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SOLVING THE SPLIT EQUALITY HIERARCHICAL FIXED POINT PROBLEM

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Abstract. This paper deals with a split equality hierarchical fixed point problem in real Hilbert spaces which is an important and natural extension of hierarchical fixed point problem and split equality fixed point problem. An iterative algorithm where the stepsizes do not depend on the operator norms, so called simultaneous Krasnoselski-Mann algorithm is suggested for solving the split equality hierarchical fixed point problem. Further we prove a weak convergence theorem for the sequence generated by this algorithm. This special aspect of the algorithm together with the convergence result makes it an interesting scheme. Furthermore, we give some examples to justify the main result. Finally, we show that our purposed iterative algorithm is more efficient than some other known iterative algorithms. On the other hand, the framework is general and allows us to treat in a unified way several iterative algorithms, recovering, developing and improving some recently known related convergence results in the literature.

Key Words and Phrases: Split equality hierarchical fixed point problem, split equality fixed point problem, maximal monotone operator, simultaneous Krasnoselski-Mann algorithm, weak convergence, weak convergence.
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1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces and let the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote, respectively, the inner product and induced norm of H_1, H_2 and H_3. Recall that a mapping $U : H_1 \to H_1$ is nonexpansive if $\|Ux - Uy\| \leq \|x - y\|$, for all $x, y \in H_1$. It is known that $\mathrm{Fix}(U) := \{x \in H_1 : Ux = x\}$ is a closed and convex subset of H_1.

Now, let $S_1, T_1 : H_1 \to H_1$ and $S_2, T_2 : H_2 \to H_2$ be nonexpansive mappings with $\mathrm{Fix}(T_1) \neq \emptyset$, $\mathrm{Fix}(T_2) \neq \emptyset$. We introduce the following new class of problems called split equality hierarchical fixed point problem (in short, $\mathrm{S_pEHFPP}$):

Find $x^* \in \mathrm{Fix}(T_1)$ and $y^* \in \mathrm{Fix}(T_2)$ such that

$$\langle x^* - S_1 x^*, x^* - x \rangle \leq 0, \ \forall x \in \mathrm{Fix}(T_1), \tag{1.1}$$

$$\langle y^* - S_2 y^*, y^* - y \rangle \leq 0, \ \forall y \in \mathrm{Fix}(T_2) \tag{1.2}$$

$$\text{and } Ax^* = By^*,$$

where $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators.

When look separately, (1.1) is called hierarchical fixed point problem (in short, HFPP), introduced and studied by Moudafi [19], and its solution set is denoted by Sol(HFPP(1.1)). We note that HFPP covers monotone variational inequality on fixed point sets, minimization problems over equilibrium constraints, hierarchical minimization problems. $\mathrm{S_pEHFPP}$(1.1)-(1.2) is governed by two pairs of mappings; in each pair, one is used to define the governing operator and the other to define the feasible set of the variational inequality. The solution set of $\mathrm{S_pEHFPP}$(1.1)-(1.2) is denoted by $\Gamma := \{(x^*, y^*) \in \mathrm{Fix}(T_1) \times \mathrm{Fix}(T_2) : x^* \in \mathrm{Sol(HFPP(1.1))}, y^* \in \mathrm{Sol(HFPP(1.2))} \text{ and } Ax^* = By^*\}$.

It is worth mentioning that when $S_1 = I_1,\ S_2 = I_2$ (I_1, I_2 are identity operators on H_1, H_2, respectively) then $\mathrm{S_pEHFPP}$(1.1)-(1.2) is reduced to the split equality fixed point problem (in short, $\mathrm{S_pEFPP}$) introduced by Moudafi [20]: Find $x^* \in H_1$ and $y^* \in H_2$ such that

$$x^* \in \mathrm{Fix}(T_1), y^* \in \mathrm{Fix}(T_2) \text{ and } Ax^* = By^*. \tag{1.3}$$

We denote the solution set of $\mathrm{S_pEFPP}$ (1.3) by Ω. Further, if take $\mathrm{Fix}(T_1) = C$, $\mathrm{Fix}(T_2) = Q$ where $C \subseteq H_1, Q \subseteq H_2$ are nonempty closed convex sets then $\mathrm{S_pEHFPP}$(1.1)-(1.2) is reduced to the split equality problem (in short, $\mathrm{S_pEP}$) introduced by Moudafi [20]:

$$x^* \in C,\ y^* \in Q \text{ and } Ax^* = By^*. \tag{1.4}$$

We note that $\mathrm{S_pEP}$ (1.4) covers many important situations, for instance in decomposition methods for partial differential equations, applications in game theory and in intensity-modulated radiation therapy (in short, IMRT). In decision sciences, this allows consideration of agents that interplay only via some components of their decision variables (see, [2]). In IMRT, this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see, [6]).

By setting $S_1 = I_1 - \gamma \mathbf{F}_1$ and $S_2 = I_2 - \gamma \mathbf{F}_2$, where for each $i \in \{1, 2\}$, $\mathbf{F}_i$ is η_i-Lipschitzian and k_i-strongly monotone with $\gamma \in \left(0, \min\left\{\frac{2k_1}{\eta_1}, \frac{2k_2}{\eta_2}\right\}\right]$, and $k_i \leq \eta_i < \frac{1}{\gamma}$, for each $i \in \{1, 2\}$, then $\mathrm{S_pEHFPP}$(1.1)-(1.2) is reduced to the split equality variational inequality problem over the fixed point sets of T_1, T_2 (in short, $\mathrm{S_pEVIP}$): Find $x^* \in \mathrm{Fix}(T_1),\ y^* \in \mathrm{Fix}(T_2)$

$$\langle x - x^*, \mathbf{F}_1(x^*)\rangle \geq 0,\ \forall x \in \mathrm{Fix}(T_1), \tag{1.5}$$
$$\langle y - y^*, \mathbf{F}_2(y^*)\rangle \geq 0,\ \forall y \in \mathrm{Fix}(T_2), \tag{1.6}$$
$$\text{and } Ax^* = By^*,$$

which is a generalization of a variational inequality studied in [27]. Now, let M, N be maximal monotone operators; by setting

$$T_1 = J_\lambda^M := (I_1 + \lambda M)^{-1},\ T_2 = J_\lambda^N := (I_2 + \lambda N)^{-1},$$
$$S_1 = I_1 - \gamma \nabla \psi_1 \text{ and } S_2 = I_2 - \gamma \nabla \psi_2,$$

where for each $i \in \{1, 2\}$, ψ_i is a convex function such that $\nabla \psi_i$ is η_i-Lipschitzian with $\gamma \in \left(0, \min\limits_{1 \leq i \leq 2} \frac{2}{\eta_i}\right]$, and using the fact that $\mathrm{Fix}(J_\lambda^M) = M^{-1}(0)$ and $\mathrm{Fix}(J_\lambda^N) = N^{-1}(0)$, the $\mathrm{S_pEHFPP}$(1.1)-(1.2) is reduced to the following new mathematical program with generalized equation constraints:

$$\begin{aligned} &\min_{0 \in M(x^*)} \psi_1(x^*), \\ &\min_{0 \in N(y^*)} \psi_2(y^*), \\ &\text{and } Ax^* = By^*, \end{aligned} \tag{1.7}$$

which is a generalization of the problem considered in [18].

Next, it is easy to observe that $\mathrm{S_pEHFPP}$(1.1)-(1.2) is equivalent to the fixed point problem: Find $x^* \in Fix(T_1)$ and $y^* \in Fix(T_2)$ such that

$$0 \in (I_1 - S_1)x^* + N_{\mathrm{Fix}(T_1)}(x^*), \tag{1.8}$$
$$0 \in (I_2 - S_2)y^* + N_{\mathrm{Fix}(T_2)}(y^*), \tag{1.9}$$
$$\text{and } Ax^* = By^*,$$

where $N_{\mathrm{Fix}(T_1)}$ denotes the normal cone to the closed convex set $\mathrm{Fix}(T_1)$.

In 2014, Moudafi [20] studied the weak convergence theorem for a new CQ algorithm for $\mathrm{S_pEP}$ (1.4). However, to employ Moudafi's CQ algorithm, one needs to know priori norms (or at least an estimate of the norms) of the bounded linear operators A and B which is in general not an easy work in practice. To overcome this difficulty, Lopez *et al.* [17] presented a helpful iterative method for estimating the stepsizes which do not need prior knowledge of the operator norms for solving the split feasibility problems; For recent work, see Qin and Yao [25]. Further, Dong *et al.* [9] extended this method for solving $\mathrm{S_pEP}$ (1.4); For recent work, see Eslamian et al. [11] and Cui et al.[8]. In 2015, Zhao [30] also extended the iterative method[17] for split equality fixed point problems ($\mathrm{S_pEFPP}$ (1.3)); See also [31]. Very recently, Chang et al.[7] studied $\mathrm{S_pEFPP}$ (1.3) for quasi-pseudo-contractive and L-Lipschitizan mappings.

On the other hand, it is known that some algorithms in signal processing and image reconstruction may be written as the Krasnoselski-Mann algorithm and that the main feature of its corresponding convergence theorems provides a unified frame for analysing various concrete algorithms; see for instance [5, 28]. Motivated by these work, Moudafi [19] introduced the following Krasnoselski-Mann algorithm for solving HFPP(1.1):

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\sigma_n S_1 x_n + (1-\sigma_n)T_1 x_n),\ \forall n \geq 0, \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two sequences in $(0,1)$. He proved a weak convergence theorem for solving HFPP(1.1). For further related work, see for instance [13, 14, 10, 15, 16, 22, 23, 29].

It is worth mentioning that to develop an iterative method for estimating the step sizes which do not need prior knowledge of the operator norms for solving $\mathrm{S_pEHFPP}$(1.1)-(1.2) (which is a more general problem than HFPP(1.1) and to prove a weak convergence theorem for such iterative method, looks an interesting problem, and this is what motivates our work.

Motivated by the work of Moudafi [19, 20, 21] and Dong *et al.* [9], we propose and analyze a simultaneous Krasnoselski-Mann algorithm for solving $\mathrm{S_pEHFPP}$(1.1)-(1.2), where the step sizes do not depend on the operator norms $\|A\|$ and $\|B\|$. Further, we prove the weak convergence of the sequence generated by this algorithm. Furthermore, we give some examples to justify the main result. Finally, we show that our purposed iterative algorithm is more efficient than some other known iterative algorithms. The framework is general enough and allows us to treat in a unified way several iterative algorithms, recovering, developing and improving some recently known related convergence results in the literature.

2. Preliminaries

Throughout the paper, we denote the strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let us recall the following concepts which are of common use in the context of convex and nonlinear analysis.

Definition 2.1. See [4]. An operator $M : H_1 \to 2^{H_1}$ is said to be:

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0,\ \textit{whenever}\ u \in M(x),\ v \in M(y);$$

(ii) maximal monotone if M is monotone and the graph, $\mathrm{graph}(M) := \{(x,y) \in H_1 \times H_1 : y \in M(x)\}$, is not properly contained in the graph of any other monotone operator.

Remark 2.1. It is well known that if T_1 is a nonexpansive mapping on H_1, then $I - T_1$ is a maximal monotone operator on H_1, (see Example 20.26; pp. 298 [3]).

Remark 2.2. It is also well known that if M is maximal monotone then for each $x \in H_1$ and $\lambda > 0$ there is a unique $z \in H_1$ such that $x \in (I + \lambda M)z$. The operator $J_\lambda^M := (I+\lambda M)^{-1}$ is called the resolvent of M. It is a single valued and nonexpansive mapping defined on H_1.

Lemma 2.1. (i) *For all $x, y \in H_1$, we have*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle; \tag{2.1}$$

(ii) *We have*

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \forall x, y \in H_1; \tag{2.2}$$

(iii) *Every Hilbert space H_1 satisfies the Opial condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality* $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, *holds for every $y \in H_1$ with $y \neq x$, see* [24].

Lemma 2.2. [24] *(Opial's lemma) Let H_1 be a Hilbert space and $\{\mu_n\}$ be a sequence in H_1 such that there exists a nonempty closed set $W \subset H_1$ satisfying:*

(i) *For every $\mu \in W$, $\lim_{n\to\infty} \|\mu_n - \mu\|$ exists,*

(ii) *Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W;*

Then there exists $\mu^* \in W$ such that $\{\mu_n\}$ converges weakly to μ^*.

3. Simultaneous Krasnoselski-Mann iterative algorithm

We suggest a simultaneous Krasnoselski-Mann iterative algorithm where the step-sizes do not depend on the operator norms $\|A\|$ and $\|B\|$, to approximate a solution to $\mathrm{S_pEHFPP}$(1.1)-(1.2).

Algorithm 3.1. Choose initial guesses $x_0 \in H_1, y_0 \in H_2$ arbitrarily. Let $\{\alpha_n\} \subset (0, 1)$ and $\{\sigma_n\} \subset (0, 1)$. Let the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:

$$\left.\begin{array}{lcl} u_n & = & (1-\alpha_n)x_n + \alpha_n(\sigma_n S_1 x_n + (1-\sigma_n)T_1 x_n); \\ v_n & = & (1-\alpha_n)y_n + \alpha_n(\sigma_n S_2 y_n + (1-\sigma_n)T_2 y_n); \\ x_{n+1} & = & u_n - \gamma_n A^*(Au_n - Bv_n); \\ y_{n+1} & = & v_n + \gamma_n B^*(Au_n - Bv_n), \end{array}\right\} \tag{3.1}$$

for all $n \geq 0$, where the step size γ_n is chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in (\epsilon, \mu_n - \epsilon)\,,\ n \in \Lambda, \tag{3.2}$$

otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), where

$$\mu_n := \frac{2\|Au_n - Bv_n\|^2}{\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2} \tag{3.3}$$

and the index set $\Lambda := \{n : Au_n - Bv_n \neq 0\}$.

Remark 3.1 ([30]). It follows from condition (3.2)-(3.3) that $\inf_{n\in\Lambda}\{\mu_n - \gamma_n\} > 0$. Since

$$\|A^*(Au_n - Bv_n)\| \leq \|A^*\|\|Au_n - Bv_n\|$$

and

$$\|B^*(Au_n - Bv_n)\| \leq \|B^*\|\|Au_n - Bv_n\|,$$

we observe that $\{\mu_n\}$ is bounded below by $\frac{2}{\|A\|^2+\|B\|^2}$ and so $\inf_{n\in\Lambda}\mu_n > 0$. Consequently, with an appropriate choice of $\epsilon > 0$ and $\gamma_n \in (\epsilon, \inf_{n\in\Lambda}\mu_n - \epsilon)$ for $n \in \Lambda$, we have $\sup_{n\in\Lambda}\gamma_n < +\infty$ and hence $\{\gamma_n\}$ is bounded.

4. Main result

In this section, we prove that the iterative sequence $\{(x_n, y_n)\}$ generated by Algorithm (3.1) is weakly convergent to a solution to $\mathrm{S_pEHFPP}$(1.1)-(1.2) for nonexpansive mappings.

Assume that $\Gamma \neq \emptyset$.

Theorem 4.1. *Let* H_1, H_2 and H_3 *be real Hilbert spaces and let* $A : H_1 \to H_3$, $B : H_2 \to H_3$ *be bounded linear operators. Let* $T_1, S_1 : H_1 \to H_1$ *and* $T_2, S_2 : H_2 \to H_2$ *be nonexpansive mappings. Assume that* $\Theta = (\mathrm{Fix}(S_1) \cap \mathrm{Fix}(T_1), \mathrm{Fix}(S_2) \cap \mathrm{Fix}(T_2))$ *with* $\mathrm{Fix}(S_1) \cap \mathrm{Fix}(T_1) \neq \emptyset, \mathrm{Fix}(S_2) \cap \mathrm{Fix}(T_2) \neq \emptyset$. *Let the sequences* $\{(x_n, y_n)\}$ *be generated by Algorithm* 3.1 *and the sequences of real numbers* $\{\alpha_n\} \in [c, 1),\ c \in (0, 1),\ \{\sigma_n\} \in [a, b] \subset (0, 1)$. *Then the sequence* $\{(x_n, y_n)\}$ *converges weakly to a point* $(\bar{x}, \bar{y})$ *of* Γ.

Proof. Suppose that $(x^*, y^*) \in \Theta$. We estimate

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|(1-\alpha_n)x_n + \alpha_n(\sigma_n S_1 x_n + (1-\sigma_n)T_1 x_n) - x^*\|^2 \\
&= \|(1-\alpha_n)(x_n - x^*) + \alpha_n\left(\sigma_n(S_1 x_n - x^*) + (1-\sigma_n)(T_1 x_n - x^*)\right)\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(\sigma_n\|S_1 x_n - x^*\|^2 + (1-\sigma_n)\|T_1 x_n - x^*\|^2 \\
&\quad -\sigma_n(1-\sigma_n)\|S_1 x_n - T_1 x_n\|^2) \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(\sigma_n\|x_n - x^*\|^2 + (1-\sigma_n)\|x_n - x^*\|^2 \\
&\quad -\sigma_n(1-\sigma_n)\|S_1 x_n - T_1 x_n\|^2) \\
&\leq \|x_n - x^*\|^2 - \alpha_n\sigma_n(1-\sigma_n)\|S_1 x_n - T_1 x_n\|^2. \qquad (4.1)
\end{aligned}$$

Similarly, we get

$$\|v_n - y^*\|^2 \leq \|y_n - y^*\|^2 - \alpha_n\sigma_n(1-\sigma_n)\|S_2 y_n - T_2 y_n\|^2. \qquad (4.2)$$

Adding (4.1) and (4.2), we get

$$\begin{aligned}
\|u_n - x^*\|^2 \quad + \quad & \|v_n - y^*\|^2 \leq (\|x_n - x^*\|^2 + \|y_n - y^*\|^2) \\
& -\alpha_n\sigma_n(1-\sigma_n)(\|S_1 x_n - T_1 x_n\|^2 + \|S_2 y_n - T_2 y_n\|^2). \qquad (4.3)
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|u_n - \gamma_n A^*(Au_n - Bv_n)) - x^*\|^2 \\
&= \|u_n - x^*\|^2 - 2\gamma_n\langle u_n - x^*, A^*(Au_n - Bv_n)\rangle + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \\
&= \|u_n - x^*\|^2 - 2\gamma_n\langle Au_n - Ax^*, Au_n - Bv_n\rangle + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2 \qquad (4.4) \\
&\leq \|u_n - x^*\|^2 + 2\gamma_n\|Au_n - Ax^*\|\|Au_n - Bv_n\| + \gamma_n^2\|A^*(Au_n - Bv_n)\|^2. \qquad (4.5)
\end{aligned}$$

Now, using (2.2) in (4.4), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|u_n-x^*\|^2-\gamma_n\|Au_n-Ax^*\|^2-\gamma_n\|Au_n-Bv_n\|^2\\ &\quad+\gamma_n\|Bv_n-Ax^*\|^2+\gamma_n^2\|A^*(Au_n-Bv_n)\|^2. \end{aligned}\tag{4.6}$$

In a similar way as (4.6), we obtain

$$\begin{aligned}\|y_{n+1}-y^*\|^2 &= \|v_n-y^*\|^2-\gamma_n\|Bv_n-By^*\|^2-\gamma_n\|Au_n-Bv_n\|^2\\ &\quad+\gamma_n\|Au_n-By^*\|^2+\gamma_n^2\|B^*(Au_n-Bv_n)\|^2. \end{aligned}\tag{4.7}$$

Adding (4.6) and (4.7), and using the fact that $Ax^*=By^*$, we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &= \|u_n-x^*\|^2+\|v_n-y^*\|^2\\ &\quad-\gamma_n[2\|Au_n-Bv_n\|^2-\gamma_n(\|A^*(Au_n-Bv_n)\|^2\\ &\quad+\|B^*(Au_n-Bv_n)\|^2)]. \end{aligned}\tag{4.8}$$

Using (4.3) in (4.8), we get

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\leq (\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &\quad-\alpha_n\sigma_n(1-\sigma_n)(\|S_1x_n-T_1x_n\|^2+\|S_2y_n-T_2y_n\|^2)\\ &\quad-\gamma_n[2\|Au_n-Bv_n\|^2-\gamma_n(\|A^*(Au_n-Bv_n)\|^2\\ &\quad+\|B^*(Au_n-Bv_n)\|^2)]. \end{aligned}\tag{4.9}$$

Now, setting $\rho_n(x^*,y^*):=\|x_n-x^*\|^2+\|y_n-y^*\|^2$ in (4.9), we obtain

$$\begin{aligned}\rho_{n+1}(x^*,y^*) &\leq \rho_n(x^*,y^*)-\alpha_n\sigma_n(1-\sigma_n)(\|S_1x_n-T_1x_n\|^2+\|S_2y_n-T_2y_n\|^2)\\ &\quad-\gamma_n[2\|Au_n-Bv_n\|^2-\gamma_n(\|A^*(Au_n-Bv_n)\|^2\\ &\quad+\|B^*(Au_n-Bv_n)\|^2)]. \end{aligned}\tag{4.10}$$

From the condition (3.2)-(3.3) on γ_n, we observe that the sequence $\{\rho_n(x,y)\}$ being decreasing and lower bounded by 0, therefore it converges to some finite limit, say $\rho(x,y)$. Thus condition (i) of Lemma 2.2 is satisfied with $\mu_n=(x_n,y_n)$, $\mu^*=(x,y)$ and $W=\Theta$.

Since $\|x_n-x^*\|^2\leq\rho_n(x^*,y^*)$, $\|y_n-y^*\|^2\leq\rho_n(x^*,y^*)$ and $\lim\limits_{n\to\infty}\rho_n(x^*,y^*)$ exists, we observe that $\{x_n\}$ and $\{y_n\}$ are bounded and $\limsup\limits_{n\to\infty}\|x_n-x^*\|$ and $\limsup\limits_{n\to\infty}\|y_n-y^*\|$ exist. From (4.1) and (4.2), we have that $\limsup\limits_{n\to\infty}\|u_n-x^*\|$ and $\limsup\limits_{n\to\infty}\|v_n-y^*\|$ also exist. Now, let $\bar{x}$ and $\bar{y}$ be weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. From Lemma 2.1(i), we have

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|x_{n+1}-x^*-x_n+x^*\|^2\\ &= \|x_{n+1}-x^*\|^2-\|x_n-x^*\|^2-2\langle x_{n+1}-x_n,x_n-x^*\rangle\\ &= \|x_{n+1}-x^*\|^2-\|x_n-x^*\|^2-2\langle x_{n+1}-\bar{x},x_n-x^*\rangle\\ &\quad+2\langle x_n-\bar{x},x_n-x^*\rangle. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.11}$$

Similarly, we have

$$\limsup_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{4.12}$$

Further, it follows from (4.11) and (4.12) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{4.13}$$

and

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{4.14}$$

Since (4.10) holds and $\lim_{n\to\infty} \rho_n(x^*, y^*)$ exists, it follows from (3.2)-(3.3) that

$$\lim_{n\to\infty} (\|A^*(Au_n - Bv_n)\|^2 + \|B^*(Au_n - Bv_n)\|^2) = 0$$

and hence

$$\lim_{n\to\infty} \|A^*(Au_n - Bv_n)\| = \lim_{n\to\infty} \|B^*(Au_n - Bv_n)\| = 0. \tag{4.15}$$

Similarly, from assumption on $\{\alpha_n\}$, $\{\sigma_n\}$ and (4.10), we observe that

$$\lim_{n\to\infty} \|S_1x_n - T_1x_n\| = \lim_{n\to\infty} \|S_2y_n - T_2y_n\| = 0. \tag{4.16}$$

Further, it follows from (4.10), (4.15), (4.16) and the facts that $\lim_{n\to\infty} \rho_n(x^*, y^*)$ exists and $\{\gamma_n\}$ is bounded, that

$$\lim_{n\to\infty} \|Au_n - Bv_n\| = 0. \tag{4.17}$$

Again, since $\{\gamma_n\}$ is bounded and

$$\|u_n - x_{n+1}\| = \gamma_n\|A^*(Au_n - Bv_n)\|,$$

we have

$$\lim_{n\to\infty} \|u_n - x_{n+1}\| = 0. \tag{4.18}$$

Since

$$\|u_n - x_n\| \le \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\|, \tag{4.19}$$

Letting $n \to \infty$, and using (4.13) and (4.18) in the above inequalities, we get

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.20}$$

The relation

$$u_n = x_n - \alpha_n(x_n - T_1x_n) + \alpha_n\sigma_n(S_1x_n - T_1x_n)$$

implies that

$$\|x_n - T_1x_n\| \le \frac{\|x_n - u_n\|}{\alpha_n} + \sigma_n\|S_1x_n - T_1x_n\|. \tag{4.21}$$

Now, taking the limit as $n \to \infty$, using (4.16) and (4.21) in the above inequality, we obtain

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = 0. \tag{4.22}$$

From (4.16) and (4.22), we have

$$\lim_{n\to\infty} \|S_1x_n - x_n\| = 0. \tag{4.23}$$

Similarly, we obtain

$$\lim_{n\to\infty} \|v_n - y_{n+1}\| = 0, \tag{4.24}$$

$$\lim_{n\to\infty} \|v_n - y_n\| = 0, \tag{4.25}$$

and

$$\lim_{n\to\infty} \|y_n - T_2 y_n\| = 0. \tag{4.26}$$

$$\lim_{n\to\infty} \|S_2 y_n - y_n\| = 0. \tag{4.27}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ and $y_{n_i} \rightharpoonup \bar{y}$. Since $x_{n_i} \rightharpoonup \bar{x}$, if $T_1\bar{x} \neq \bar{x}$, by Lemma 2.1(iii) and (4.22), we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_{n_i} - \bar{x}\| &< \liminf_{n\to\infty} \|x_{n_i} - T_1\bar{x}\| \\
&\leq \liminf_{n\to\infty} (\|x_{n_i} - T_1 x_{n_i}\| + \|T_1 x_{n_i} - T_1\bar{x}\|) \\
&\leq \liminf_{n\to\infty} \|x_{n_i} - \bar{x}\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $\bar{x} \in \mathrm{Fix}(T_1)$. Similarly, we can obtain $\bar{y} \in \mathrm{Fix}(T_2)$. Since $\{x_n\}$ and $\{y_n\}$ have the same asymptotic behaviour as the sequences $\{u_n\}$ and $\{v_n\}$, respectively, there exist subsequences $\{u_{n_i}\}$ of $\{u_n\}$ and $\{v_{n_i}\}$ of $\{v_n\}$ such that $u_{n_i} \rightharpoonup \bar{x}$ and $v_{n_i} \rightharpoonup \bar{y}$.

Further, since $\|\cdot\|^2$ is weakly lower semicontinuous, it follows from (4.17) that

$$\|A\bar{x} - B\bar{y}\|^2 \leq \liminf_{n\to\infty} \|Au_{n_i} - Bv_{n_i}\|^2 = 0, \tag{4.28}$$

i.e., $A\bar{x} = B\bar{y}$. Thus, $(\bar{x}, \bar{y}) \in \Theta$ and hence $w_w(x_n, y_n) \subset \Theta$. Now, it follows from Lemma 2.2 that the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges weakly to $(\bar{x}, \bar{y}) \in \Theta$.

Next, we show that $(\bar{x}, \bar{y}) \in \Gamma$. Since

$$u_n - x_n = \alpha_n(\sigma_n(S_1 x_n - x_n) + (1 - \sigma_n)(T_1 x_n - x_n)) \tag{4.29}$$

and hence

$$\frac{1}{\alpha_n \sigma_n} (x_n - u_n) = (I - S_1)x_n + \left(\frac{1 - \sigma_n}{\sigma_n}\right)(I - T_1)x_n, \tag{4.30}$$

and hence for all $z \in \mathrm{Fix}(T_1)$ and using monotonicity of $I - S_1$, we have

$$\begin{aligned}
\left\langle \frac{x_n - u_n}{\alpha_n \sigma_n}, x_n - z \right\rangle &= \langle (I - S_1)x_n - (I - S_1)z, x_n - z\rangle + \langle (I - S_1)z, x_n - z\rangle \\
&\quad + \frac{1 - \sigma_n}{\sigma_n} \langle x_n - T_1 x_n, x_n - z\rangle \\
&\geq \langle (I - S_1)z, x_n - z\rangle + \frac{1 - \sigma_n}{\sigma_n} \langle x_n - T_1 x_n, x_n - z\rangle. \qquad (4.31)
\end{aligned}$$

Using (4.20), (4.22), conditions on parameters α_n and σ_n in (4.31), we have

$$\overline{\lim_{n\to\infty}} \langle z - Sz, x_n - z\rangle \leq 0 \ \forall z \in \mathrm{Fix}(T_1). \tag{4.32}$$

Due to the fact that x_n weakly converges to $\bar{x}$, we have

$$\langle (I - S_1)z, \bar{x} - z\rangle \leq 0 \ \forall z \in \mathrm{Fix}(T_1). \tag{4.33}$$

Since $\mathrm{Fix}(T_1)$ is convex, $\lambda z + (1-\lambda)\bar{x} \in \mathrm{Fix}(T_1)$ for $\lambda \in (0,1)$ and hence

$$\begin{aligned} & \langle (I - S_1)(\lambda z + (1-\lambda)\hat{x}), \bar{x} - (\lambda z + (1-\lambda)\bar{x})\rangle && (4.34)\\ = \quad & \lambda\langle (I - S_1)(\lambda z + (1-\lambda)\bar{x}), \bar{x} - z\rangle && (4.35)\\ \leq \quad & 0 \ \forall z \in \mathrm{Fix}(T_1), && (4.36) \end{aligned}$$

which implies

$$\langle (I - S_1)(\lambda z + (1-\lambda)\bar{x}), \bar{x} - z\rangle \leq 0 \ \forall z \in \mathrm{Fix}(T_1).$$

On taking limits $\lambda \to 0_+$, we have

$$\langle (I - S_1)\bar{x}, \bar{x} - z\rangle \leq 0 \ \forall z \in \mathrm{Fix}(T_1). \tag{4.37}$$

That is $\bar{x}$ solves (1.1). Similarly, we can show that $\bar{y}$ solves (1.2). Thus, $(\bar{x}, \bar{y}) \in \Gamma$. This completes the proof. □

5. Consequences and applications

Besides some applications that were mentioned in the introduction, we now present some other applications and consequences of Theorems 4.1.

5.1. **Applications to maximal monotone operators and optimization.** Let H_1, H_2 and H_3 be real Hilbert spaces and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Let $F_1 : D(F_1) \subseteq H_1 \rightrightarrows H_1$ and $F_2 : D(F_2) \subseteq H_2 \rightrightarrows H_2$ be two maximal monotone operators.
We consider the following problem:

$$find\, x^* \in F_1^{-1}(0)\,, y^* \in F_2^{-1}(0) \ such\, that \ \ Ax^* = By^*. \tag{5.1}$$

We denote the solution set of (5.1) by Θ_1. Let $\lambda > 0$ be an arbitrary positive number. Denote by $T_1 := J_\lambda^{F_1}$ and $T_2 := J_\lambda^{F_2}$ the resolvent of F_1 and F_2, respectively. It is known that T_1 and T_2 are nonexpansive.

Theorem 5.1. *Let H_1, H_2 and H_3 be real Hilbert spaces and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Let $F_1 : D(F_1) \subseteq H_1 \rightrightarrows H_1$ and $F_2 : D(F_2) \subseteq H_2 \rightrightarrows H_2$ be two maximal monotone operators and let $T_1 := J_\lambda^{F_1}$ and $T_2 := J_\lambda^{F_2}$. Assume that $\Theta_1 \neq \emptyset$. Let the sequences $\{(x_n, y_n)\}$ be generated by Algorithm* 3.1 *with $S_1 = I_1, S_2 = I_2$, and the sequences of real numbers $\{\alpha_n\} \in [c,1)$, $c \in (0,1)$, $\{\sigma_n\} \in [a,b] \subset (0,1)$. Then the sequence $\{(x_n, y_n)\}$ converges weakly to a point $(\bar{x}, \bar{y})$ of Θ_1.*

Proof. Since the zero set of F_1 and F_2 coincides with the fixed point set of the resolvent of F_1 and F_2, respectively and the set of the fixed points of T_1 and T_2 coincides with the solution set of (5.1). The result follows from Theorem 4.1 with $S_1 = I_1$, $S_2 = I_2$. □

Remark 5.1. Let $f_1 : H_1 \longrightarrow (-\infty, +\infty]$ and $f_2 : H_2 \longrightarrow (-\infty, +\infty]$ be proper, convex and lower semicontinuous functions. We know that $F_1 = \partial f_1$ and $F_2 = \partial f_2$ are maximal monotone operators. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. We consider the following problem:

$$find\ x^* \in argminf_1\,, y^* \in argminf_2\ such\,that\ \ Ax^* = By^*. \tag{5.2}$$

The solution to the above problem solves the following problem:

$$\min_{x,y}\{f_1(x) + f_2(y) : Ax = By\}. \tag{5.3}$$

Therefore Theorem 5.1 provides a solution to the above problem.

5.2. **Applications to common fixed point problem.** Let H be a real Hilbert space and $T_1, T_2 : H \longrightarrow H$ be two nonexpansive mappings. By taking $H_1 = H_2 = H_3 = H$, and $S_1 = S_2 = A = B = I$, we have $\Theta_2 = \{(\bar{x}, \bar{x}) : \bar{x} \in Fix(T_1) \cap Fix(T_2)\}$ and we obtain the following common fixed point theorem.

Theorem 5.2. *Let H be a real Hilbert space and let $T_1, T_2 : H \to H$ be nonexpansive mappings. Assume that $\Theta_2 = Fix(T_1) \bigcap Fix(T_2) \neq \emptyset$. Let the sequences $\{(x_n, y_n)\}$ be generated by following algorithm:*

$$\begin{cases} u_n &= (1-\alpha_n)x_n + \alpha_n(\sigma_n x_n + (1-\sigma_n)T_1 x_n); \\ v_n &= (1-\alpha_n)y_n + \alpha_n(\sigma_n y_n + (1-\sigma_n)T_2 y_n); \\ x_{n+1} &= u_n - \gamma_n(u_n - v_n); \\ y_{n+1} &= v_n + \gamma_n(u_n - v_n), \end{cases}$$

where $\{\alpha_n\} \in [c, 1)$, $c \in (0, 1)$, $\{\sigma_n\} \in [a, b] \subset (0, 1)$ and step size γ_n is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in (\epsilon, 1-\epsilon)$. Then the sequence $\{(x_n, y_n)\}$ converges weakly to a point $(\bar{x}, \bar{x})$ of Θ_2.

5.3. **Applications to variational inequalities.** Let D_1 and D_2 be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2 respectively, and let $F_0 : D_1 \longrightarrow H_1$ and $G_0 : D_2 \longrightarrow H_2$ be two single-valued, monotone and hemicontinuous (i.e. continuous along each line segment in H_i with respect to the weak topology) mappings. Let $N_{D_i}(z)$ $(i = 1, 2)$ denote the normal cone to D_i at z:

$$N_{D_i}(z) := \{w \in H_i : \langle w, z - u\rangle \geq 0, \forall u \in D_i\},$$

and let $F : H_1 \longrightarrow H_1$ and $G : H_2 \longrightarrow H_2$ be defined by:

$$F(z) := \begin{cases} F_0(z) + N_{D_1}(z), & if z \in D_1, \\ \emptyset, & if z \notin D_1, \end{cases}$$

and

$$G(z) := \begin{cases} G_0(z) + N_{D_2}(z), & if z \in D_2, \\ \emptyset, & if z \notin D_2. \end{cases}$$

The maximal monotonicity of these multivalued mappings were proved by Rockafellar [26]. The relation $0 \in F(z)$ and $0 \in G(w)$ reduces to $-F_0(z) \in N_{D_1}(z)$ and $-G_0(w) \in N_{D_2}(w)$, or the so called variational inequality: find $(z, w) \in D_1 \times D_2$ such that

$$\langle z - u, F_0(z)\rangle \leq 0,\ \langle w - v, G_0(w)\rangle \leq 0, \forall u \in D_1 \text{ and } v \in D_2.$$

We define $VI(F_0, G_0, D)$ as follows:

$$VI(F_0, G_0, D) := \{(z, w) \in D_1 \times D_2 : \langle z - u, F_0(z)\rangle \leq 0,$$
$$\langle w - v, G_0(w)\rangle \leq 0, \ \ \forall u \in D_1, v \in D_2\}.$$

If D_1 and D_2 are cones, this condition can be written as

$$(z, w) \in D_1 \times D_2, -F_0(z) \in D_1^\circ, -G_0(w) \in D_2^\circ \ (\textit{the polar sets of} D_1 \textit{ and } D_2),$$
$$\textit{and} \ \langle z, F_0(z)\rangle = 0, \langle w, G_0(w)\rangle = 0,$$

and the problem of finding such z and w is an important instance of the well-known complementarity problem of mathematical programming. Then Theorem 4.1 provide an approximation scheme for a solution to the variational inequality for the single-valued, monotone and hemicontinuous maps $F_0 : D_1 \longrightarrow H_1$ and $G_0 : D_2 \longrightarrow H_2$.

Theorem 5.3. *Let H_1, H_2 and H_3 be real Hilbert spaces and D_1 and D_2 be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2 respectively, and let $F_0 : D_1 \longrightarrow H_1$ and $G_0 : D_2 \longrightarrow H_2$ be two single-valued, monotone and hemicontinuous mappings, and $N_{D_i}(z)$ be the normal cone to D_i at z. Also let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Suppose that $T_1 := J_\lambda^F$ and $T_2 := J_\lambda^G$ where F and G are defined as above and $\Theta_3 = \{(z, w) \in VI(F_0, G_0, D) : Az = Bw\} \neq \emptyset$. Let the sequences $\{(x_n, y_n)\}$ be generated by Algorithm 3.1 and the sequences of real numbers $\{\alpha_n\} \in [c, 1),\ c \in (0, 1),\ \{\sigma_n\} \in [a, b] \subset (0, 1)$. Then the sequence $\{(x_n, y_n)\}$ converges weakly to a point $(\bar{x}, \bar{x})$ of Θ_3.*

The result follows from Theorem 4.1 with $S_1 = I_1$, $S_2 = I_2$ and $T_1 := J_\lambda^F$, $T_2 := J_\lambda^G$.

Remark 5.2. By taking $H_1 = H_2 = H_3 = H$, $D_1 = D_2 = D$ and $A = B = I$ in the above theorem, we have

$$\Theta = VI(F_0, G_0, D) := \{(z, z) \in D \times D : \langle z - u, F_0(z)\rangle \leq 0,$$
$$\langle z - u, G_0(z)\rangle \leq 0, \ \ \forall u \in D\}$$

and

$$\Gamma := \{(x^*, x^*) \in VI(F_0, G_0, D) : x^* \in \mathrm{Sol(HFPP(1.1))}, x^* \in \mathrm{Sol(HFPP(1.2))}\}.$$

Therefore, the above theorem provides an approximation scheme to the solution of the following common variational inequality problem:

$$\textit{find } z \in D \textit{ such that } \langle z - u, F_0(z)\rangle \leq 0, \langle z - u, G_0(z)\rangle \leq 0, \ \textit{for all } u \in D.$$

6. Numerical examples

Now, we give some examples which justify Theorem 4.1.

Example 6.1. Let $H_1 = H_2 = H_3 = \ell_2$ be the space of all square summable sequences of real numbers, i.e.,

$$\ell_2 = \{x : x := \{x_1, x_2, \cdots, x_i, \cdots\} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\},$$

when an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by

$$\langle \cdot, \cdot \rangle = \sum_{i=1}^{\infty} x_i y_i,$$

where $x := \{x_i\}_{i=1}^{\infty},\ y := \{y_i\}_{i=1}^{\infty} \in \ell_2$ and $\| \cdot \| : \ell_2 \to \mathbb{R}$ defined by

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

Let the mappings $A : \ell_2 \to \ell_2$ and $B : \ell_2 \to \ell_2$ be defined by

$$A(x) = \{2x_1, 2x_2, \cdots, 2x_i, \cdots\},\ \forall x = \{x_i\}_{i=1}^{\infty} \in \ell_2$$

and

$$B(y) = \{-2y_1, -2y_2, \cdots, -2y_i, \cdots\},\ \forall y = \{y_i\}_{i=1}^{\infty} \in \ell_2,$$

respectively. Let the mappings $T_1 : \ell_2 \to \ell_2$, $S_1 : \ell_2 \to \ell_2$ be defined by

$$T_1 x = \left\{ \frac{x_1 + 2}{7}, \frac{x_2 + 2}{7}, \cdots, \frac{x_i + 2}{7}, \cdots \right\},$$

$$S_1 x = \{x_1, x_2 \cdots, x_i, \cdots\},\ \forall x = \{x_i\}_{i=1}^{\infty} \in \ell_2$$

and $T_2 : \ell_2 \to \ell_2$, $S_2 : \ell_2 \to \ell_2$ be defined by

$$T_2 y = \left\{ \frac{y_1 - 2}{7}, \frac{y_2 - 2}{7}, \cdots, \frac{y_i - 2}{7}, \cdots \right\},$$

$$S_2 y = \left\{ \frac{y_1 - 1}{4}, \frac{y_2 - 1}{4} \cdots, \frac{y_i - 1}{4}, \cdots \right\},\ \forall y = \{y_i\}_{i=1}^{\infty} \in \ell_2,$$

respectively. It is easy to observe that A and B are bounded linear operators on ℓ_2 with their adjoint operators $A^*,\ B^*$ and $\|A^*\| = \|A\| = 2, \|B^*\| = \|B\| = 2$. Further, the mappings $S_1, T_1, S_2,\ T_2$ are nonexpansive mappings with

$$\mathrm{Fix}(S_1) = \ell_2,\ \mathrm{Fix}(T_1) = \left\{ \frac{1}{3} \right\} = \left\{ \frac{1}{3}, \frac{1}{3}, \cdots, \frac{1}{3}, \cdots \right\},$$

$$\mathrm{Fix}(S_2) = \mathrm{Fix}(T_2) = \left\{ -\frac{1}{3} \right\} = \left\{ -\frac{1}{3}, -\frac{1}{3}, \cdots, -\frac{1}{3}, \cdots \right\}.$$

Thus the operators A, B, S_1, S_2, T_1, T_2 satisfy all conditions of Theorem 4.1. Now, from (1.1)-(1.2), we have to find that $x^* \in \mathrm{Fix}(T_1)$ and $y^* \in \mathrm{Fix}(T_2)$ such that

$$\begin{gathered} \langle x^* - x^*, x^* - x \rangle \leq 0,\ \forall x \in \mathrm{Fix}(T_1), \\ \langle 3y^* + 1, y^* - y \rangle \leq 0,\ \forall y \in \mathrm{Fix}(T_2) \\ \text{and } A\left\{ \frac{1}{3} \right\} = B\left\{ -\frac{1}{3} \right\}. \end{gathered}$$

This implies that $\Gamma = \mathrm{Sol}(\mathrm{S_pEHFPP}(1.1) - (1.2)) = \left(\frac{1}{3}, -\frac{1}{3}\right)$. In this case, Algorithm 3.1 is reduced to the following iterative algorithm:

Algorithm 6.1. Given initial value $x_1 = \{x_1^1, x_1^2, \cdots, x_1^i, \cdots\}$, $y_1 = \{y_1^1, y_1^2, \cdots, y_1^i, \cdots\}$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ be generated by the following schemes:

$$\begin{array}{lcl} u_n &=& (1-\alpha_n)x_n + \alpha_n(\sigma_n x_n + (1-\sigma_n)\frac{x_n+2}{7}); \\ v_n &=& (1-\alpha_n)y_n + \alpha_n(\sigma_n \frac{y_n-1}{4} + (1-\sigma_n)\frac{y_n-2}{7}); \\ x_{n+1} &=& u_n - 2\gamma_n(2u_n + 2v_n); \\ y_{n+1} &=& v_n - 2\gamma_n(2u_n + 2v_n), \end{array} \tag{6.1}$$

Now, setting $\alpha_n = \frac{0.99}{n^2}$ and $\sigma_n = \frac{0.99}{n}$, $\forall n \geq 1$ and $\gamma_n = 0.02 + \frac{0.02}{n}$, $\forall n$, then Theorem 4.1 implies that the iterative sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 6.1 converge to

$$x^* = \left\{\frac{1}{3}\right\} = \left\{\frac{1}{3}, \frac{1}{3}, \cdots, \frac{1}{3}, \cdots\right\},$$

$$y^* = \left\{-\frac{1}{3}\right\} = \left\{-\frac{1}{3}, -\frac{1}{3}, \cdots, -\frac{1}{3}, \cdots\right\},$$

respectively, such that $(x^*, y^*) = (\{\frac{1}{3}\}, \{-\frac{1}{3}\}) \in \Gamma$. We shall perform the computer programming in the next example in the setting of finite dimensional space.

Example 6.2. Let $H_1 = H_2 = H_3 = \mathbb{R}$, the set of all real numbers, with the induced usual norm $|\cdot|$. Let the mappings $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}$ be defined by

$$A(x) = 2x, \forall x \in \mathbb{R} \text{ and } B(y) = -2y, \forall y \in \mathbb{R},$$

respectively. Let the mappings $T_1 : \mathbb{R} \to \mathbb{R}$, $S_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_1 x = \frac{x+2}{7},\ S_1 x = x,\ \forall x \in \mathbb{R}$$

and $T_2 : \mathbb{R} \to \mathbb{R}$, $S_2 : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_2 y = \frac{y-2}{7},\ S_2 y = \frac{y-1}{4},\ \forall y \in \mathbb{R}$$

respectively. Setting $\{\alpha_n\} = \{\frac{0.99}{n^2}\}$ and $\{\sigma_n\} = \{\frac{0.99}{n}\}$, $\forall n \geq 1$. Then the sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 3.1 converge to $x^* = \frac{1}{3}$, $y^* = -\frac{1}{3}$, respectively, such that $(x^*, y^*) = (\frac{1}{3}, -\frac{1}{3}) \in \Gamma$.

Proof. It is easy to observe that A and B are bounded linear operators on $\mathbb{R}$ with adjoint operators A^*, B^* and $\|A\| = \|A^*\| = 2$, $\|B\| = \|B^*\| = 2$, and hence $\gamma_n \in \left(\epsilon, \frac{1}{9} - \epsilon\right)$. Therefore, for $\epsilon = \frac{1}{100}$, we choose $\gamma_n = 0.02 + \frac{0.02}{n}$, $\forall n$. Further, it is easy to observe that S_1, S_2 are nonexpansive mappings with

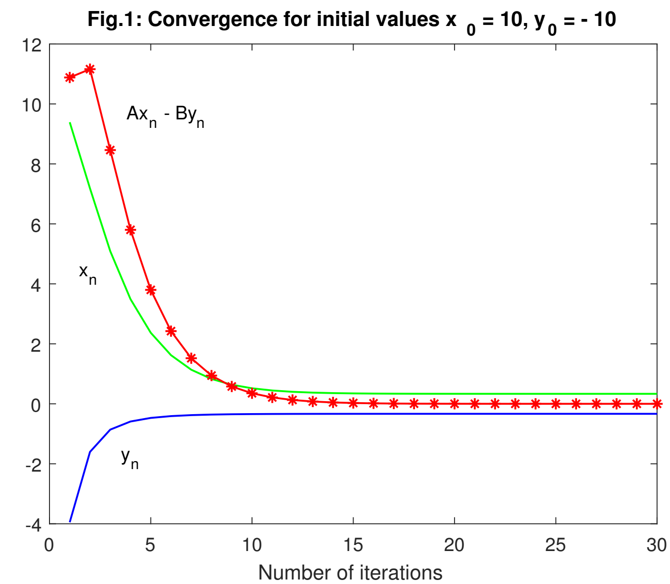
$$\text{Fix}(S_1) = H_1,\ \text{Fix}(S_2) = \left\{-\frac{1}{3}\right\}$$

and T_1, T_2 are nonexpansive mappings with

$$\text{Fix}(T_1) = \left\{\frac{1}{3}\right\},\ \text{Fix}(T_2) = \left\{-\frac{1}{3}\right\}.$$

Furthermore, it is easy to prove that $\Gamma = \text{Sol}(\text{S}_\text{p}\text{EHFPP}(1.1) - (1.2)) = \left(\frac{1}{3}, -\frac{1}{3}\right)$. Next, using the software Matlab 7.8, we have the following figures (Fig.1, Fig.2) and

Table 1, which show that the sequences $\{x_n\}, \{y_n\}$ converge to $x^* = \frac{1}{3}$, $y^* = -\frac{1}{3}$, respectively, such that $(x^*, y^*) = (\frac{1}{3}, -\frac{1}{3}) \in \Gamma$. $\square$



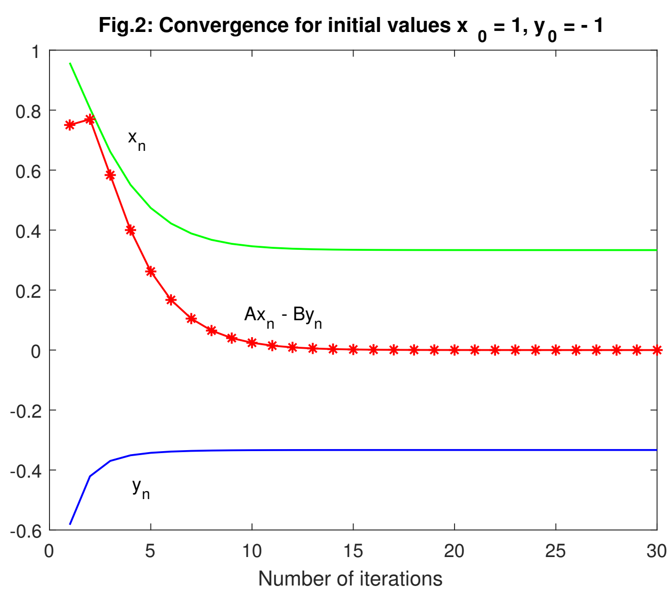


Table 1

No. of iterations	x_n $x_0 = 10$	y_n $y_0 = -10$	$Ax_n - By_n$	x_n $x_0 = 1$	y_n $y_0 = -1$	$Ax_n - By_n$
1	9.390528	-3.949162	10.882731	0.957967	-0.582701	0.750533
2	7.183582	-1.603348	11.160468	0.805764	-0.420921	0.769687
3	5.091249	-0.860112	8.462273	0.661465	-0.369663	0.583605
4	3.490858	-0.589547	5.802621	0.551094	-0.351003	0.400181
5	2.370129	-0.470102	3.800054	0.473802	-0.342766	0.262073
6	1.621913	-0.409685	2.424455	0.422201	-0.338599	0.167204
7	1.137101	-0.376858	1.520486	0.388766	-0.336335	0.104861
8	0.829323	-0.358402	0.941843	0.367540	-0.335062	0.064955
9	0.636816	-0.347850	0.577932	0.354263	-0.334334	0.039857
10	0.517760	-0.341765	0.351990	0.346052	-0.333915	0.024275
15	0.347597	-0.333898	0.027398	0.334317	-0.333372	0.001890
20	0.334362	-0.333371	0.001983	0.333404	-0.333336	0.000137
25	0.333405	-0.333336	0.000138	0.333338	-0.333334	0.000009
29	0.333342	-0.333334	0.000016	0.333334	-0.333333	0.000001
30	0.333338	-0.333333	0.000009	0.333334	-0.333333	0.000001

It is worth mentioning that if we set $S_1 = I_1,\ S_2 = I_2$ (I_1, I_2 are identity operators on H_1, H_2, respectively) in Theorem 4.1 then the sequence $\{(x_n, y_n)\}$ converges weakly to a point $(x^*, y^8) \in \Omega$, a solution of split equality fixed point problem, $\mathrm{S_pEFPP}$(1.3). In this case, Algorithm 3.1 with $\sigma_n = 0, \forall n$, reduces to the following algorithm:

Algorithm 6.2. Choose initial guesses $x_0 \in H_1, y_0 \in H_2$ arbitrarily.
Let $\{\alpha_n\} \subset (0,1)$. Let the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:

$$\begin{cases} u_n &= (1-\alpha_n)x_n + \alpha_n T_1 x_n; \\ v_n &= (1-\alpha_n)y_n + \alpha_n T_2 y_n; \\ x_{n+1} &= u_n - \gamma_n A^*(Au_n - Bv_n); \\ y_{n+1} &= v_n + \gamma_n B^*(Au_n - Bv_n), \end{cases} \tag{6.2}$$

for all $n \geq 0$, where the step size γ_n is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in (\epsilon, \mu_n - \epsilon)\,,\ n \in \Lambda$, otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), where μ_n is given by (3.3)

Now, we demonstrate that Algorithm 6.2 with conditions given in Theorem 4.1, approximate a solution to $\mathrm{S_pEFPP}$(1.3) for nonexpansive mappings. We also observe that this is more efficient than the iterative algorithm (3.1) due to [30] and the iterative algorithm (3.1) due to [31] for nonexpansive mappings which are as follows:

Algorithm 6.3. [30] Choose initial guesses $x_0 \in H_1, y_0 \in H_2$ arbitrarily.
Let $\{\alpha_n\} \subset (0,1)$. Let the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:

$$\begin{aligned} u_n &= x_n - \gamma_n A^*(Ax_n - By_n); \\ v_n &= y_n + \gamma_n A^*(Ax_n - By_n); \\ x_{n+1} &= \alpha_n u_n + (1-\alpha_n)T_1 u_n; \\ y_{n+1} &= \alpha_n v_n + (1-\alpha_n)T_2 v_n. \end{aligned} \tag{6.3}$$

for all $n \geq 0$, where the step size γ_n is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in (\epsilon, \mu_n - \epsilon)\,,\ n \in \Lambda$, otherwise $\gamma_n = \gamma$ ($\gamma \geq 0$), where μ_n is given by (3.3).

Algorithm 6.4. [31] Choose initial guesses $x_0 \in H_1, y_0 \in H_2$ arbitrarily. Let $\{\alpha_n\} \subset (0,1)$. Let the iteration sequence $\{(x_n, y_n)\}$ be generated by the scheme:

$$\begin{aligned} u_n &= x_n - \gamma_n A^*(Ax_n - By_n); \\ v_n &= y_n + \gamma_n A^*(Ax_n - By_n); \\ x_{n+1} &= T_1(\alpha_n v + (1-\alpha_n)u_n); \\ y_{n+1} &= T_2(\alpha_n v + (1-\alpha_n)v_n). \end{aligned} \tag{6.4}$$

for all $n \geq 0$, where the step size γ_n is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in (\epsilon, \mu_n - \epsilon)$, $n \in \Lambda$, otherwise $\gamma_n = \gamma$ $(\gamma \geq 0)$, where μ_n is given by (3.3)

If the operators A, B, T_1, T_2 and the control sequences $\{\alpha_n\}$, $\{\gamma_n\}$ are same as in Example 6.2, we can easily observe that the sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 6.2, Algorithm 6.3 and Algorithm 6.4 converge to $x^* = \frac{1}{3}$, $y^* = -\frac{1}{3}$, such that $(x^*, y^*) = (\frac{1}{3}, -\frac{1}{3})$ is a solution to $\mathrm{S_pEFPP}(1.3)$.
Finally, using the software Matlab 7.8, we have following figure (Fig.3) and Table 2, which show that the sequences $\{x_n\}, \{y_n\}$ converge to $(x^*, y^*) = (\frac{1}{3}, -\frac{1}{3})$. It is evident from figures and table that the sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 6.2 converge faster than the sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 6.3 and Algorithm 6.4.

Fig.3: Convergence for initial values x 0 = 1, y 0 = - 1

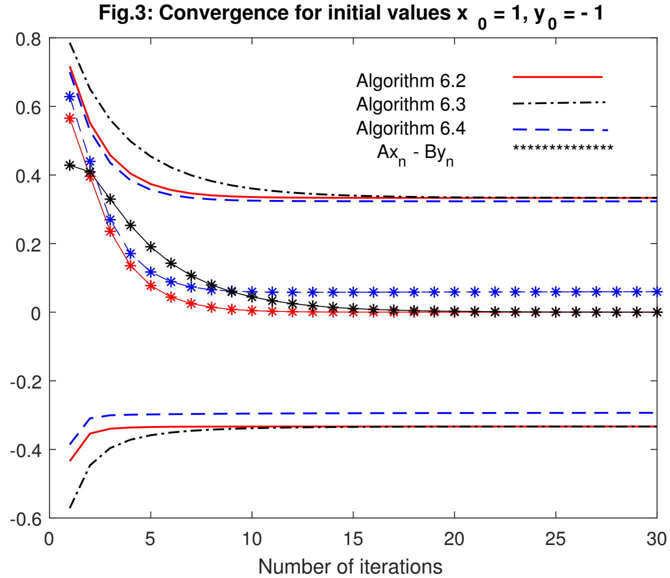


Table 2

No.of iterations	x_n for (6.2)	y_n for (6.2)	$Ax_n - By_n$ for (6.2)	x_n for (6.3)	y_n for (6.3)	$Ax_n - By_n$ for (6.3)	x_n for (6.4)	y_n for (6.4)	$Ax_n - By_n$ for (6.4)
1	0.717143	-0.434286	0.565714	0.785714	-0.571429	0.428571	0.700000	-0.385714	0.628571
2	0.551728	-0.353760	0.395935	0.650952	-0.446395	0.409116	0.529156	-0.309333	0.439646
3	0.457467	-0.339624	0.235686	0.560811	-0.395957	0.329708	0.435369	-0.300525	0.269687
4	0.403907	-0.336070	0.135673	0.498313	-0.371758	0.253111	0.384337	-0.298889	0.170895
5	0.373470	-0.334734	0.077473	0.454031	-0.358620	0.190821	0.356650	-0.298033	0.117233
6	0.356167	-0.334093	0.044148	0.422200	-0.350778	0.142845	0.341638	-0.297352	0.088573
7	0.346326	-0.333754	0.025145	.399086	-0.345749	0.106675	0.333490	-0.296783	0.073414
8	0.340728	-0.333568	0.014319	0.382174	-0.342351	0.079645	0.329054	-0.296308	0.065492
9	0.337542	-0.333465	0.008154	0.369726	-0.339971	0.059510	0.326626	-0.295910	0.061432
10	0.335729	-0.333408	0.004643	0.360521	-0.338262	0.044519	0.325284	-0.295575	0.059418
15	0.333477	-0.333338	0.000278	0.339835	-0.334514	0.010643	0.323587	-0.294496	0.058183
20	0.333342	-0.333334	0.000017	0.334933	-0.333626	0.002613	0.323362	-0.293925	0.058874
25	0.333334	-0.333333	0.000001	0.333733	-0.333407	0.000652	0.323263	-0.293576	0.059375
29	0.333333	-0.333333	0.000000	.333466	-0.333358	0.000217	0.323210	-0.293380	0.059660
30	0.333333	-0.333333	0.000000	0.333434	-0.333352	0.000165	0.323199	-0.293339	0.059720

Remark 6.1. It is of further research effort to study the split equality hierarchical fixed point problem, $\mathrm{S_pEHFPP}$(1.1)-(1.2), in the setting of Banach spaces.

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