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ON MODIFIED $\mathcal{L}$-CONTRACTION VIA BINARY RELATION WITH AN APPLICATION

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Abstract. In this paper, we introduce the idea of $\mathcal{L}_{\mathcal{R}}$-contraction by employing an amorphous binary relation on $\mathcal{L}$-contraction in a metric space. We prove an existence and corresponding uniqueness fixed point results for $\mathcal{L}_{\mathcal{R}}$-contraction employing an S-transitive binary relation on metric spaces without completeness and also furnish an illustrative example to demonstrate the utility of our main results. Finally, we apply our newly obtained results to show the existence of a non-negative solution of the first-order ordinary differential equation.
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1. Introduction

In 1986, Turinici [21] utilized the idea of order-theoretic notion in fixed point considerations. Later, in 2004, Ran and Reurings [17] established a relatively more natural order-theoretic version of Banach contraction principle besides presenting an application of their result to matrix equations. Thereafter, Nieto and Rodríguez-López [15] slightly modified Ran-Reurings theorem and gave the application to solve boundary value problems in differential equations. Later, Samet and Turinici [19] obtained fixed point results under symmetric closure of an amorphous binary relation for nonlinear contractions. In 2014, Ben-El-Mechaiekh [4] extends Ran-Reurings fixed point theorem for uniform local contraction employing a transitive binary relation instead of a partial order relation. Thereafter, Alam and Imdad [1] obtained a relation-theoretic analog of Banach contraction principle under arbitrary binary relation which unifies several well-known order-theoretic fixed point results.

In 2014, Jleli and Samet [10] introduced the notion of θ-contraction and obtained a notable generalization of Banach contraction principle in the setting of generalized metric space often known as Branciari distance spaces [5]. Thereafter, Ahmad et al. [9] modified the conditions on the auxiliary functions θ and obtained a natural analogous result in a metric space. On the other hand, using a family of control functions (known as simulation functions), Khojasteh et al. [13] introduced the notion of $\mathcal{Z}$-contraction and unified several types of linear as well as nonlinear contractions of the existing literature. There is already a vast literature exists in this direction (see [18, 12, 16, 20, 7]) and references cited therein. Inspired by these ideas, Cho [6] introduced a new kind of contraction, called $\mathcal{L}$-contraction and proved some fixed point results in generalized metric space for such contraction.

In this paper, we introduce the notion of $\mathcal{L}_{\mathcal{R}}$-contraction involving an amorphous binary relation $\mathcal{R}$ and utilize the same to prove an existence and corresponding uniqueness fixed point results on metric space without completeness. Our newly obtained results unify, generalize and extend many well-known results of the existing literature. An example is adopted to demonstrate the utility of our newly proved results. Finally, we show the applicability of our main results to discuss the existence of a non-negative solution of the first-order ordinary differential equation.

2. Preliminaries

To make our paper self-contained, we recall the following terminological and notational conventions. In what follows $\mathbb{N}$,$\mathbb{Q}$ and $\mathbb{R}$ respectively denote the sets of natural numbers, rational numbers and real numbers wherein $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
Following [10], let Θ be the set of all function $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:
(θ_1) θ is nondecreasing;
(θ_2) for each sequence $\{\beta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\beta_n) = 1 \Longleftrightarrow \lim_{n\to\infty} \beta_n = 0$;
(θ_3) there exist $\kappa \in (0, 1)$ and $\gamma \in (0, \infty]$ such that $\lim_{\beta\to 0^+} \frac{\theta(\beta)-1}{\beta^\kappa} = \gamma$.
After that, Ahmad et al. [9] replaced the condition (θ_3) by the following:
(θ_4) θ is continuous.
Let Θ^* denotes the family of all functions satisfying $(\theta_1), (\theta_2)$ and (θ_4). Here, for the sake of convenience we provide some examples of such functions.

Example 2.1. Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\sqrt{\beta}}}}$, then $\theta \in \Theta^*$.

Example 2.2. [10] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{\sqrt{\beta}}$, then $\theta \in \Theta$ as well as $\theta \in \Theta^*$.

Example 2.3. [8] Define $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(\beta) = \begin{cases} e^{\sqrt{\beta}} & \beta \leq k; \\ e^{2(k+1)} & \beta > k, \end{cases}$$

where $k \geq 1$ (a fixed real number). Then $\theta \in \Theta$ but $\theta \notin \Theta^*$.

Example 2.4. [8] Define $\theta : (0, \infty) \to (1, \infty)$ by $\theta(\beta) = e^{e^{-\frac{1}{\beta}}}$, then $\theta \in \Theta^*$ but $\theta \notin \Theta$.

From above examples, we can conclude that $\Theta \cap \Theta^* \neq \emptyset$, $\Theta \not\subseteq \Theta^*$, $\Theta^* \not\subseteq \Theta$. Using Θ^* (instead of Θ), authors in [9] have proved the following fixed point result:

Theorem 2.1. *On a complete metric space, every θ-contraction mapping (with $\theta \in \Theta^*$) possesses a unique fixed point.*

In recent past, Cho [6] initiated the idea of $\mathcal{L}$-simulation functions as follows:

Definition 2.1. A mapping $\zeta : [1, \infty) \times [1, \infty) \to \mathbb{R}$ is said to be a $\mathcal{L}$-simulation function if the following conditions are satisfied:

(ζ_1) $\zeta(1, 1) = 1$;
(ζ_2) $\zeta(x, y) < \frac{y}{x}$ for all $x, y > 1$;
(ζ_3) if $\{x_n\}, \{y_n\}$ are sequences in $(1, \infty)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 1$, then $\limsup_{n\to\infty} \zeta(x_n, y_n) < 1$.

The family of $\mathcal{L}$-simulation functions will be denoted by $\mathcal{L}$. Some examples of $\mathcal{L}$-simulation functions are as under:

Example 2.5. [6] We define the mappings $\zeta_i : [1, \infty) \times [1, \infty) \to \mathbb{R}$ for $i = 1, 2, 3$, as follows:

- $\zeta_1(x, y) = \frac{y^k}{x}$ for all $x, y \in [1, \infty)$, where $k \in (0, 1)$.
- $\zeta_2(x, y) = \frac{y}{x\varphi(y)}$ for all $x, y \in [1, \infty)$, where $\varphi : [1, \infty) \to [1, \infty)$ is a lower semi continuous and non-decreasing function such that $\varphi^{-1}(\{1\}) = \{1\}$.
- $$\zeta_3(x, y) = \begin{cases} 1 & if\ (x, y) = (1, 1); \\ \frac{x}{2y} & if\ x < y; \\ \frac{y^k}{x} & elsewhere, \end{cases}$$
  for all $x, y \in [1, \infty)$ and $k \in (0, 1)$.

Then ζ_i are $\mathcal{L}$-simulation functions for $i = 1, 2, 3$.

Utilizing $\mathcal{L}$-simulation functions, Cho [6] introduced $\mathcal{L}$-contraction in generalized metric space without using (θ_4). Here, we mention the definition of $\mathcal{L}$-contraction for $\theta \in \Theta^*$ in the setting of metric space.

Definition 2.2. Let (M, d) be a metric space and $S : M \to M$. Then S is said to be a $\mathcal{L}$-contraction w.r.t. ζ if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that

$$\zeta\big(\theta(d(Sr, Ss)), \theta(d(r, s))\big) \geq 1 \tag{2.1}$$

for all $r, s \in M$ with $d(Sr, Ss) > 0$.

If we take $\zeta(x, y) = \frac{y^k}{x}$ for all $x, y \in [1, \infty)$ with $k \in (0, 1)$ then $\mathcal{L}$-contraction takes the form of θ-contraction which was extensively used in getting many results in the literature.

Remark 2.1. Due to the condition (ζ_2), we have $\zeta(x, x) < 1$, for all $x > 1$. Therefore, if a mapping S is a $\mathcal{L}$-contraction then it cannot be an isometry (*i.e.*, distance does not preserve under such mappings).

Using $\mathcal{L}$-contraction, Cho [6] obtained some fixed point results in Branciari distance spaces while author presumed the continuity condition of θ without mentioning it. Authors in [11] demonstrated that there is a gap in the proof of ([6], Theorem 4) and modified the assumption and presented a new proof. In this paper, we restrict ourselves in metric space setting and metrical version of the theorem due to Cho [[6], Theorem 4] (whose modified version appears as a corollary in [11]) is as under:

Theorem 2.2. [6] *Let (M, d) be a complete metric space and $S : M \to M$ be a $\mathcal{L}$-contraction w.r.t. some ζ. Then S has a unique fixed point.*

Before proceeding further, we recollect some basic relation-theoretic notions, definitions and relevant results described in the following:
Any subset $\mathcal{R}$ of $M \times M$ is said to be a binary relation on a non-empty set M. Trivially, $\emptyset$ and $M \times M$ are known as the empty relation and the universal relation on M, respectively. From now on, a non-empty binary relation will be denoted by $\mathcal{R}$. If $(r, s) \in \mathcal{R}$ and $(s, t) \in \mathcal{R}$ imply $(r, t) \in \mathcal{R}$, for all $r, s, t \in M$ then $\mathcal{R}$ is said to be transitive relation on M. Furthermore, if S is a self mapping on M, then $\mathcal{R}$ is said to be S-transitive if it is transitive on $S(M)$. The inverse of $\mathcal{R}$ is denoted by $\mathcal{R}^{-1}$ and is defined as $\mathcal{R}^{-1} := \{(r, s) \in M \times M : (s, r) \in \mathcal{R}\}$ and $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$. Two elements r and s of M are said to be $\mathcal{R}$-comparable if $(r, s) \in \mathcal{R}$ or $(s, r) \in \mathcal{R}$ and is denoted by $[r, s] \in \mathcal{R}$.

Proposition 2.1. [1] *For a binary relation $\mathcal{R}$ defined on a non-empty set M,*

$$(r, s) \in \mathcal{R}^s \text{ if and only if } [r, s] \in \mathcal{R}.$$

Definition 2.3. [1] Let $\mathcal{R}$ be a binary relation on a non-empty set M. A sequence $(r_n) \subset M$ is called $\mathcal{R}$-preserving if

$$(r_n, r_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0.$$

Definition 2.4. [1] Let $\mathcal{R}$ be a binary relation on a non-empty set M and $S : M \to M$. Then $\mathcal{R}$ is said to be S-closed if for any $r, s \in M$,

$$(r, s) \in \mathcal{R} \text{ implies } (Sr, Ss) \in \mathcal{R}.$$

Definition 2.5. [3] Let (M, d) be a metric space and $\mathcal{R}$ be binary relation on M. If every $\mathcal{R}$-preserving Cauchy sequence converges to some point of M, then we say (M, d) is $\mathcal{R}$-complete.

Remark 2.2. Every complete metric space is $\mathcal{R}$-complete for arbitrary binary relation $\mathcal{R}$. Particularly, under the universal relation $\mathcal{R}$-completeness turn into the usual completeness.

Definition 2.6. [1] Let (M, d) be a metric space equipped with a binary relation $\mathcal{R}$. We say $\mathcal{R}$ is d-self-closed if whenever $\mathcal{R}$-preserving sequence (r_n) converges to r, then there exists a subsequence $(r_{n(l)})$ of (r_n) with $[r_{n(l)}, r] \in \mathcal{R}$, for all $l \in \mathbb{N}_0$.

Definition 2.7. [2] Let (M, d) be a metric space endowed with a binary relation $\mathcal{R}$ and $S : M \to M$. Then S is said to be $\mathcal{R}$-continuous at $r \in M$ if for any $\mathcal{R}$-preserving sequence $(r_n) \subset M$ with $r_n \xrightarrow{d} r$, implies $Sr_n \xrightarrow{d} Sr$. If S is $\mathcal{R}$-continuous for every point of M then we say that S is $\mathcal{R}$-continuous.

Remark 2.3. Every continuous mapping can be treated as $\mathcal{R}$-continuous mapping (irrespective of a binary relation $\mathcal{R}$). On the other hand, $\mathcal{R}$-continuity turn into with the usual continuity under the universal relation.

Definition 2.8. [14] For $r, s \in M$, a path (of length n, $n \in \mathbb{N}$) in $\mathcal{R}$ from r to s is a sequence (finite) $\{r_0, r_1, r_2, \cdots, r_n\} \subseteq M$ such that $r_0 = r,\ r_n = s$ with $(r_i, r_{i+1}) \in \mathcal{R}$, for each $i \in \{0, 1, \cdots, n-1\}$.

It is worth mentioning here that a path of length n involves $n+1$ elements of M (not necessarily distinct).

Definition 2.9. [2] A subset $N \subseteq M$ is said to be $\mathcal{R}$-connected if for each $r, s \in N$, there exists a path from r to s in $\mathcal{R}$.

The following notations will be used in the forthcoming discussions.

(•) $M(S; \mathcal{R}) := \{r \in M : (r, Sr) \in \mathcal{R}\}$, where $S : M \to M$ be any given mapping;
(•) $\Upsilon(r, s, \mathcal{R}) :=$ the family of all paths from r to s in $\mathcal{R}$, where $r, s \in M$.

3. Main results

We start this section by defining the notion of $\mathcal{L}_{\mathcal{R}}$-contraction in metric space as follows:

Definition 3.1. Let $\mathcal{R}$ be a binary relation on metric space (M, d) and $S : M \to M$. We say that S is $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. $\zeta \in \mathcal{L}$, if there exist $\zeta \in \mathcal{L}$ and $\theta \in \Theta^*$ such that the following condition holds:

$$\zeta(\theta(d(Sr, Ss)), \theta(d(r, s))) \geq 1, \quad \forall\, r, s \in M \text{ with } (r, s) \in \mathcal{R}^* \tag{3.1}$$

where $(r, s) \in \mathcal{R}^* := \{(r, s) \in \mathcal{R} : Sr \neq Ss\}$.

By symmetricity of the metric d, we obtain the following proposition.

Proposition 3.1. *Let (M, d) be a metric space equipped with a binary relation $\mathcal{R}$, and $S : M \to M$. For a given $\zeta \in \mathcal{L}$, $\theta \in \Theta^*$, the following are equivalent:*

(i) $\forall\, r, s \in M$ *with* $(r, s) \in \mathcal{R}^* \implies \zeta(\theta(d(Sr, Ss)), \theta(d(r, s))) \geq 1$;
(ii) $\forall\, r, s \in M$ *with* $[r, s] \in \mathcal{R}^* \implies \zeta(\theta(d(Sr, Ss)), \theta(d(r, s))) \geq 1$.

Now, we state and prove our main result.

Theorem 3.1. *Let (M, d) be a metric space, $\mathcal{R}$ be a binary relation on M and $S : M \to M$. Suppose that the following conditions hold:*

(i) $M(S; \mathcal{R})$ is non-empty;
(ii) $\mathcal{R}$ is S-closed and S-transitive;
(iii) S is $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. some $\zeta \in \mathcal{L}$;
(iv) (M, d) is $\mathcal{R}$-complete;
(v) either S is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed.

Then S has a fixed point. Moreover, for each $r_0 \in M(S; \mathcal{R})$, the Picard sequence $S^n(r_0)$ for all $n \in \mathbb{N}$, converges to a fixed point of S.

Proof. Since $M(S; \mathcal{R}) \neq \emptyset$, let r_0 be an arbitrary point such that $r_0 \in M(S; \mathcal{R})$. Now define a sequence (r_n) by $r_n = S^n r_0$, for all $n \in \mathbb{N}_0$. Since $(r_0, Sr_0) \in \mathcal{R}$, then due to the S-closedness of $\mathcal{R}$, we have

$$(r_n, Sr_n) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{3.2}$$

Now, if there exists some $n_0 \in \mathbb{N}_0$ such that $d(r_{n_0}, Sr_{n_0}) = 0$, then the result follows immediately. Otherwise, for all $n \in \mathbb{N}_0$, $r_n \neq r_{n+1}$ *i.e.*, $d(Sr_{n-1}, Sr_n) > 0$ which enable us to conclude that $(r_{n-1}, r_n) \in \mathcal{R}^*$. As S is $\mathcal{L}_{\mathcal{R}}$-contraction, we have

$$\begin{aligned} 1 &\leq \zeta\big(\theta(d(Sr_{n-1}, Sr_n)), \theta(d(r_{n-1}, r_n))\big) \\ &< \frac{\theta(d(r_{n-1}, r_n))}{\theta(d(Sr_{n-1}, Sr_n))} \\ \theta(d(Sr_{n-1}, Sr_n)) &< \theta(d(r_{n-1}, r_n)), \end{aligned} \tag{3.3}$$

then due to (θ_1), we deduce $d(r_n, r_{n+1}) < d(r_{n-1}, r_n)$ for all $n \in \mathbb{N}$. Therefore, $\{d(r_n, r_{n+1})\}_{n=0}^{\infty}$ is a monotonically decreasing sequence of positive real numbers, and hence there exists $a \geq 0$, such that $\lim\limits_{n\to\infty} d(r_n, r_{n+1}) = a$.

Now, we show that $a = 0$. On contrary, suppose that $a > 0$ then by using (θ_4), we obtain

$$\lim_{n\to\infty} \theta(d(r_n, r_{n+1})) = \lim_{n\to\infty} \theta(d(r_{n+1}, r_{n+2})) = \theta(a).$$

Now, if we set $x_n = \theta(d(r_n, r_{n+1}))$, $y_n = \theta(d(r_{n+1}, r_{n+2}))$ then $y_n < x_n$, for all $n \in \mathbb{N}$ (by (3.3)) and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n > 1$. Then by (ζ_3), we obtain

$$1 \leq \limsup_{n\to\infty} \zeta\big(\theta(d(Sr_n, Sr_{n+1})), \theta(d(r_n, r_{n+1}))\big) < 1,$$

which is a contradiction and hence $l = 0, i.e.$,

$$\lim_{n\to\infty} d(r_n, r_{n+1}) = 0, \tag{3.4}$$

by (θ_2), we also have

$$\lim_{n\to\infty} \theta(d(r_n, r_{n+1})) = 1.$$

Next, we show that (r_n) is Cauchy. To do this, on contrary let (r_n) is not Cauchy, then there exists $\epsilon > 0$ and $l_0 \in \mathbb{N}_0$ with $m(l) > n(l) > l \geq l_0$, such that

$$d(r_{m(l)}, r_{n(l)}) \geq \epsilon \;\text{ and }\; d(r_{m(l)-1}, r_{n(l)}) < \epsilon.$$

Thus, we can have

$$\epsilon \leq d(r_{m(l)}, r_{n(l)}) \leq d(r_{m(l)}, r_{m(l)-1}) + d(r_{m(l)-1}, r_{n(l)}) < d(r_{m(l)}, r_{m(l)-1}) + \epsilon$$

taking $l \to \infty$ and using (3.4), we get

$$\lim_{l\to\infty} d(r_{m(l)}, r_{n(l)}) = \epsilon \;\text{ or }\; \lim_{l\to\infty} \theta(d(r_{m(l)}, r_{n(l)})) = \theta(\epsilon), \tag{3.5}$$

and hence

$$\lim_{l\to\infty} d(r_{m(l)+1}, r_{n(l)+1}) = \epsilon \text{ or } \lim_{l\to\infty} \theta(d(r_{m(l)+1}, r_{n(l)+1})) = \theta(\epsilon). \tag{3.6}$$

As the sequence (r_n) is $\mathcal{R}$-preserving and $\mathcal{R}$ is S-transitive, therefore $(r_{m(l)}, r_{n(l)}) \in \mathcal{R}^*$ and we obtain

$$\zeta\big(\theta(d(Sr_{m(l)}, Sr_{n(l)})), \theta(d(r_{m(l)}, r_{n(l)}))\big) \geq 1.$$

Now taking $l \to \infty$ and on using (3.5), (3.6) and (ζ_3), we get

$$1 \leq \limsup_{l\to\infty} \zeta\big(\theta(d(Sr_{m(l)}, Sr_{n(l)})), \theta(d(r_{m(l)}, r_{n(l)}))\big) < 1,$$

which is a contradiction. Thus, the sequence (r_n) is an $\mathcal{R}$-preserving Cauchy sequence in M. Owing to the $\mathcal{R}$-completeness of M, there exists $r^* \in M$ such that $r_n \xrightarrow{d} r^*$. If S is $\mathcal{R}$-continuous, then we have

$$r^* = \lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} Sr_n = S(\lim_{n\to\infty} r_n) = Sr^*,$$

and hence r^* is a fixed point of S.

Otherwise, suppose that $\mathcal{R}$ is d-self-closed. Then, there exists a subsequence $(r_{n(l)})$ of (r_n) with $[r_{n(l)}, r^*] \in \mathcal{R}$, for all $l \in \mathbb{N}_0$. Now, without loss of generality, we may assume that $r_{n(l)} \neq r^*$, for all $l \in \mathbb{N}$. Since S is $\mathcal{L}_{\mathcal{R}}$-contraction then from Eq. (3.1) and Proposition 3.1, we have

$$\zeta\big(\theta(d(Sr_{n(l)}, Sr^*)), \theta(d(r_{n(l)}, r^*))\big) \geq 1, \quad \forall\, l \in \mathbb{N}_0. \tag{3.7}$$

We show that r^* is a fixed point of S. On contrary, suppose that it is not the case then $d(Sr^*, r^*) > 0$ which gives rise $\theta(d(Sr^*, r^*)) > 1$. By using (ζ_2) and (3.7), we obtain

$$\begin{array}{rcl} 1 & \leq & \zeta(\theta(d(Sr_{n(l)}, Sr^*)), \theta(d(r_{n(l)}, r^*))) \\ & < & \dfrac{\theta(d(r_{n(l)}, r^*))}{\theta(d(Sr_{n(l)}, Sr^*))} \\ \limsup\limits_{l\to\infty} \theta(d(Sr_{n(l)}, Sr^*)) & \leq & \limsup\limits_{l\to\infty} \theta(d(r_{n(l)}, r^*)) = 1 \\ \text{or, } \limsup\limits_{l\to\infty} \theta(d(r_{n(l)+1}, Sr^*)) & \leq & 1 \\ \Longrightarrow \quad \theta(d(Sr^*, r^*)) & \leq & 1, \end{array}$$

a contradiction, which ensures that r^* is a fixed point of S. □

Next, we prove corresponding uniqueness fixed point result.

Theorem 3.2. *In addition to the assumptions of Theorem 3.1, if $\Upsilon(r, s; \mathcal{R}|_{S(M)})$ is non-empty for all $r, s \in S(M)$ then S admits a unique fixed point.*

Proof. On the lines of the proof of Theorem 3.1, one can show that $Fix(S)$ is non-empty. Now, if $Fix(S)$ is singleton then the proof is obvious. Otherwise, there exist two distinct elements $r^*, s^* \in Fix(S)$. As $\Upsilon(r, s; \mathcal{R}|_{S(M)})$ is non-empty for all $r, s \in S(M)$, there exists a path of some finite length n from r^* to s^* in $\mathcal{R}|_{S(M)}$ say $\{Sr_0, Sr_1, Sr_2, \cdots, Sr_n\}$ such that $r^* = Sr_0, s^* = Sr_n$ with $(Sr_i, Sr_{i+1}) \in \mathcal{R}|_{S(M)}$ for each $i = 0, 1, 2, \cdots, n-1$. As $\mathcal{R}$ is S-transitive, we obtain

$$(r^*, Sr_1) \in \mathcal{R}, (Sr_1, Sr_2) \in \mathcal{R}, \cdots, (Sr_{n-1}, s^*) \in \mathcal{R} \text{ implies } (r^*, s^*) \in \mathcal{R}.$$

Now, as S is $\mathcal{L}_{\mathcal{R}}$-contraction, we have

$$1 \leq \zeta\big(\theta(d(Sr^*, Ss^*)), \theta(d(r^*, s^*))\big) < \frac{\theta(d(r^*, s^*))}{\theta(d(Sr^*, Ss^*))} = 1,$$

a contradiction. Therefore, the fixed point of S is unique. □

Remark 3.1. In Theorem 3.1 and Theorem 3.2, if we consider $\mathcal{R}$ to be a transitive binary relation then the conclusions still hold as every transitive binary relation is S-transitive for a given self-map S on M.

Now, we provide an explanatory example in the support of our newly obtained results while many related results in the existing literature are not applicable.

Example 3.1. Let $(M=(0,\infty),d)$ be a metric space endowed with a binary relation $\mathcal{R}:=\{(\frac{1}{2},1),(1,1),(1,2),(1,5),(2,1),(2,2),(2,4),(2,5),(1,4)\}$, where $d(r,s)=|r-s|$, for all $r,s\in M$. Define a mapping $S:M\to M$ by

$$Sr=\begin{cases} 2 & if\ r\in(0,4);\\ 1 & if\ r=4;\\ r-1 & if\ r>4,\end{cases}$$

then $M(S;\mathcal{R})\neq\emptyset$ as $(1,S1)=(1,2)\in\mathcal{R}$, S is $\mathcal{R}$-continuous which is not continuous in usual sense. Also, M is $\mathcal{R}$-complete, $\mathcal{R}$ is S-closed and $\mathcal{R}$ is S-transitive but not transitive.

Now, choose $\theta(\beta)=e^{\beta}$ for all $\beta>0$ and if we take $\zeta^*(x,y)=\frac{y}{x\varphi(y)}$ for all $x,y\in[1,\infty)$, where $\varphi:[1,\infty)\to[1,\infty)$ is defined by

$$\varphi(y)=\begin{cases} 1 & if\ y\leq e^2;\\ \sqrt[3]{y} & if\ y>e^2.\end{cases}$$

Then it is easy to verify that S is $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. $\zeta^*\in\mathcal{L}$. Since, we have $\mathcal{R}^*=\{(1,5),(2,4),(2,5),(1,4)\}$, then the following four cases are arise:

Case:(I) If we take $r=1,s=5$, then we have

$$\zeta(\theta(d(S1,S5)),\theta(d(1,5)))=\frac{e^4}{e^{\frac{10}{3}}}=e^{\frac{2}{3}}>1.$$

Case:(II) If $r=2,s=4$, then we have

$$\zeta(\theta(d(S2,S4)),\theta(d(2,4)))=\frac{e^2}{e\cdot 1}=e>1.$$

Case: (III) For $r=2$ and $s=5$, we have

$$\zeta(\theta(d(S2,S5)),\theta(d(2,5)))=\frac{e^3}{e^2\cdot e}=1.$$

Case: (IV) For $r=1$ and $s=4$, we get

$$\zeta(\theta(d(S1,S4)),\theta(d(1,4)))=\frac{e^3}{e\cdot e}=e>1.$$

Hence, S is $\mathcal{L}_{\mathcal{R}}$-contraction w.r.t. $\zeta^*\in\mathcal{L}$. Therefore, all the required conditions of Theorems 3.1 and 3.2 are fulfilled and consequently S has a unique fixed point. It is worth mentioning here that S is not a $\mathcal{L}$-contraction w.r.t. any $\theta\in\Theta^*$ and $\zeta\in\mathcal{L}$ (by Remark 2.1, as S is an isometry for $r,s>4$), so we can not apply Theorem 2.2. The present example demonstrates the utility of our results over the known relevant results especially in the context of contraction condition.

If we take $\zeta(x,y)=\frac{y^k}{x}$ for all $x,y\in[1,\infty)$, where $k\in(0,1)$ in Theorems 3.1 and 3.2, then we get the following corollary which is a sharpened version as well as relation theoretic analog of the main result due to Jamshaid et al. [9] (see [9], Theorem 2.2):

Corollary 3.1. *Let (M,d) be a metric space endowed with a binary relation $\mathcal{R}$ and $S:M\to M$. Suppose that the following conditions hold:*

(i) $M(S;\mathcal{R})$ *is non-empty;*
(ii) $\mathcal{R}$ *is* S*-closed and* S*-transitive;*
(iii) *there exists* $\theta \in \Theta^*$ *such that*

$$\theta(d(Sr,Ss)) \leq \theta(d(r,s))^k$$

for all $r,s \in M$ *with* $(r,s) \in \mathcal{R}^*$ *and* $k \in (0,1)$*;*
(iv) (M,d) *is* $\mathcal{R}$*-complete;*
(v) *either* S *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* d*-self-closed.*

Then S *has a fixed point. Moreover, for each* $r_0 \in M(S;\mathcal{R})$*, the Picard sequence* $S^n(r_0)$ *for all* $n \in \mathbb{N}$*, converges to a fixed point of* S*. In addition, if* $\Upsilon(r,s;\mathcal{R}|_{S(M)})$ *is non-empty for all* $r,s \in S(M)$ *then* S *admits a unique fixed point.*

Also, by taking $\zeta = \zeta_2$ (as defined in Example 2.5) in Theorems 3.1 and 3.2, we deduce the following:

Corollary 3.2. *Let* (M,d) *be a metric space endowed with a binary relation* $\mathcal{R}$ *and* $S : M \to M$*. Suppose that the following conditions hold:*

(i) $M(S;\mathcal{R})$ *is non-empty;*
(ii) $\mathcal{R}$ *is* S*-closed and* S*-transitive;*
(iii) *there exists* $\theta \in \Theta^*$ *such that*

$$\theta(d(Sr,Ss)) \leq \frac{\theta(d(r,s))}{\varphi(\theta(d(r,s)))}$$

for all $r,s \in M$ *with* $(r,s) \in \mathcal{R}^*$ *and* $\varphi : [1,\infty) \to [1,\infty)$ *is a lower semi continuous and nondecreasing function such that* $\varphi^{-1}(\{1\}) = \{1\}$;
(iv) (M,d) *is* $\mathcal{R}$*-complete;*
(v) *either* S *is* $\mathcal{R}$*-continuous or* $\mathcal{R}$ *is* d*-self-closed.*

Then S *has a fixed point. Moreover, if* $\Upsilon(r,s;\mathcal{R}|_{S(M)})$ *is non-empty for all* $r,s \in S(M)$ *then* S *admits a unique fixed point.*

If we choose $\mathcal{R} = \{(r,s) \in M \times M \mid r \preceq s\}$ (where, '$\preceq$' stands for natural ordering on the set M) in Theorem 3.1 and Theorem 3.2 then we deduce the following corollaries which seems to be new in the existing literature.

Corollary 3.3. *Let* $(M,d,\preceq)$ *be an ordered metric space and* $S : M \to M$*. Suppose that the following conditions hold:*

(i) *there exists* $r_0 \in M$ *such that* $r_0 \preceq Sr_0$*;*
(ii) S *is increasing (i.e.,* $r \preceq s$ *implies* $Sr \preceq Ss$*);*
(iii) S *is* $\mathcal{L}_{\preceq}$*-contraction w.r.t. some* $\zeta \in \mathcal{L}$*;*
(iv) (M,d) *is* $\preceq$*-complete;*
(v) *either* S *is* $\preceq$*-continuous or* $\preceq$ *is* d*-self-closed.*

Then S *has a fixed point. Moreover, for each* $r_0 \in M$ *such that* $r_0 \preceq Sr_0$*, the Picard sequence* $S^n(r_0)$ *for all* $n \in \mathbb{N}$ *converges to a fixed point of* S*.*

Corollary 3.4. *In addition to the assumptions of Corollary 3.3, if* $\Upsilon(r,s;\preceq|_{S(M)})$ *is non-empty for all* $r,s \in S(M)$ *then the fixed point of* S *is unique.*

4. Application to ordinary differential equation

In this section, we apply one of our main results to obtain a non-negative solution under some suitable conditions for the following ordinary differential equation:

$$\begin{cases} u'(t) = g(t, u(t)), & a.e.\ t \in I := [0,1], \\ u(0) = a, & a \geq 0, \end{cases} \tag{4.1}$$

where $g : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function.

Consider $M = \{u \in C(I, \mathbb{R}) : u(t) \geq 0,\ for\ all\ t \in I\}$, where $C(I, \mathbb{R})$ is the space of all continuous functions $u : I \to \mathbb{R}$ equipped with the supremum norm

$$\|u\| = \sup_{t \in I} \|u(t)\|.$$

Now, let us consider the metric d induced by supremum norm on M, *i.e.*, $d(x, y) = \|x - y\|$, for all $x, y \in M$. Clearly, the metric space (M, d) is complete.
Also, let Φ be the collection of all mappings $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(φ_1) φ is non-decreasing;
(φ_2) $\varphi(r) \leq r$, for all $r \in [0, \infty)$.

Theorem 4.1. *Consider the differential Eq.* (4.1) *and if g satisfies the following conditions:*

(c_1) g is non-decreasing in the second variable with $g(r, p) \geq 0$, for all $p \geq 0$ and $r \in I$;
(c_2) there exists a positive number h and $\varphi \in \Phi$ such that

$$|g(r, x(r)) - g(r, y(r))| \leq \varphi\Big(\frac{|x(r) - y(r)|}{(1 + h\sqrt{\|x - y\|})^2}\Big),$$

for each $x, y \in M$ with $x(r) \leq y(r)$ for all $r \in I$.

Then the differential Eq. (4.1) *has a non-negative solution.*

Proof. Consider a mapping $\mathcal{S} : M \to M$ defined by

$$(\mathcal{S}u)(t) = \int_0^t g(r, u(r))dr + a, \quad u \in M.$$

Clearly, the solutions of (4.1) are nothing but fixed points of $\mathcal{S}$. Now, define a binary relation $\mathcal{R}$ on M as follows:

$$\mathcal{R} := \{(x, y) \in \mathcal{R} \Leftrightarrow x(t) \leq y(t),\ \text{for all } x, y \in M \text{ and } t \in I\}.$$

Therefore, for all $x, y \in M$ with $(x, y) \in \mathcal{R}$ and $t \in I$, we have

$$\begin{aligned} (\mathcal{S}x)(t) = \quad & \int_0^t g(r, x(r))dr + a \leq \int_0^t g(r, y(r))dr + a = (\mathcal{S}y)(t) \\ & \Rightarrow (\mathcal{S}x)(t) \leq \mathcal{S}(y(t)). \end{aligned}$$

Thus, we have $(\mathcal{S}x, \mathcal{S}y) \in \mathcal{R}$, *i.e.*, $\mathcal{R}$ is $\mathcal{S}$-closed. By the definition of $\mathcal{R}$, it is clear that $\mathcal{R}$ is transitive and hence $\mathcal{S}$-transitive. Also, we have $\mathbf{0} \in M$ such that $(\mathbf{0}, \mathcal{S}\mathbf{0}) \in \mathcal{R}$ and hence $M(\mathcal{S}; \mathcal{R}) \neq \emptyset$. It is easy to see that $\mathcal{S}$ is $\mathcal{R}$-continuous and being the space

(M, d) complete, it is $\mathcal{R}$-complete.
Now, let $(x, y) \in \mathcal{R}^* = \{(x, y) \in \mathcal{R} : Sx \neq Sy\}$ and $t \in I$. Then

$$\begin{aligned}
|(\mathcal{S}x)(t) - (\mathcal{S}y)(t)| &= \left| \int_0^t (g(r, x(r)) - g(r, y(r)))dr \right| \\
&\leq \int_0^t |(g(r, x(r)) - g(r, y(r)))| \, dr \\
&\leq \int_0^t \varphi\Big(\frac{|x(r) - y(r)|}{(1 + h\sqrt{\|x - y\|})^2} \Big) dr \\
&\leq \int_0^t \varphi\Big(\frac{\|x - y\|}{(1 + h\sqrt{\|x - y\|})^2} \Big) dr \quad \text{(by using condition } (\varphi_1)) \\
&\leq \int_0^t \frac{\|x - y\|}{(1 + h\sqrt{\|x - y\|})^2} dr. \qquad \text{(by using condition } (\varphi_2))
\end{aligned}$$

Thus, we obtain

$$|(\mathcal{S}x)(t) - (\mathcal{S}y)(t)| \leq \frac{\|x - y\|}{(1 + h\sqrt{\|x - y\|})^2}, \ \forall t \in I.$$

Taking supremum over both the sides, we have

$$\begin{aligned}
\|\mathcal{S}x - \mathcal{S}y\| &\leq \frac{\|x - y\|}{(1 + h\sqrt{\|x - y\|})^2} \\
\implies \Big(h + \frac{1}{\sqrt{\|x - y\|}} \Big)^2 &\leq \frac{1}{\|\mathcal{S}x - \mathcal{S}y\|} \\
\implies \quad h + \frac{1}{\sqrt{\|x - y\|}} &\leq \frac{1}{\sqrt{\|\mathcal{S}x - \mathcal{S}y\|}} \\
\implies \quad \frac{-1}{\sqrt{d(\mathcal{S}x, \mathcal{S}y)}} &\leq \frac{-1}{\sqrt{d(x, y)}} - h.
\end{aligned}$$

Taking $\theta(\beta) = e^{e^{-\frac{1}{\sqrt{\beta}}}}$ and $\zeta(r, s) = \frac{s^k}{r}$ for all $r, s \in [1, \infty)$ such that $k \in (0, 1)$, it follows that $\mathcal{S}$ is $\mathcal{L}_{\mathcal{R}}$-contraction, *i.e.*, $\mathcal{S}$ satisfies (3.1) with this θ and ζ (here $k = e^{-h}, h > 0$). Therefore, by Theorem 3.1, S has a fixed point and consequently the differential Eq. (4.1) has a non-negative solution. □

5. Conclusions

In this paper, we have obtained an existence as well as uniqueness fixed point results for newly introduced $\mathcal{L}_{\mathcal{R}}$-contraction employing a binary relation on metric spaces without completeness condition which in turn generalize, extend and unify several results in the existing literature. Also, we have utilized our main results to show the existence of a non-negative solution of the first-order differential equation. On the similar lines we can undertake the investigation of the existence of a common fixed point for two or more maps under suitable conditions. Also, these results can be extended to more general spaces such as partial metric spaces, b-metric spaces, semi-metric spaces and similar other abstract distance spaces.

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