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EXISTENCE OF SOLUTION OF FUNCTIONAL INTEGRAL EQUATIONS BY MEASURE OF NONCOMPACTNESS AND SINC INTERPOLATION TO FIND SOLUTION

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Abstract. In this article, we discuss the existence of solution for the functional integral equations in the Banach space $BC(\mathbb{R}^+)$ of real-valued continuous and bounded functions, using the method associated with the technique of measure of noncompactness and generalized Darbo fixed point theorem. We provide an example to illustrate our results, and we make an iterative algorithm by the Sinc interpolation to find solution to the above problem with acceptable accuracy.
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1. Introduction

The functional integral equations are important; and usually occur in different branches of sciences such as mechanics, mathematical physics, etc. The measure of noncompactness is an effective tool for operator theory and metric fixed point theory in Banach spaces to prove the existence of the solution of integral equations. For samples, see [1, 18, 19, 20, 21, 22] to prove the existence of solution for singular Volterra integral equations, functional integral equations, nonlinear quadratic integral equations, and nonlinear Volterra type integral equations. Therefore, measures of noncompactness to generalize the Darbo fixed point theorem were studied by many other authors [5, 6, 15, 27, 28]. We decide to generalize Darbo fixed point theorem [10] and extend some results that obtained by Aghajani et al. [3]. Also, we prove existence of solution of functional integral equations as follows

$$x(t) = h(t) + u(t, x(\alpha_1(t))) + g(t, x(\alpha_2(t))) \int_0^{\alpha_3(t)} k(t, s, x(\alpha_4(s)))ds, \qquad (1.1)$$

for $t \in \mathbb{R}^+, x \in \mathcal{F} = BC(\mathbb{R}^+)$ the space of real valued continuous and bounded functions on $\mathbb{R}^+$.

2. Preliminaries

We recall some preliminary concepts. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ = [0, +\infty)$, $(E, \|\cdot\|)$ be a real Banach space with zero element θ, and $\overline{B}(x, r)$ be closed ball. Suppose $\overline{B}(\theta, r)$ is shown by $\overline{B}_r$ briefly. Also for $\emptyset \neq X \subseteq E$, $\overline{X}$ and $\mathrm{Conv}X$ are closure and convex closure of X, respectively. Moreover, let $\emptyset \neq \mathfrak{M}_E$ be the family of all nonempty bounded sets of E, $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets, and $\emptyset \neq \mathfrak{F}_E$ be a family of nonempty, bounded, closed and convex subsets of the Banach space E.

Definition 2.1. [10] A mapping $\eta : \mathfrak{M}_E \to \mathbb{R}^+$ is a measure of noncompactness in E if it satisfies the following conditions:

(1^0) The family $ker\eta = \{X_1 \in \mathfrak{M}_E : \eta(X_1) = 0\}$ is nonempty and $\ker\eta \subset \mathfrak{N}_E$,
(2^0) $X_1 \subset X_2 \implies \eta(X_1) \leq \eta(X_2)$,
(3^0) $\eta(\bar{X_1}) = \eta(X_1)$,
(4^0) $\eta(\mathrm{Conv}X_1) = \eta(X_1)$,
(5^0) $\eta(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\eta(X_1) + (1-\lambda)\eta(X_2)$ for $\lambda \in [0, 1]$,
(6^0) if (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n (n = 1, 2, ...)$ and $\lim\limits_{n\to\infty} \eta(X_n) = 0$, then $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty,

where $\ker\eta$ is called the kernel of measure of noncompactness η and inequality $\eta(X_\infty) \leq \eta(X_n)$ for $n = 1, 2, 3, ...,$ concludes that $\eta(X_\infty) = 0$, thus $X_\infty \in \ker\eta$. To find some other properties see [9, 10].

Darbo fixed point theorem introduced in [14] is a generalization of Schauder fixed point theorem, which includes a kind of Banach fixed point theorem.

Theorem 2.2. [1] *(Schauder fixed point theorem) Let C be a closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

Theorem 2.3. [14] *Let $\zeta \in \mathfrak{F}_E$ and $\tau : \zeta \to \zeta$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\eta(\tau\zeta) \leq k\eta(\zeta)$. Then τ has a fixed-point in the set ζ.*

Now, we consider a fixed point theorem of Darbo type was introduced by Banaś and Goebel [10].

Theorem 2.4. *Let $\zeta \in \mathfrak{F}_E$ and $\tau : \zeta \to \zeta$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\eta(\tau X) \leq k\eta(X)$ for any X nonempty subset of ζ. Then τ has a fixed point in the set ζ.*

Also, some generalizations of Darbo fixed point theorem were constructed by Aghajani et al in [3], as follows:

Theorem 2.5. *Let $\zeta \in \mathfrak{F}_E$ and let $\tau : \zeta \longrightarrow \zeta$ be a continuous mapping such that*

$$\forall\ \emptyset \neq X \subseteq \zeta,\ \eta(\tau X) \leq \varphi(\eta(X)) \tag{2.1}$$

where η *is an arbitrary measure of noncompactness and* $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a nondecreasing function such that* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for each* $t \geq 0$*. Then* τ *has at least one fixed point in the set* ζ*.*

Definition 2.6. [17] Let S be a class of functions such as $\alpha : [0,\infty) \longrightarrow [0,1)$ with condition $\alpha(t_n) \longrightarrow 1$ implies $t_n \longrightarrow 0$.

The main result in [2] is the following fixed point theorem.

Theorem 2.7. *Let* $\zeta \in \mathfrak{F}_E$ *and let* $\tau : \zeta \longrightarrow \zeta$ *be a continuous mapping such that*

$$\forall\ \emptyset \neq X \subseteq \zeta,\ \ \eta(\tau X) \leq \alpha(\eta(X))\ \eta(X) \tag{2.2}$$

where η *is an arbitrary measure of noncompactness and* $\alpha \in S$*. Then* τ *has at least one fixed point.*

Altun and Turkoglu [4] introduced the concept of the following function $\Lambda(f;.)$ and its examples.

Definition 2.8. Let $\tau([0,\infty))$ be the class of all functions $u : [0,\infty) \longrightarrow [0,\infty)$ and Θ be class of all operators

$$\Lambda(\bullet;.) : \tau([0,\infty)) \longrightarrow \tau([0,\infty)),\ u \rightarrow \Lambda(u;.)$$

satisfying the following conditions:

(i) $\Lambda(u;t) > 0$ for $t > 0$ and $\Lambda(u;0) = 0$,
(ii) $\Lambda(u;t) \leq \Lambda(u;s)$ for $t \leq s$,
(iii) $\lim_{n\to\infty} \Lambda(u;t_n) = \Lambda(u; \lim_{n\to\infty} t_n)$,
(iv) $\Lambda(u;\max\{t,s\}) = \max\{\Lambda(u;t), \Lambda(u;s)\}$ for some $u \in \tau([0,\infty))$.

Example 2.1. If $u : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue integrable mapping which for each $t > 0$, $\int_0^t u(s)ds > 0$, then the operator defined by

$$\Lambda(u;t) = \int_0^t u(s)ds,$$

satisfies in $(i) - (iv)$ conditions.

3. A few generalizations of Darbo fixed point theorem

In this section, we prove a few generalizations of Darbo fixed point theorem.

Theorem 3.1. *Let* $\zeta \in \mathfrak{F}_E$ *and* $T : \zeta \longrightarrow \zeta$ *be a continuous operator such that*

$$\forall X \subseteq \zeta, \Lambda(\bullet;.) \in \Theta,\ \ \Lambda(u;\eta(T(X))) \leq \varphi(\Lambda(u;\eta(X)))), \tag{3.1}$$

where η *is an arbitrary measure of noncompactness,* $\varphi : [0,\infty) \rightarrow [0,\infty)$ *is a nondecreasing function such that* $\lim_{n\to\infty} \varphi^n(t) = 0$ *for each* $t \geq 0$*. Then* T *has at least one fixed point in* ζ*.*

Proof. Let $\zeta_0 = \zeta$. We make a sequence $\{\zeta_n\}$ such that $\zeta_{n+1} = \mathrm{Conv}(T\zeta_n)$, for $n \geq 0$. $T\zeta_0 = T\zeta \subseteq \zeta = \zeta_0, \zeta_1 = \mathrm{Conv}(T\zeta_0) \subseteq \zeta = \zeta_0$, and iterating the above process implies that

$$\zeta_0 \supseteq \zeta_1 \supseteq \ldots \supseteq \zeta_n \supseteq \zeta_{n+1} \supseteq \cdots$$

If there exists $N \in \mathbb{N}$ such that $\eta(\zeta_N) = 0$, then ζ_N is compact and Theorem 2.2 concludes that T has a fixed point. Otherwise we assume $\eta(\zeta_n) \neq 0$ for $n = 0, 1, 2, \ldots$

From (3.1) we have

$$\begin{aligned}\Lambda(u;\eta(\zeta_{n+1})) &= \Lambda(u;\eta(\mathrm{Conv}(T\zeta_n))) = \Lambda(u;\eta(T\zeta_n)) \\ &\leq \varphi(\Lambda(u;\eta(\zeta_n))) \\ &\leq \varphi^2(\Lambda(u;\eta(\zeta_{n-1}))) \\ &\ \vdots \\ &\leq \varphi^n(\Lambda(u;\eta(\zeta_0))) = \varphi^n(\Lambda(u;\eta(\zeta))).\end{aligned} \tag{3.2}$$

By the properties of φ and (iii) in Definition 2.8 we can write,

$$\lim_{n\to\infty} \Lambda(u;\eta(\zeta_{n+1})) = \Lambda(u;\lim_{n\to\infty} \eta(\zeta_{n+1})) = 0,$$

then by the property (i) we conclude that

$$\lim_{n\to\infty} \eta(\zeta_{n+1})) = 0.$$

Because $\zeta_n \supseteq \zeta_{n+1}$ and $T\zeta_n \subseteq \zeta_n$ for all $n \in \mathbb{N}$, it follows from (6^0) that $\zeta_\infty = \bigcap_{n=1}^{\infty} \zeta_n$ is a nonempty, convex, and closed set, invariant under T and belongs to $ker\eta$. Therefore Theorem 2.2 completes the proof.

Remark 3.2. Obviously, Theorem 2.4 and Theorem 2.5 are special cases of Theorem 3.1.

Corollary 3.3. *Let $\zeta \in \mathfrak{F}_E$ and E be a Banach space, also $k \in (0,1)$ and $T : \zeta \longrightarrow \zeta$ be a continuous operator such that,*

$$\forall X \subseteq \zeta,\ \int_0^{\eta(T(X))} u(s)\ ds \leq k\ \int_0^{\eta(X)} u(s)\ ds,$$

where η is an arbitrary measure of noncompactness and $u : [0,\infty) \longrightarrow [0,\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, and for each $\epsilon > 0$, $\int_0^\epsilon u(s)\ ds > 0$. Then T has at least one fixed point in ζ.

Theorem 3.4. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \longrightarrow \zeta$ be a continuous operator such that*

$$\Lambda(u;\eta(T(X))) \leq \alpha(\eta(X))\Lambda(u;\eta(X)), \tag{3.3}$$

for each $X \subseteq \zeta$, $\alpha \in S$ and $\Lambda(\bullet;.) \in \Theta$, where η is an arbitrary measure of noncompactness, then T has at least one fixed point in ζ.

Proof. We define the same sequence $\{\zeta_n\}$ as in the proof of Theorem 3.1, so T has at least one fixed point in ζ. But if we assume $\eta(\zeta_n) \neq 0$ for $n = 0, 1, 2, \cdots$ From (3.3) we can write

$$\begin{aligned}\Lambda(u;\eta(\zeta_{n+1})) &= \Lambda(u;\eta(\mathrm{Conv}(T\zeta_n))) = \Lambda(u;\eta(T\zeta_n)) \\ &\leq \alpha(\eta(\zeta_n))\Lambda(u;\eta(\zeta_n)) \\ &\leq \Lambda(u;\eta(\zeta_n)).\end{aligned} \tag{3.4}$$

From inequality (3.4) and property (i) in Definition 2.8, indicate that $\{\Lambda(u;\eta(\zeta_n))\}$ is a non-increasing and nonnegative sequence, then it is a convergent sequence. Thus we can give,

$$\lim_{n\to\infty} \Lambda(u;\eta(\zeta_n)) = \Lambda(u;\lim_{n\to\infty}\eta(\zeta_n)) = r, r \geqslant 0$$

moreover, (3.4) and $\alpha \in S$ lead to,

$$\frac{\Lambda(u;\eta(\zeta_{n+1}))}{\Lambda(u;\eta(\zeta_n))} \leq \alpha(\eta(\zeta_n)) < 1.$$

If $r = 0$, then from (i) in Definition 2.8 concludes that $\lim_{n\to\infty}\eta(\zeta_n) = 0$.
In otherwise for $r > 0$ implies that,

$$\alpha(\eta(\zeta_n)) \to 1 \text{ } as \text{ } n \to \infty,$$

since $\alpha \in S$, then $\lim_{n\to\infty}\eta(\zeta_n) = 0$ and it leads to $r = 0$, which this is a contradiction. So $r = 0$ and there exists $N \in \mathbb{N}$ such that $\eta(\zeta_N) = 0$. On the other hand $\zeta_n \supseteq \zeta_{n+1}$ and $T\zeta_n \subseteq \zeta_n$ for $n \in \mathbb{N}$ then (6^0) in Definition 2.1 concludes that, $\zeta_\infty = \bigcap_{n=1}^{\infty}\zeta_n$ is a nonempty convex closed set, invariant under T and belongs to $ker\eta$. Also Theorem 2.2 completes the proof.

Remark 3.5. Obviously, Theorem 2.7 is a special case of Theorem 3.4.

Theorem 3.6. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \longrightarrow \zeta$ be a continuous operator such that*

$$\psi(\Lambda(u;\eta(T(X)))) \leq \psi(\Lambda(u;\eta(X))) - \phi(\Lambda(u;\eta(X))), \tag{3.5}$$

for any subset X of ζ and $\Lambda(\bullet;.) \in \Theta$, where η is an arbitrary measure of noncompactness, $\psi, \phi : [0,\infty) \to [0,\infty)$ are given functions such that ψ is continuous and ϕ is lower semi-continuous and $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has at least one fixed point in the set ζ.

Proof. We construct a sequence $\{\zeta_n\}$ similar to Theorem 3.1 with assumption, if there exists $N \in \mathbb{N}$ such that $\eta(\zeta_N) = 0$, then T has a fixed point. But by assuming $\eta(\zeta_n) \neq 0$ for $n = 0, 1, 2, ...\cdot$ From (3.5) we have

$$\begin{aligned}\psi(\Lambda(u;\eta(\zeta_{n+1}))) &= \psi(\Lambda(u;\eta(\text{Conv}(T\zeta_n)))) = \psi(\Lambda(u;\eta(T\zeta_n)))\\ &\leq \psi(\Lambda(u;\eta(\zeta_n))) - \phi(\Lambda(u;\eta(\zeta_n)))\\ &\leq \psi(\Lambda(u;\eta(\zeta_n))).\end{aligned} \tag{3.6}$$

It shows that $\{\Lambda(u;\eta(\zeta_n))\}$ is a monotone decreasing sequence of positive real numbers, hence there exists $\delta > 0$ such that

$$\lim_{n\to\infty} \Lambda(u;\eta(\zeta_n)) = \Lambda(u;\lim_{n\to\infty}\eta(\zeta_n)) = \delta. \tag{3.7}$$

Let $\delta > 0$, by taking upper limit as $n \to \infty$ and using (3.6) and (3.7) we obtain

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta),$$

which is a contradiction. Thus $\delta = 0$, and

$$\lim_{n\to\infty} \Lambda(u;\eta(\zeta_n)) = \Lambda(u;\lim_{n\to\infty}\eta(\zeta_n)) = 0,$$

by (i) we conclude that

$$\lim_{n\to\infty}\eta(\zeta_n) = 0.$$

Because $\zeta_n \supseteq \zeta_{n+1}$ and $T\zeta_n \subseteq \zeta_n$ for $n \in \mathbb{N}$, it follows from (6^0) that $\zeta_\infty = \bigcap_{n=1}^{\infty} \zeta_n$ is a nonempty convex closed set, invariant under T and belongs to $ker\eta$. Also Theorem 2.2 completes the proof.

Corollary 3.7. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \longrightarrow \zeta$ be a continuous operator such that*

$$\psi(\eta(T(X))) \leq \psi(\eta(X)) - \phi(\eta(X)),$$

for any subset X of ζ, where η is an arbitrary measure of noncompactness, $\psi, \phi : [0, \infty) \to [0, \infty)$ are given functions such that ψ is continuous and ϕ is lower semi-continuous and $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has at least one fixed point in the set ζ.

Remark 3.8. Corollary3.7 is a generalization of Theorem 2.2 in [3].

Corollary 3.9. *Let $\zeta \in \mathfrak{F}_E$, and $T : \zeta \longrightarrow \zeta$ be a continuous operator such that for any $X \subseteq \zeta$ one has*

$$\psi\left(\int_0^{\eta(T(X))} u(s)\ ds\right) \leq \psi\left(\int_0^{\eta(X)} u(s)\ ds\right) - \phi\left(\int_0^{\eta(X)} u(s)\ ds\right),$$

where η is an arbitrary measure of noncompactness and if $u : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, \infty)$, and for each $\epsilon > 0$, $\int_0^\epsilon u(s)\ ds > 0$. Also let $\psi, \phi : [0, \infty) \to [0, \infty)$ are given functions such that ψ is continuous and ϕ is lower semi-continuous and $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has at least one fixed point in the set ζ.

Theorem 3.10. [10] *Suppose $\eta_1, \eta_2, ..., \eta_n$ are measures of noncompactness in $E_1, E_2, ..., E_n$, respectively. Moreover assume that the function $\tau : [0, \infty)^n \to [0, \infty)$ is convex and $\tau(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, ..., n$. Then*

$$\eta(X) = \tau(\eta_1(X_1), \eta_2(X_2), ..., \eta_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times ... \times E_n$, where X_i denotes the natural projection of X into E_i for $i = 1, 2, ..., n$.

Now, as results from Theorem 3.10, we present the following example.

Example 3.1 [10] Let η be a measure of noncompactness. We define $\tau(x, y) = x + y$ for any $x, y \in [0, \infty)$. Then τ has all the properties mentioned in Theorem 3.10. Thus $\widetilde{\eta}(X) = \eta(X_1) + \eta(X_2)$ is a measure of noncompactness in the space $E \times E$ where $X_i, i = 1, 2$ are the natural projections of X.

Definition 3.11. [13] An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Theorem 3.12. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \times \zeta \longrightarrow \zeta$ be a continuous function such that*

$$\Lambda(u; \eta(T(X_1 \times X_2))) \leq \frac{1}{2}\varphi(\Lambda(u; \eta(X_1) + \eta(X_2))), \tag{3.8}$$

for any subset X_1, X_2 of ζ, where η is an arbitrary measure of noncompactness in E and $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Also $\Lambda(\bullet; .) \in \Theta$ and $\Lambda(u; t + s) \leq \Lambda(u; t) + \Lambda(u; s)$ for all $t, s \geq 0$. Then T has at least a coupled fixed point.

Proof. *First note that, from Example 3.1, $\widetilde{\eta}(X) = \eta(X_1) + \eta(X_2)$ for any bounded subset $X \subseteq E \times E$ defines a measure of noncompactness on $E \times E$ where X_1 and X_2 denote the natural projections of X. we define a mapping $\widetilde{T} : \zeta \times \zeta \longrightarrow \zeta \times \zeta$ by*

$$\widetilde{T}(x, y) = (T(x, y), T(y, x)).$$

It is obvious that $\widetilde{T}$ is continuous. Now we claim that $\widetilde{T}$ satisfies all the conditions of Theorem 3.1. To prove this, let $X \subseteq \zeta \times \zeta$ be any nonempty subset. Then by (2^0), (3.8) and (ii) we obtain

$$\begin{aligned}
\Lambda(u; \widetilde{\eta}(\widetilde{T}(X))) &\leq \Lambda(u; \widetilde{\eta}(T(X_1 \times X_2), T(X_2 \times X_1))) \\
&= \Lambda(u; \eta(T(X_1 \times X_2)) + \eta(T(X_2 \times X_1))) \\
&\leq \Lambda(u; \eta(T(X_1 \times X_2)) + \Lambda(u; \eta(T(X_2 \times X_1))) \\
&\leq \frac{1}{2}\varphi[\Lambda(u; \eta(X_1) + \eta(X_2))] + \frac{1}{2}\varphi[\Lambda(u; \eta(X_1) + \eta(X_2))] \\
&= \varphi[\Lambda(u; \eta(X_1) + \eta(X_2))] \\
&= \varphi[\Lambda(u; \widetilde{\eta}(X)].
\end{aligned}$$

Hence, from Theorem 3.1, $\widetilde{T}$ has at least one fixed point in $\zeta \times \zeta$. Now the conclusion of theorem follows from the fact that every fixed point of $\widetilde{T}$ is a coupled fixed point of T.

As applications for Theorem 3.12, one can get the following Corollaries 13 and 14.

Corollary 3.13. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \times \zeta \to \zeta$ be a continuous function such that*

$$\eta(T(X_1 \times X_2)) \leq \frac{1}{2}\varphi(\eta(X_1) + \eta(X_2)),$$

for any subset X_1, X_2 of ζ, where η is an arbitrary measure of noncompactness and $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing functions such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \geq 0$. Then T has at least a coupled fixed point.

Corollary 3.14. *Let $\zeta \in \mathfrak{F}_E$ and $T : \zeta \times \zeta \to \zeta$ be a continuous function. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\eta(T(X_1 \times X_2)) \leq \frac{k}{2}(\eta(X_1) + \eta(X_2)),$$

for any subset X_1, X_2 of ζ, where η is an arbitrary measure of noncompactness. Then T has at least a coupled fixed point.

4. Application

Consider $\mathcal{F} = BC(\mathbb{R}^+)$, equipped with $||x|| = \sup\{|x(t)| : t \geq 0\})$ norm is a Banach space of all bounded and continuous functions on $\mathbb{R}^+$.

Let X be a fixed nonempty and bounded subset of $\mathcal{F}$ and fix a positive number T. For $x \in X$ and $\epsilon > 0$ denote by $\omega^T(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, T]$; i.e.

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : x \in X\},$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon)$$

and

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

Moreover for a fixed number $t \in \mathbb{R}^+$, let us denote

$$X(t) = \{x(t) : x \in X\},$$
$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Also, consider the function η on the family $\mathcal{F}$ by the following formula,

$$\eta(X) = \omega_0(X) + \limsup_{t \to \infty} \operatorname{diam} X(t).$$

Similar to [10] (cf. also [11]), it can be shown that the function η is a measure of noncompactness.

As an application of our results we are going to study the existence of solutions for nonlinear integral equations (1.1). Consider the following assumptions

(a_1) $\alpha_i : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, nondecreasing and $\lim_{t\to\infty} \alpha_i(t) = \infty$, $1 \le i \le 4$;

(a_2) The function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and bounded;

(a_3) $k : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant M such that

$$M = \sup\left\{ \int_0^{\alpha_3(t)} |k(t, s, x(\alpha_4(s)))| ds : t, s \in \mathbb{R}^+, x \in \mathcal{F}) \right\}. \tag{4.1}$$

Moreover,

$$\lim_{t\to\infty} \left| \int_0^{\alpha_3(t)} [k(t, s, x(\alpha_4(s))) - k(t, s, y(\alpha_4(s)))] ds \right| = 0 \tag{4.2}$$

uniformly with respect to $x, y \in \mathcal{F}$;

(a_4) The functions $u, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists an upper semi-continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \ge 0$. Also there exist two bounded functions $a, b : \mathbb{R}^+ \to \mathbb{R}$ with bound $K = \max\left\{ \sup_{t\in\mathbb{R}^+} a(t), \sup_{t\in\mathbb{R}^+} b(t) \right\}$ and a positive constant D such that

$$|u(t, x) - u(t, y)| \le \frac{a(t)\varphi(|x - y|)}{D + \varphi(|x - y|)},$$

and

$$|g(t, x) - g(t, y)| \le \frac{b(t)\varphi(|x - y|)}{D + \varphi(|x - y|)},$$

for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$. Additionally we assume that φ is super-additive i.e., $\varphi(t) + \varphi(s) \le \varphi(t + s)$ for all $t, s \in \mathbb{R}^+$. Moreover, we assume that $K(1 + M) \le D$.

(a_5) The functions $H_1, H_2 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $H_1(t) = |u(t, 0)|$ and $H_2(t) = |g(t, 0)|$ are bounded on $\mathbb{R}^+$ with $H_0 = \max\left\{ \sup_{t\in\mathbb{R}^+} H_1(t), \sup_{t\in\mathbb{R}^+} H_2(t) \right\}$.

Theorem 4.1. *Under assumptions* $(a_1)-(a_5)$, *equation (1.1) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{F} = BC(\mathbb{R}^+)$.

Let G and τ be two operators defined on the space $\mathcal{F} = BC(\mathbb{R}^+)$ by

$$\tau(x)(t) = \int_0^{\alpha_3(t)} k(t, s, x(\alpha_4(s)))ds \tag{4.3}$$

and

$$(Gx)(t) = h(t) + u(t, x(\alpha_1(t))) + g(t, x(\alpha_2(t)))\tau(x)(t). \tag{4.4}$$

Solving Eq.(1.1) is equivalent to finding a fixed point of the operator G defined on the space $\mathcal{F}$. For better readability, the proof is divided into steps.

Step 1: G transforms the space $\mathcal{F}$ into itself.

The conditions of theorem imply that $G(x)$ is continuous on $\mathbb{R}^+$. Now we prove that $G(x) \in \mathcal{F}$ for any $x \in \mathcal{F}$. For arbitrarily fixed $t \in \mathbb{R}^+$ we have

$$\begin{aligned} |(Gx)(t)| &\leq |h(t)| + |u(t, x(\alpha_1(t)))| + |g(t, x(\alpha_2(t)))||\tau(x)(t)| \\ &\leq ||h|| + \frac{K\varphi(|x(\alpha_1(t))|)}{D + \varphi(|x(\alpha_1(t))|)} + H_0 + \Big[\frac{K\varphi(|x(\alpha_2(t))|)}{D + \varphi(|x(\alpha_2(t))|)} + H_0\Big]M. \end{aligned} \tag{4.5}$$

Indeed,

$$\begin{aligned} |u(t, x(\alpha_1(t)))| &\leq |u(t, x(\alpha_1(t))) - u(t, 0)| + |u(t, 0)| \leq \frac{a(t)\varphi(|x(\alpha_1(t))|)}{D + \varphi(|x(\alpha_1(t))|)} + |u(t, 0)| \\ &\leq \frac{K\varphi(|x(\alpha_1(t))|)}{D + \varphi(|x(\alpha_1(t))|)} + H_0, \\ |g(t, x(\alpha_2(t)))| &\leq |g(t, x(\alpha_2(t))) - g(t, 0)| + |g(t, 0)| \leq \frac{b(t)\varphi(|x(\alpha_2(t))|)}{D + \varphi(|x(\alpha_2(t))|)} + |g(t, 0)| \\ &\leq \frac{K\varphi(|x(\alpha_2(t))|)}{D + \varphi(|x(\alpha_2(t))|)} + H_0, \\ |(\tau x)(t)| &= \Big|\int_0^{\alpha_3(t)} k(t, s, x(\alpha_4(t)))ds\Big| \leq \int_0^{\alpha_3(t)} |k(t, s, x(\alpha_4(t)))|ds \leq M. \end{aligned}$$

According to (a_4) function φ is nondecreasing and some computing implies that,

$$\begin{aligned} ||Gx|| &\leq ||h|| + \Big[\frac{K\varphi(||x||)}{D + \varphi(||x||)} + H_0\Big](1 + M) \\ &\leq ||h|| + (K + H_0)(1 + M). \end{aligned} \tag{4.6}$$

Thus G maps the space $\mathcal{F}$ into itself. More precisely, from (4.6) we obtain that the operator G is a self mapping of the ball B_{r_0}, where $r_0 = ||h|| + (K + H_0)(1 + M)$.

Step 2: We show that the operator G is continuous on the ball $\overline{B}_{r_0}$.

To do this, let us fix arbitrarily $\epsilon > 0$ and take $x, y \in \overline{B}_{r_0}$ such that $||x - y|| \leq \epsilon$.

Then

$$\begin{aligned}
&|(Gx)(t)-(Gy)(t)|\\
&=|u(t,x(\alpha_1(t)))+g(t,x(\alpha_2(t)))\tau(x)(t)-u(t,y(\alpha_1(t)))-g(t,y(\alpha_2(t)))\tau(y)(t)|\\
&\le|u(t,x(\alpha_1(t)))-u(t,y(\alpha_1(t)))|+|g(t,x(\alpha_2(t)))||(\tau x)(t)-(\tau y)(t)|\\
&+|g(t,x(\alpha_2(t)))-g(t,y(\alpha_2(t)))||(\tau y)(t)|\\
&\le\frac{a(t)\varphi(|x(\alpha_1(t))-y(\alpha_1(t))|)}{D+\varphi(|x(\alpha_1(t))-y(\alpha_1(t))|)}+\Big[\frac{K\varphi(|x(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))|)}+H_0\Big]|(\tau x)(t)-(\tau y)(t)|\\
&+\Big[\frac{b(t)\varphi(|x(\alpha_2(t))-y(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))-y(\alpha_2(t))|)}\Big]M\\
&\le\frac{K(1+M)\varphi(||x-y||)}{D+\varphi(||x-y||)}+\Big[\frac{K\varphi(||x||)}{D+\varphi(||x||)}+H_0\Big]|(\tau x)(t)-(\tau y)(t)|,
\end{aligned} \tag{4.7}$$

furthermore, by the condition (a_3) there exists $T>0$ such that for $t>T$ we have

$$|(\tau x)(t)-(\tau y)(t)|=\Big|\int_0^{\alpha_3(t)}[k(t,s,x(\alpha_4(s)))-k(t,s,y(\alpha_4(s)))]ds\Big|<\epsilon. \tag{4.8}$$

Suppose that $t>T$, then by (4.7) and super-additivity of φ we obtain

$$|(Gx)(t)-(Gy)(t)|<\epsilon.$$

If $t\in[0,T]$, then

$$|(\tau x)(t)-(\tau y))(t)|\le\alpha_T\omega_1(k,\epsilon), \tag{4.9}$$

where

$$\alpha_T=\sup\{\alpha_i(t):t\in[0,T],1\le i\le 4\},$$

and

$$\begin{aligned}
&\omega_1(k,\epsilon)=\\
&\quad\sup\{|k(t,s,x)-k(t,s,y)|\quad:t\in[0,T],s\in[0,\alpha_T],x,y\in[-r_0,r_0],||x-y||\le\epsilon\}.
\end{aligned}$$

The continuity of k on $[0,T]\times[0,\alpha_T]\times[-r_0,r_0]$, implies that $\omega_1(k,\epsilon)\to 0$ and $\varphi(\epsilon)\to 0$ as $\epsilon\to 0$. Now, from the inequalities (4.7) and (4.9) we deduce that

$$|(Gx)(t)-(Gy)(t)|\le D\frac{\varphi(\epsilon)}{D+\varphi(\epsilon)}+[K+H_0]\alpha_T\omega_1(k,\epsilon), \tag{4.10}$$

so,

$$|(Gx)(t)-(Gy)(t)|\le\epsilon.$$

Thus, the operator G is continuous on the ball $\overline{B}_{r_0}$.

Step 3: In the sequel, we show that for any nonempty set $X\subseteq\overline{B}_{r_0}$,

$$\eta(G(X))\le\varphi(\eta(X)).$$

Indeed, by virtue of assumptions $(a_1)-(a_5)$, (4.9) and (4.10),we conclude that for any $x, y \in X$ and $t \in \mathbb{R}^+$,

$$\begin{aligned}
|(Gx)(t)-(Gy)(t)| \leq & \frac{K\varphi(|x(\alpha_1(t))-y(\alpha_1(t))|)}{D+\varphi(|x(\alpha_1(t))-y(\alpha_1(t))|)} + \Big[\frac{K\varphi(|x(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))|)} + H_0\Big]\beta(t) \\
& + \Big[\frac{K\varphi(|x(\alpha_2(t))-y(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))-y(\alpha_2(t))|)}\Big]M \\
& \leq \frac{1}{1+M}\varphi(|x(\alpha_1(t))-y(\alpha_1(t))|) + \Big[\frac{K\varphi(|x(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))|)} + H_0\Big]\beta(t) \\
& + \frac{M}{1+M}\varphi(|x(\alpha_2(t))-y(\alpha_2(t))|),
\end{aligned}$$

where

$$\beta(t) = \sup\left\{\int_0^{\alpha_3(t)} |k(t,s,x(\alpha_4(s)))-k(t,s,y(\alpha_4(s)))|ds : x, y \in E\right\}.$$

This estimate allows us to derive the following one

$$\begin{aligned}
\operatorname{diam}(G(X))(t) \leq & \frac{1}{1+M}\varphi(\operatorname{diam}X(\alpha_1(t))) + \Big[\frac{K\varphi(|x(\alpha_2(t))|)}{D+\varphi(|x(\alpha_2(t))|)} + H_0\Big]\beta(t) \\
& + \frac{M}{1+M}\varphi(\operatorname{diam}X(\alpha_2(t))).
\end{aligned} \tag{4.11}$$

Consequently, in view of the upper semi-continuity of the function φ and from (4.11) and assumption (4.2), we have

$$\begin{aligned}
\limsup_{t\longrightarrow\infty} \operatorname{diam}(G(X))(t) \leq & \frac{1}{1+M}\varphi(\limsup_{t\longrightarrow\infty} \operatorname{diam}X(\alpha_1(t))) \\
& + \frac{M}{1+M}\varphi(\limsup_{t\longrightarrow\infty} \operatorname{diam}X(\alpha_2(t))) \\
& \leq \varphi(\limsup_{t\longrightarrow\infty} \operatorname{diam}X(t)).
\end{aligned} \tag{4.12}$$

Next, fix arbitrarily $T > 0$ and $\epsilon > 0$. Let us choose $t_1, t_2 \in [0, T]$, with $|t_2 - t_1| \leq \epsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then for $x \in X$,

$$\begin{aligned}
|u(t_2, x(\alpha_1(t_2))) - u(t_1, x(\alpha_1(t_1)))| \leq & |u(t_2, x(\alpha_1(t_2))) - u(t_2, x(\alpha_1(t_1)))| \\
& + |u(t_2, x(\alpha_1(t_1))) - u(t_1, x(\alpha_1(t_1)))| \\
& \leq \frac{K\varphi(|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|}{D+\varphi(|x(\alpha_1(t_2)) - x(\alpha_1(t_1))|)} \\
& + |u(t_2, x(\alpha_1(t_1))) - u(t_1, x(\alpha_1(t_1)))| \\
& \leq \frac{1}{1+M}\varphi(\omega^T(x, \omega^T(\alpha_1, \epsilon))) + \omega^T(u, \epsilon),
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}|(\tau x)(t_2)-(\tau x)(t_1)| &\leq \int_0^{\alpha_3(t_2)} |k(t_2,s,x(\alpha_4(s)))-k(t_1,s,x(\alpha_4(s)))|ds \\ &+ \int_{\alpha_3(t_1)}^{\alpha_3(t_2)} |k(t_1,s,x(\alpha_4(s)))|ds \\ &\leq \int_0^{\alpha_3(t_2)} \omega^T(k,\epsilon)ds + \int_{\alpha_3(t_1)}^{\alpha_3(t_2)} K^T ds \\ &\leq \alpha_T\omega^T(k,\epsilon)+\omega^T(\alpha_3,\epsilon)K^T,\end{aligned} \tag{4.14}$$

and applying (a_2), (4.13) and (4.14),

$$\begin{aligned}&|g(t_2,x(\alpha_2(t_2)))(\tau x)(t_2)-g(t_1,x(\alpha_2(t_1)))(\tau x)(t_1)| \\ &\leq |g(t_2,x(\alpha_2(t_2)))(\tau x)(t_2)-g(t_2,x(\alpha_2(t_1)))(\tau x)(t_2)| \\ &+ |g(t_2,x(\alpha_2(t_1)))(\tau x)(t_2)-g(t_2,x(\alpha_2(t_1)))(\tau x)(t_1)| \\ &+ |g(t_2,x(\alpha_2(t_1)))(\tau x)(t_1)-g(t_1,x(\alpha_2(t_1)))(\tau x)(t_1)| \\ &\leq \frac{K\varphi(|x(\alpha_2(t_2))-x(\alpha_2(t_1))|}{D+\varphi(|x(\alpha_2(t_2))-x(\alpha_2(t_1))|)}|(\tau x)(t_2)| \\ &+ \Big[\frac{K\varphi(|x(\alpha_2(t_1))|)}{D+\varphi(|x(\alpha_2(t_1))|)}+H_0\Big]|(\tau x)(t_2)-(\tau x)(t_1)| \\ &+ |g(t_2,x(\alpha_2(t_1)))-g(t_1,x(\alpha_2(t_1)))|\ |(\tau x)(t_1)| \\ &\leq \frac{M}{1+M}\varphi(\omega^T(x,\omega^T(\alpha_2,\epsilon))) \\ &+ (K+H_0)\Big[\alpha_T\omega^T(k,\epsilon)+\omega^T(\alpha_3,\epsilon)K^T\Big] \\ &\quad + \omega^T(g,\epsilon)\alpha_T K^T.\end{aligned} \tag{4.15}$$

Therefore

$$\begin{aligned}&|(Gx)(t_2)-(Gx)(t_1)| \\ &\leq |h(t_2)-h(t_1)|+|u(t_2,x(\alpha_1(t_2)))-u(t_1,x(\alpha_1(t_1)))| \\ &+ |g(t_2,x(\alpha_2(t_2)))(\tau x)(t_2)-g(t_1,x(\alpha_2(t_1)))(\tau x)(t_1)| \\ &\leq \omega^T(h,\epsilon)+\frac{1}{1+M}\varphi(\omega^T(x,\omega^T(\alpha_1,\epsilon)))+\omega^T(u,\epsilon) \\ &+ \frac{M}{1+M}\varphi(\omega^T(x,\omega^T(\alpha_2,\epsilon))) \\ &+ (K+H_0)[\alpha_T\omega^T(k,\epsilon)+\omega^T(\alpha_3,\epsilon)K^T]+\omega^T(g,\epsilon)M,\end{aligned} \tag{4.16}$$

where we defined

$$
\begin{aligned}
&\omega^T(g,\epsilon) = \sup\{|g(t_2,x)-g(t_1,x)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \epsilon, x \in [-r_0,r_0]\},\\
&\omega^T(u,\epsilon) = \sup\{|u(t_2,x)-u(t_1,x)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \epsilon, x \in [-r_0,r_0]\},\\
&\omega^T(h,\epsilon) = \sup\{|h(t_2)-h(t_1)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \epsilon\},\\
&\omega^T(k,\epsilon) = \sup\{|k(t_2,s,x)-k(t_1,s,x)| : t_1,t_2 \in [0,T], |t_2-t_1| \le \epsilon, s \in [0,\alpha_T],\\
&\qquad\qquad x \in [-r_0,r_0]\},\\
&\omega^T(\alpha_i,\epsilon) = \sup\{|\alpha_i(t_1)-\alpha_i(t_2)| : t_1,t_2 \in [0,T], |t_1-t_2| \le \epsilon, i=1,2\},\\
&\omega^T(x,\omega^T(\alpha_i,\epsilon)) = \sup\{|x(t_2)-x(t_1)| : t_1,t_2 \in [0,\alpha_T], |t_2-t_1| \le \omega^T(\alpha_i,\epsilon)\},\\
&K^T = \sup\{|k(t,s,x)| : t \in [0,T], s \in [0,\alpha_T], x \in [-r_0,r_0]\}.
\end{aligned}
\tag{4.17}
$$

Since x was an arbitrary element of X, the inequality (4.16) implies that

$$
\begin{aligned}
\omega^T(G(X),\epsilon) \le\ & \omega^T(h,\epsilon) + \frac{1}{1+M}\varphi(\omega^T(X,\omega^T(\alpha_1,\epsilon)))\\
&+ \omega^T(u,\epsilon) + \frac{M}{1+M}\varphi(\omega^T(X,\omega^T(\alpha_2,\epsilon)))\\
&+ (K+H_0)[\alpha_T\omega^T(k,\epsilon) + \omega^T(\alpha_3,\epsilon)K^T] + \omega^T(g,\epsilon)M,
\end{aligned}
\tag{4.18}
$$

Because of the uniform continuity of the functions u, h and k on $[0,T]\times[-r_0,r_0]$, $[0,T]$ and $[0,T]\times[0,\alpha_T]\times[-r_0,r_0]$, respectively, we have $\omega^T(h,\epsilon) \to 0$, $\omega^T(u,\epsilon) \to 0$ and $\omega^T(k,\epsilon) \to 0$. Moreover, it is obvious that the constant K^T is finite and $\omega^T(\alpha_1,\epsilon) \to 0$, $\omega^T(\alpha_2,\epsilon) \to 0$ and $\omega^T(\alpha_3,\epsilon) \to 0$ as $\epsilon \to 0$. Thus linking the facts with the estimate (4.18) we get

$$
\omega_o(G(X)) \le \varphi(\omega_o(X)). \tag{4.19}
$$

Finally, from (4.12), (4.19), and taking into account the super-additivity of the function φ and the definition of the measure of noncompactness η, we obtain

$$
\begin{aligned}
\eta(G(X)) &= \omega_0(G(X)) + \limsup_{t\to\infty} \operatorname{diam}(G(X)(t))\\
&\le \varphi(\omega_o(X)) + \varphi(\limsup_{t\to\infty} \operatorname{diam} X(t))\\
&\le \varphi(\omega_o(X) + \limsup_{t\to\infty} \operatorname{diam} X(t))\\
&= \varphi(\eta(X)).
\end{aligned}
\tag{4.20}
$$

Consequently, operator G has a fixed point and then functional integral equations (1.1) has a solution in $\mathcal{F} = BC(\mathbb{R}^+)$ space.

Remark 4.2. If $h(t) = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = I$ for any $t \in \mathbb{R}^+$, then Theorem 4.1 reduces to Theorem 3.1 of Liu and Kang [23].

Remark 4.3. If $u(t,x) = 0$ for any $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then Theorem 4.1 reduces to Theorem 3.1 of Dhage and Bellale [16].

Remark 4.4. If $h(t) = 0$, $u(t,x) = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = I$ for any $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then Theorem 4.1 reduces to Theorem 2 of Banaś and Rzepka [12].

Remark 4.5. If $h(t) = 0$, $g(t,x) = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = I$ for any $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then Theorem 4.1 reduces to Theorem 2 of Banaś and Rzepka [11].

Example 4.1. Let us consider now the following integral equation

$$\begin{aligned} x(t) =& e^{-2t} + \frac{t}{1+t} + \frac{|x(t)|}{9|x(t)|+9} \\ &+ \Big(e^{-2t} + \frac{|x(\sqrt{t})|}{10|x(\sqrt{t})|+10}\Big)\Big(\int_0^{\sqrt{t}} \frac{\ln(1+s|x(s^2)|)+s(1+x^2(s^2)}{(1+t^2)(1+x^2(s^2))}ds\Big), \end{aligned} \tag{4.21}$$

where $t \in [0, +\infty)$.
It can be easily seen that equation (4.21) is a particular case of the equation (1.1), where

$$\begin{aligned} h(t) &= e^{-2t},\ \alpha_1(t) = t,\ \alpha_2(t) = \alpha_3(t) = \sqrt{t},\ \alpha_4(t) = t^2, \\ u(t,x) &= \frac{t}{1+t} + \frac{|x|}{9|x|+9}, \\ g(t,x) &= e^{-2t} + \frac{|x|}{10|x|+10}, \\ k(t,s,x) &= \frac{\ln(1+s|x|)+s(1+x^2)}{(1+t^2)(1+x^2)}. \end{aligned}$$

The functions α_i $(i = 1, 2, 3, 4)$ are continuous, nondecreasing and $\lim_{t\to\infty} \alpha_i(t) = \infty$. Also, the function h satisfies assumption (a_2). Moreover, the function k satisfies assumption (a_3). Indeed,

$$\begin{aligned} \int_0^{\sqrt{t}} |k(t,s,x(s^2))|ds &= \int_0^{\sqrt{t}} \frac{\ln(1+s|x(s^2)|)}{(1+t^2)(1+x^2(s^2))}ds + \int_0^{\sqrt{t}} \frac{s}{1+t^2}ds \\ &\le \int_0^{\sqrt{t}} \frac{s|x(s^2)|}{(1+t^2)(1+x^2(s^2))}ds + \int_0^{\sqrt{t}} \frac{s}{1+t^2}ds \\ &\le \frac{1}{2}. \end{aligned}$$

Here we have

$$M = \sup\left\{\int_0^{\sqrt{t}} |k(t,s,x(s^2))|ds : t,s \in \mathbb{R}^+, x \in \mathcal{F})\right\} = \frac{1}{2}.$$

Also,

$$\begin{aligned} &\Big|\int_0^{\sqrt{t}} [k(t,s,x(s^2)) - k(t,s,y(s^2))]ds\Big| \\ &\quad\le \int_0^{\sqrt{t}} \Big|\frac{\ln(1+s|x(s^2)|)+s(1+x^2(s^2))}{(1+t^2)(1+x^2(s^2))} - \frac{\ln(1+s|y(s^2)|)+s(1+y^2(s^2))}{(1+t^2)(1+y^2(s^2))}\Big|ds \\ &\quad\le \int_0^{\sqrt{t}} \frac{s|x(s^2)|+s(1+x^2(s^2))}{(1+t^2)(1+x^2(s^2))} + \frac{s|y(s^2)|+s(1+y^2(s^2))}{(1+t^2)(1+y^2(s^2))}ds \\ &\quad\le \frac{3}{2}\frac{t}{1+t^2}, \end{aligned}$$

uniformly with respect to $x, y \in \mathcal{F}$. This implies that

$$\lim_{t\to\infty}\Big|\int_0^{\sqrt{t}}[k(t,s,x(s^2))-k(t,s,y(s^2))]ds\Big|=0.$$

Assume that $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$. Then we get

$$\begin{aligned}|u(t,x)-u(t,y)| &= \Big|\frac{t}{1+t}+\frac{|x|}{9|x|+9}-\frac{t}{1+t}-\frac{|y|}{9|y|+9}\Big|\\ &\le \frac{1}{9}\Big|\frac{|x|-|y|}{(1+|x|)(1+|y|)}\Big|\\ &\le \frac{1}{9}\frac{|x-y|}{1+|x-y|}=\frac{1}{9}\frac{\frac{2}{3}|x-y|}{\frac{2}{3}(1+|x-y|)}\\ &= \frac{1}{9}\frac{\varphi(|x-y|)}{\frac{2}{3}+\varphi(|x-y|)}.\end{aligned}$$

Here we have $a(t)=\frac{1}{9}$, $D=\dfrac{2}{3}$ and $\varphi(t)=\dfrac{2}{3}t$.
Similarly,

$$|g(t,x)-g(t,y)|\le \frac{1}{10}\frac{\varphi(|x-y|)}{\frac{2}{3}+\varphi(|x-y|)},$$

which $b(t)=\frac{1}{10}$. Also, $K=\max\left\{\sup\limits_{t\in\mathbb{R}^+}a(t), \sup\limits_{t\in\mathbb{R}^+}b(t)\right\}=\frac{1}{9}$.
Since $K(1+M)=\dfrac{1}{9}\left(1+\dfrac{1}{2}\right)=\dfrac{1}{6}<\dfrac{2}{3}=D$. Hence, applying Theorem 4.1 we infer that Eq.(4.21) has a solution $x=x(t)$ in the space $\mathcal{F}=BC(\mathbb{R}^+)$.

5. An iterative algorithm by sinc interpolation to solve (4.21)

Because finding of the exact solution of the nonlinear integral equations such as Example 4.1 is difficult, so numerical techniques are affective to estimate of solutions. Numerical methods to solve nonlinear integral and integro-differential equations were introduced by some authors in [24, 25, 29, 30]. Also, the numerical methods for functional integral equations based on the collocation method in [8, 26], modified homotopy perturbation in [20, 34], and Lagrange and Chebyshev interpolation in [32] were used. In this section, we apply the sinc-quadrature formula to make an iterative algorithm to find approximation of the solution of equation (4.21), where this the method does not consist of reducing the solution to a set of algebraic equations by expanding $x(t)$ in terms of Sinc functions with unknown coefficients, thus this scheme has less computations and exponential accuracy (for exponential accuracy see [33]).

To this end, we consider the Sinc function and some of its properties [33].

$$\operatorname{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, \ x \neq 0 \\ 1, \ x = 0. \end{cases} \tag{5.1}$$

For $h > 0$ and integer k the shifted Sinc function named k'th Sinc function with step size h is introduced as follows:

$$S(k,h)(x) = \operatorname{Sinc}\left(\frac{x - kh}{h}\right), \tag{5.2}$$

therefore, it is easily concluded that

$$S(k,h)(jh) = \delta_{kj} = \begin{cases} 1, k = j \\ 0, k \neq j. \end{cases} \tag{5.3}$$

Definition 5.1. [33] Let u be a function defined on real line, then for $h > 0$ the series,

$$C(u,h)(x) = \sum_{k=-\infty}^{\infty} u(kh) S(k,h)(x), \tag{5.4}$$

is called the Whittaker cardinal expansion of u function, wherever this series converges.

By (5.4-5.3) cardinal function interpolates u at the points $\{kh\}_{k=-\infty}^{\infty}$.
According to Example 4.1, the integrating is over $[0, \sqrt{t}]$ and $t \in [0, \infty)$, so we introduce conformal map in the following form,

$$\begin{aligned} \phi : [0, \sqrt{t}] &\longrightarrow (-\infty, \infty) \\ x &\longrightarrow \ln\left(\frac{x}{\sqrt{t} - x}\right) \end{aligned} \tag{5.5}$$

easily $\lim\limits_{x \to 0} \phi(x) = -\infty$ and $\lim\limits_{x \to \sqrt{t}} \phi(x) = \infty$.
By (5.2) and (5.5) combination of $S(k,h)$ and ϕ functions is $S(k,h) o \phi$ function with $[0, \sqrt{t}]$ domain, so the integrand function of Example 4.1 can be approximated by $S(k,h) o \phi$ interpolation. Let u be an integrand function, then with the help of cardinal function (5.4) and the above explanations we can write,

$$u_n(x) = \sum_{k=-N}^{N} u(kh) S(k,h) o \phi(x). \tag{5.6}$$

Considering to (5.6) and (5.3) if $\phi(x) = kh$ for $k = -N, \dots, N$, then $u_n(kh) = u(kh)$. In other word (5.6) is interpolation of u function such that the interpolating points can be given by,

$$\begin{cases} x_k = \phi^{-1}(kh) = \dfrac{\sqrt{t} e^{kh}}{1 + e^{kh}}, \ k = -N + 1, \dots, N + 1 \\ x_{-N} = 0, \ x_N = \sqrt{t}. \end{cases} \tag{5.7}$$

Similar to [7, 31], the following approximation with the help of Sinc function to compute integral on $[0, \sqrt{t}]$ is introduced,

$$\int_0^{\sqrt{t}} u(x)dx \approx h \sum_{k=-N}^{N} \frac{u(x_k)}{\phi'(x_k)}, \tag{5.8}$$

$$where,\ \phi'(x_k) = \frac{\sqrt{t}}{x_k(\sqrt{t} - x_k)}. \tag{5.9}$$

Now, with the help of (5.6-5.9), we can give an iterative algorithm to solve Equation (4.21).

Algorithm

$x_0(t) = 1,$

$$\begin{aligned} x_{n+1}(t) &= e^{-2t} + \frac{t}{1+t} + \frac{|x_n(t)|}{9|x_n(t)|+9} + \\ &\Big(e^{-2t} + \frac{|x_n(\sqrt{t})|}{10|x_n(\sqrt{t})|+10}\Big)\frac{1}{1+t^2}\Big(h\sum_{k=-N}^{N} \frac{s_k(\sqrt{t}-s_k)}{\sqrt{t}} \frac{\ln(1+s_k|x_n(s_k^2)|)}{(1+x_n^2(s_k^2))} + \frac{t}{2}\Big), \end{aligned} \tag{5.10}$$

$n = 1, 2, 3, \ldots$

where s_k for $k = -N, \ldots, N$ are given by (5.7).

For $h = \frac{\pi}{200}$, $N = 10$, $x_i(t)$ for $i = 1, 2$ are shown in Figure 1.

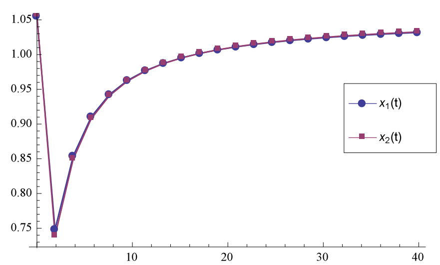

Figure 1. $x_1(t)$ and $x_2(t)$ are approximations of solution for $h = \frac{\pi}{200}$, $N = 10$.

Therefore by substituting $x_1(t)$ and $x_2(t)$ in (4.21) and comparing both sides in the sample of the interval as $[0, 40]$, absolute errors are shown in Table 1. Moreover, with the help of more iterations, the accuracy can be increased.

6. Conclusions

In this research, we proved the existence of solution for functional integral equations, and some examples to show the efficiency of our results was presented. Therefore, an iterative algorithm to approximate of the solution of the above equations was introduced with acceptable accuracy.

Table 1. Absolute errors

t	Absolute errors for $x_1(t)$	Absolute errors for $x_2(t)$
0	1.5×10^{-3}	3.9×10^{-5}
4	2.8×10^{-3}	1.5×10^{-3}
8	7.6×10^{-4}	7.6×10^{-4}
12	5.1×10^{-5}	4.9×10^{-4}
16	3.1×10^{-4}	3.6×10^{-4}
20	5.3×10^{-4}	2.8×10^{-4}
24	6.8×10^{-4}	2.3×10^{-4}
28	7.9×10^{-4}	2.0×10^{-4}
32	8.8×10^{-4}	1.7×10^{-4}
36	9.4×10^{-4}	1.5×10^{-4}
40	9.9×10^{-4}	1.4×10^{-4}

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