

Similarly for $x > 1$,

$$O_T(x) = \left\{x, \frac{1}{3}, 2, \frac{1}{3}, 2, \dots\right\}.$$

Therefore for all $x \in (0,1) \cup (1,\infty)$, $O_T(x)$ contains two subsequences. However, the subsequence $(y_n) = \{\frac{1}{3}\}$ is the only Cauchy O-sequence which converges in X.
Case-II: For $x = 1, O_T(x) = \{1,1,1,\dots\}$ contains a constant sequence which is a Cauchy O-sequence.

From the above two cases we deduce that $(X, \perp, d)$ is a T-orbitally O-complete metric space. Now we consider a sequence (x_n) in X such that $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly, this sequence is a Cauchy O-sequence, but not convergent in X. Therefore, $(X, \perp, d)$ is not an O-complete metric space.

In the sequel, we extend the above notions in weak orthogonal metric spaces.

Definition 2.16. Let $(X, \perp, d)$ be an O_w-metric space and let T be a self-mapping on X. Then T is said to be orbitally O_w-continuous at $z \in X$ if for every O_w-sequence (y_n) in $O_T(x)$ for any $x \in X$,

$$y_n \to z \Rightarrow Ty_n \to Tz.$$

Definition 2.17. Let $(X, \perp, d)$ be an O_w-metric space and let T be a self-mapping on X. Then X is said to be T-orbitally O_w-complete if every Cauchy O_w-sequence (y_n) in $O_T(x)$ for any $x \in X$, converges in X.

3. Main results

This section comes up with the definition of generalized quasi-orthogonal contractions in a weak orthogonal metric space and it presents a fixed point result concerning such kind of maps. We also illustrate an example to validate our findings.

Definition 3.1. Let $(X, \perp, d)$ be an O_w-metric space and let T be a self-map on X. Then T is a generalized quasi $\perp$-contraction if

$$d(Tx, Ty) \le kM(x,y)$$

holds for all orthogonally related elements $x, y \in X$, where, $0 \le k < 1$ and

$$\begin{aligned} M(x,y) = \max\{&d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(Tx,y), \\ &d(T^2x,x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\}. \end{aligned}$$

Now we discuss one fixed point result related to such kind of contractions.

Theorem 3.2. Let T be a self-map on a weak orthogonal metric space $(X, \perp, d)$ and let X be a T-orbitally O_w-complete metric space. If T is a weak $\perp$-preserving, orbitally O_w-continuous and generalized quasi $\perp$-contraction, then T owns a unique fixed point.

Proof. Since X is a weak orthogonal set, there exists at least one element $x_0 \in X$ such that

$$\forall y \in X, (x_0 \perp y \text{ or } y \perp x_0).$$

This implies that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Let us consider the iterated sequence (x_n) where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Since T is a weak $\perp$-preserving map, we must have either $T^n x_0 \perp T^{n+1} x_0$ or $T^{n+1} x_0 \perp T^n x_0$ for all $n \in \mathbb{N}$, which implies that (x_n) is an O_w-sequence. Furthermore, $T^n x_0$ and $T^m x_0$ are orthogonally related for all $m, n \in \mathbb{N}$. Hence, as in the proof of Theorem 3.1 of [12], we get, for all $n < m,\ n, m \in \mathbb{N}$

$$d(T^n x_0, T^m x_0) \leq \frac{k^n}{1-k} d(x_0, Tx_0). \tag{3.1}$$

This shows that (x_n) is a Cauchy O_w-sequence. Since X is T-orbitally O_w-complete, there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. We claim that z is a fixed point of T. Since T is an orbitally O_w-continuous mapping, and also, the sequence (x_n) is itself an O_w-sequence converging to z, we have that $Tx_n \to Tz$ as $n \to \infty$. Therefore,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z,$$

which implies that z is a fixed point of T. Finally, we prove the uniqueness of the obtained fixed point. Let us consider, w is another fixed point of T. Then we have either $x_0 \perp w$ or $w \perp x_0$. As T is a weak orthogonality preserving map for all $n \in \mathbb{N},\ x_n \perp w$ or $w \perp x_n$. Then

$$\begin{aligned}
d(x_n, w) &= d(Tx_{n-1}, Tw) \\
&\leq kM(x_{n-1}, w) \\
&= k \cdot \max\{d(x_{n-1}, w), d(w, w), d(x_{n-1}, x_n), d(x_{n-1}, w), d(x_n, w), \\
&\quad d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n), d(x_{n+1}, w), d(x_{n+1}, w)\} \\
&= k \cdot \max\{d(x_{n-1}, w), d(x_{n-1}, x_n), d(x_n, w), \\
&\quad d(x_{n+1}, x_{n-1}), d(x_{n+1}, x_n), d(x_{n+1}, w)\}.
\end{aligned}$$

Now, taking the limit as $n \to \infty$, we get

$$d(z, w) \leq kd(z, w),$$

that is, $z = w$.

The existence of a fixed point of the mapping T in the above theorem can be established under the following condition instead of orbitally O_w-continuity of T.

(O1) Suppose (x_n) is an O_w-sequence in $O_T(x)$, where $x_n = T^n x$ for some $x \in X$, converging to $z \in X$. Then (x_n) has a subsequence (x_{n_r}) such that for all $r \in \mathbb{N}$,

$$x_{n_r} \perp z \text{ or } z \perp x_{n_r}.$$

Theorem 3.3. Let T be a self-mapping defined on a weak orthogonal metric space $(X, \perp, d)$ and also, let X be T-orbitally O_w-complete. If T is a weak orthogonality preserving, generalized quasi $\perp$-contraction and satisfies condition (O1), then T owns a unique fixed point.

Proof. Continuing in a similar fashion of the proof of the above theorem, let us consider that the Cauchy O_w-sequence converges to z. We prove that z is a fixed point of T. By the property (O1), $x_{n_r} \perp z$ or $z \perp x_{n_r}$ for all $r \in \mathbb{N}$ which implies that $Tx_{n_r} \perp Tz$ or $Tz \perp Tx_{n_r}$ for all $r \in \mathbb{N}$. Thus,

$$\begin{aligned} d(Tx_{n_r}, Tz) &\leq kM(x_{n_r}, z) \\ &= k \cdot \max\big\{d(x_{n_r}, z), d(x_{n_r}, x_{n_r+1}), d(z, Tz), d(x_{n_r}, Tz), d(x_{n_r+1}, z), \\ &\quad d(x_{n_r+2}, x_{n_r}), d(x_{n_r+2}, x_{n_r+1}), d(x_{n_r+2}, z), d(x_{n_r+2}, Tz)\big\}. \end{aligned}$$

Taking the limit $r \to \infty$ in both the sides of the inequality, we get

$$d(z, Tz) \leq kd(z, Tz),$$

that is, $Tz = z$. Therefore, z is a fixed point of T. The uniqueness of fixed point can be proved in a similar way to that of Theorem 3.2.

Remark 3.4. Obviously from Theorem 3.2, we can obtain Theorem 1.8, Theorem 1.10 and further, one can deduce Theorem 1.9. Moreover, it is worth noting that the contraction condition which we consider here is more general than the contraction condition due to Ćirić [6]. Therefore, one can easily access the fixed point result for the mapping satisfying Ćirić contraction condition from our results in weak orthogonal metric spaces. Additionally, we can obtain fixed point results for the mappings satisfying Kannan contraction [10] (see also [13, 19]) and Chatterjea contraction [5] (see also [13, 19]).

In support of our main result, we present the following example.

Example 3.5. Let us set $X = \{0, 1, 2, 3, 4\}$ and consider an arbitrary binary relation $\mathcal{R}$ on X as

$$\mathcal{R} = \{(0,0), (1,0), (0,2), (3,4), (3,0), (4,0)\}.$$

For any two elements $x, y \in X$, $x \perp y$ if $(x,y) \in \mathcal{R}$. Clearly, $(X, \perp)$ is not an orthogonal set but it is a weak orthogonal set, as for all $x \in X$, there exists $y = 0$ such that $(0,x) \in \mathcal{R}$ or $(x,0) \in \mathcal{R}$. Now, we define a mapping $T : X \to X$ by

$$T0 = 0, T1 = 0, T2 = 1, T3 = 0, T4 = 2.$$

We check whether T satisfies all the hypotheses of Theorem 3.2 or not. First of all, note that except $(x,y) = (4,0)$, for all $x, y \in X$ with $x \perp y$, we have $Tx \perp Ty$. For $(x,y) = (4,0)$, we have $T0 \perp T4$. This implies T is a weak $\perp$-preserving mapping.

Again, for all $x, y \in X$ with $x \perp y$, we have $d(Tx,Ty) \leq kd(x,y)$ for some $k \in [0,1)$ except $(x,y) = (3,4)$. We show that T satisfies the generalized contraction condition.

For $(x, y) = (3, 4)$,

$$\begin{aligned} M(3,4) = \max \Big\{ & d(3,4), d(3,T3), d(4,T4), \frac{d(3,T4)+d(T3,4)}{2}, \\ & \frac{d(T^2 3,3)+d(T^2 3,T4)}{2}, d(T^2 3,T3), d(T^2 3,4), d(T^2 3,T4) \Big\} \\ = \max \Big\{ & d(3,4), d(3,0), d(2,4), \frac{d(3,2)+d(0,4)}{2}, \frac{d(0,3)+d(0,2)}{2}, \\ & d(0,4), d(0,2) \Big\} \\ = 4&. \end{aligned}$$

Therefore, for all $x, y \in X$ with $x \perp y$, we have

$$d(Tx, Ty) \leq kM(x, y),$$

i.e., T is a generalized quasi $\perp$-contraction mapping. Now we have

$$\begin{aligned} O_T(0) &= \{0, 0, 0, \dots\}, \\ O_T(1) &= \{1, 0, 0, \dots\}, \\ O_T(2) &= \{2, 1, 0, 0, \dots\}, \\ O_T(3) &= \{3, 0, 0, \dots\}, \\ O_T(4) &= \{4, 2, 1, 0, 0, \dots\}. \end{aligned}$$

Observe that, for all $x \in X$, $O_T(x)$ contains a constant sequence. This implies that X is a T-orbitally O_w-complete metric space and T is also an O_w-continuous map. Therefore, all the conditions of Theorem 3.2 are satisfied. Here $x = 0$ is the unique fixed point of T.

We close this section with the following result related to Ćirić [7]. Our result is a generalization of the pioneering result (Theorem 1, [7]), where he firstly proposed the concept of a non-unique fixed point.

Theorem 3.6. Let T be a self-map defined on a weak orthogonal metric space $(X, \perp, d)$ and let X be a T-orbitally O_w-complete metric space. Suppose that T is a weak $\perp$-preserving, orbitally O_w-continuous map such that there is $k \in [0, 1)$ and

$$\min C(x, y) - \min D(x, y) \leq kd(x, y) \tag{3.2}$$

for all orthogonally related $x, y \in X$, where

$$C(x, y) = \{d(Tx, Ty), d(x, Tx), d(y, Ty), d(T^2x, Tx), d(T^2x, y)\}$$

and

$$D(x, y) = \{d(x, Ty), d(Tx, y)\}.$$

Then for each weak orthogonal element $x_0 \in X$, the sequence $(T^n x_0)$ converges to a fixed point of T.

Proof. Let $x_0 \in X$ be a weak orthogonal element in X, and let us consider the iterated sequence (x_n) where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Since T is a weak $\perp$-preserving map, we must have either $T^n x_0 \perp T^{n+1} x_0$ or $T^{n+1} x_0 \perp T^n x_0$ for all $n \in \mathbb{N}$, which implies that (x_n) is an O_w-sequence. Furthermore, $T^n x_0$ and $T^m x_0$ are orthogonally related for $n, m \in \mathbb{N}$. Now, putting $x = x_n$ and $y = x_{n+1}$ in (3.2), we get,

$$\min C(x_n, x_{n+1}) - \min D(x_n, x_{n+1}) \leq kd(x_n, x_{n+1}), \tag{3.3}$$

where

$$C(x_n, x_{n+1}) = \{d(Tx_n, Tx_{n+1}), d(x_n, Tx_n)\}$$

and

$$D(x_n, x_{n+1}) = \{d(x_n, Tx_{n+1}), d(Tx_n, x_{n+1})\}$$

for all $n \in \mathbb{N}$. Now $\min D(x_n, x_{n+1}) = 0$, and (3.3) implies

$$\min\{d(Tx_n, Tx_{n+1}), d(x_n, Tx_n)\} \leq kd(x_n, x_{n+1}),$$

for all $n \in \mathbb{N}$. Since $d(x_n, Tx_n) \leq kd(x_n, x_{n+1})$ is impossible, we have to only consider the case

$$d(Tx_n, Tx_{n+1}) \leq kd(x_n, x_{n+1}).$$

Further, one can easily verify that (x_n) is a Cauchy O_w-sequence. Since X is T-orbitally O_w-complete, there exists some $z \in X$ such that $x_n \to z$ as $n \to \infty$. Since T is an orbitally O_w-continuous mapping, and the sequence (x_n) is itself an O_w-sequence converging to z, we have that $Tx_n \to Tz$ as $n \to \infty$. Therefore,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z,$$

which implies that z is a fixed point of T.

The subsequent results are some corollaries which can be obtained from our findings.

Corollary 3.7. Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). If there is $k \in [0, 1)$ such that

$$\min C(x, y) - \min D(x, y) \leq kd(x, y),$$

for all $x, y \in X$, where

$$C(x, y) = \{d(Tx, Ty), d(x, Tx), d(y, Ty), d(T^2x, Tx), d(T^2x, y)\}$$

and

$$D(x, y) = \{d(x, Ty), d(Tx, y)\},$$

then for each $x_0 \in X$, the sequence $(T^n x_0)$ converges to a fixed point of T.

Corollary 3.8. (Non-unique fixed point theorem of Ćirić [7]**)** Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). If there is $k \in [0, 1)$ such that

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y),$$

for all $x, y \in X$, then for each $x_0 \in X$, the sequence $(T^n x_0)$ converges to a fixed point of T.

In the subsequent example, we show that Corollary 3.7 is a proper generalization of Corollary 3.8. We use Example 2.5 of [12] where the authors used it to show that a generalized quasi-contraction is not a quasi-contraction, generally.

Example 3.9. Let $X = \{1, 2, 3, 4, 5\}$ and let us define metric $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \in X; \\ 2, & \text{if } (x,y) \in \{(1,4),(1,5),(4,1),(5,1)\}; \\ 1, & \text{otherwise.} \end{cases}$$

Let $T : X \to X$ be defined by

$$T1 = T2 = T3 = 1, T4 = 2, T5 = 3.$$

Then, we have $\min\{d(T2,T4), d(2,T2), d(4,T4)\} = 1$, $\min\{d(2,T4), d(T2,4)\} = 0$ and $d(2,4) = 1$. So all the assumptions in Corollary 3.8 are not satisfied. However, one can note that all the conditions of Corollary 3.7 are fulfilled. This follows by $d(Tx,Ty) \leq 1$ for all $x, y \in X$. Hence $\min C(x,y) \leq 1$ for all $x, y \in X$. But if $\min C(x,y) = 1$, then $\min D(x,y) = 1$.

4. Answer to an open question posed in [9]

We have already mentioned that the authors of [9] defined the concept of O-continuity. They proved that every continuous function is O-continuous but the reverse implication does not hold in general. In that connection, they raised the following question on inner product spaces.

Problem 4.1. Let X be an inner product space with the inner product $\langle .,. \rangle$. We define an orthogonal relation $\perp$ on X as $x \perp y$ if $\langle x, y \rangle = 0$. Let $f : X \to X$ be an O-continuous function. Is f continuous?

The authors of [22] tried to answer this question and claimed that in an inner product space, every O-continuous function is continuous. Here we reinvestigate that problem and observe that their claim was not right, that is, there may exist an O-continuous function which is not necessarily continuous in the inner product spaces. In this purpose, we construct the following example of an O-continuous function in the standard inner product space $\mathbb{R}^2$ which is not continuous.

Example 4.2. Let $\big(X, \langle .,. \rangle\big)$ be a standard inner product space, where $X = \mathbb{R}^2$ and let $\langle x, y \rangle = x_1 y_1 + x_2 y_2$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. An orthogonal relation on X is defined as

$$x \perp y \text{ if } \langle x, y \rangle = 0.$$

Clearly, $(X, \perp)$ is an orthogonal set as for all $x \in X, \langle \theta, x \rangle = 0$ where $\theta = (0,0)$.
Let us define a function $F : X \to X$ by

$$F(x_1,x_2) = \begin{cases} \left(\dfrac{x_1 x_2}{x_1^2 + x_2^2}, 0\right), & (x_1,x_2) = \left(\frac{1}{n}, \frac{1}{n+1}\right), \ n \in \mathbb{N}; \\ (0,0), & \text{otherwise.} \end{cases}$$

We prove that this function is O-continuous at $\theta = (0,0)$, but not continuous at that point. Before showing that, we claim the following:

(A) there exists no orthogonal sequence $(x^{(n)})$ such that $x^{(n)} = \left(\frac{1}{i+n}, \frac{1}{i+n+1}\right)$ for some $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}$;

(B) for any $k \in \mathbb{N}$, $\left(\frac{1}{k}, \frac{1}{k+1}\right)$ can not be a limit point of any orthogonal sequence.

Proof. **(A)** Let us consider that there exists an O-sequence $(x^{(n)})$ such that

$$x^{(n)} = \left(\frac{1}{i+n}, \frac{1}{i+n+1}\right)$$

for some $i \in \mathbb{N}_0$ and for all $n \in \mathbb{N}$. Then

$$\langle x^{(n)}, x^{(n+1)} \rangle = \frac{1}{i+n}\frac{1}{i+n+1} + \frac{1}{i+n+1}\frac{1}{i+n+2} \neq 0$$

for all $i \in \mathbb{N}_0$ and $n \in \mathbb{N}$ which contradicts the fact that $(x^{(n)})$ is an orthogonal sequence.

(B) If possible, let $\left(\frac{1}{k}, \frac{1}{k+1}\right)$ for some $k \in \mathbb{N}$, be a limit point of an orthogonal sequence $(x^{(n)})$. Let us choose a number $\epsilon > 0$ such that $\epsilon < \frac{1}{k+1}$. Then for every such choice of $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$x_1^{(n)} \in \left(\frac{1}{k} - \epsilon, \frac{1}{k} + \epsilon\right) \quad \text{and} \quad x_2^{(n)} \in \left(\frac{1}{k+1} - \epsilon, \frac{1}{k+1} + \epsilon\right),$$

i.e., $x_i^{(n)} > 0$ for $i = 1, 2$ and for all $n \geq n_0$. This implies that for all $n \geq n_0$, $\langle x^{(n)}, x^{(n+1)} \rangle \neq 0$, which contradicts the orthogonality of the sequence $(x^{(n)})$. Hence our assumption was wrong.

Therefore for any O-sequence $(x^{(n)})$ converging to $z = (x, y)$, we must have

$$F\left(x_1^{(n)}, x_2^{(n)}\right) = (0, 0) = F(x, y)$$

for all $n \in \mathbb{N}$. This shows that F is O-continuous at $z = (x, y)$ and also F is O-continuous at $\theta = (0, 0)$. Next, we consider a sequence $(y^{(n)})$ where

$$y^{(n)} = \left(y_1^{(n)}, y_2^{(n)}\right) = \left(\frac{1}{n}, \frac{1}{n+1}\right)$$

for all $n \in \mathbb{N}$. It is clear from (A) that this sequence is not an O-sequence. Also the sequence $\left(y^{(n)}\right)$ converges to $\theta = (0, 0)$ as $n \to \infty$ but

$$\lim_{n \to \infty} F\left(y_1^{(n)}, y_2^{(n)}\right) = \left(\frac{1}{2}, 0\right) \neq (0, 0).$$

This implies that F is not continuous at $\theta = (0, 0)$. Again, in the standard inner product space $\mathbb{R}^n$, one can consider the following O-continuous function which is not continuous. It is easy to check that the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F(x_1, \cdots, x_n) = \begin{cases} \left(\frac{x_1 x_2}{x_1^2 + x_2^2}, 0, \cdots, 0\right), & (x_1, x_2, x_3, \cdots, x_n) = \left(\frac{1}{n}, \frac{1}{n+1}, 0, \cdots, 0\right), n \in \mathbb{N}; \\ (0, \cdots, 0), & \text{otherwise} \end{cases}$$

is O-continuous at $\theta = (0, 0, \cdots, 0) \in \mathbb{R}^n$. But it is not a continuous function. Therefore in general, we can conclude that in arbitrary inner product spaces an O-continuous function may not be a continuous function.

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