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DARBO TYPE BEST PROXIMITY POINT (PAIR) RESULTS USING MEASURE OF NONCOMPACTNESS WITH APPLICATION

MOOSA GABELEH*, DEEPESH KUMAR PATEL** AND PRADIP RAMESH PATLE***

*Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
E-mail: Gabeleh@abru.ac.ir, gab.moo@gmail.com

**Department of Mathematics, Visvesvaraya National Institute of Technology,
Nagpur-400010, India
E-mail: deepesh456@gmail.com

***Department of Mathematics, Amity School of Engineering and Technology,
Amity University Madhya Pradesh, Gwalior 474020, M.P., India
E-mail: pradip.patle12@gmail.com, prpatle@gwa.amity.edu

Abstract. Primarily this work intends to investigate the existence of best proximity points (pairs) for new classes of cyclic (noncyclic) mappings via simulation functions and measure of noncompactness. Use of different classes of additional functions make it possible to generalize the contractive inequalities in this work. As an application of the main conclusions, a survey for the existence of optimal solutions of a system of integro-differential equations under some new conditions is presented. As an application of our existence results, we establish the existence of a solution for the following system of integro-differential equations

$$\begin{cases} u'(t) = F_1(t, u(t), \int_{t_0}^{t} k_1(t,s,u(s))ds), & u(t_0) = u_1, \\ v'(t) = F_2(t, v(t), \int_{t_0}^{t} k_2(t,s,v(s))ds), & v(t_0) = u_2, \end{cases}$$

in the space of all bounded and continuous real functions on $[0, +\infty[$ under suitable assumptions on F_1, F_2.

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1. Introduction

In 1910, Brouwer established a fundamental fixed point theorem which states that every continuous self-mapping defined on a closed ball in $\mathbb{R}^n$ admits a fixed point. Later on, Schauder extended Brouwer's fixed point result to Banach spaces as follows.

Theorem 1.1. ([20]) *Let K be a nonempty, compact and convex subset of a Banach space $\mathcal{X}$ and $T : K \to K$ be a continuous mapping. Then T has a fixed point.*

Schauder's fixed point theorem is a very useful tool for proving the existence of solutions to many nonlinear problems, especially problems concerning ordinary and partial differential equations, and it has a number of extensions.

Let $\mathcal{X}$ be a normed linear space and $T : K \subseteq \mathcal{X} \to \mathcal{X}$ be a mapping. Then T is called a *compact operator* provided that T is continuous and maps bounded sets into relatively compact sets.
For such mappings the Schauder's fixed point theorem was generalized as below.
Theorem 1.2. *Let K be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{X}$ and $T : K \to K$ be a compact operator. Then T has a fixed point.*

Another important improvement of Schauder's theorem was presented by Darbo [7] using the concept of measure of noncompactness. Before going into details about these generalizations, we will recall the important notion of measure of noncompactness.
Let $\mathcal{B}(\mathcal{X})$ be a collection of bounded subsets of a metric space $\mathcal{X}$.
Definition 1.1. A mapping $\mathcal{N} : \mathcal{B}(\mathcal{X}) \to [0, +\infty)$ is said to be a measure of noncompactness (MNC) on $\mathcal{X}$ if it satisfies the following axioms:

(1) $\mathcal{N}(P) = 0$ if and only if P is relatively compact,
(2) $\mathcal{N}(P) = \mathcal{N}(\overline{P})$, $P \in \mathcal{B}(\mathcal{X})$,
(3) $\mathcal{N}(P \cup Q) = \max\{\mathcal{N}(P), \mathcal{N}(Q)\}$, where $P, Q \in \mathcal{B}(\mathcal{X})$.

An MNC mapping $\mathcal{N}$ on $\mathcal{B}(\mathcal{X})$ satisfies the following properties (see [5]):

(a) $P \subseteq Q$ implies $\mathcal{N}(P) \leq \mathcal{N}(Q)$;
(b) If P is a finite set, then $\mathcal{N}(P) = 0$;
(c) $\mathcal{N}(P \cap Q) \leq \min\{\mathcal{N}(P), \mathcal{N}(Q)\}$, for all $P, Q \in \mathcal{B}(\mathcal{X})$.
(d) If $\lim\limits_{n\to\infty} \mathcal{N}(P_n) = 0$ for a nonincreasing sequence $\{P_n\}$ of nonempty, bounded and closed subsets of $\mathcal{X}$, then $P_\infty := \cap_{n\geq 1} P_n$ is nonempty and compact.

On a Banach space $\mathcal{X}$, an MNC mapping $\mathcal{N}$ on $\mathcal{B}(\mathcal{X})$ has following properties:

(i) $\mathcal{N}(\overline{con}(Q)) = \mathcal{N}(Q)$, for all $Q \in \mathcal{B}(\mathcal{X})$, where $\overline{con}(Q)$ denotes the closed and convex hull of the set $Q \in \mathcal{B}(\mathcal{X})$;
(ii) $\mathcal{N}(\lambda Q) = |\lambda|\mathcal{N}(Q)$ for any number λ and $Q \in \mathcal{B}(\mathcal{X})$;
(iii) $\mathcal{N}(P + Q) \leq \mathcal{N}(P) + \mathcal{N}(Q)$ for all $P, Q \in \mathcal{B}(\mathcal{X})$.

Here, we mention two well-known examples of MNCs.
Example 1.1. Let $(\mathcal{X}, d)$ be a metric space. The function $\alpha : \mathcal{B}(\mathcal{X}) \to [0, \infty)$ defined as

$$\alpha(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \leq \varepsilon\},$$
$$\forall B \in \mathcal{B}(\mathcal{X})$$

is called the Kuratowski measure of noncompactness.
Similarly, the function $\chi : \mathcal{B}(\mathcal{X}) \to [0, \infty)$ defined by

$$\chi(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\},$$
$$\forall B \in \mathcal{B}(\mathcal{X}),$$

is called the Hausdorff measure of noncompactness. It was introduced in [13] as a generalization of the Kuratowski measure of noncompactness. We refer to [5] for more interesting information related to measures of noncompactness.

We are now ready to state a famous generalization of Theorem 1.2.

Theorem 1.3. (Darbo, (1955)) *Let A be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{X}$ and $\mathcal{N}$ be an* MNC *on X. Suppose that $T : A \to A$ is a continuous mapping such that, for some $r \in [0, 1)$,*

$$\mathcal{N}(T(K)) \leq r\mathcal{N}(K), \tag{1.1}$$

for all nonempty and bounded $K \subseteq A$. Then T has a fixed point.

There are many extensions of Darbo's fixed point problem by considering various contractive conditions using appropriate control functions that satisfy the relation (1.1) (see [4] for more recent new generalizations).

The main purpose of this article is to study the existence of best proximity points (pairs) for new classes of cyclic (noncyclic) condensing operators by using the notion of measure of noncompactness. As an application of the main conclusions, a survey for the existence of optimal solutions of a system of integro-differential equations under some new conditions is presented.

2. Preliminaries

2.1. **Proximal Pairs.** Let us take two nonempty subsets P and Q of a normed linear space $\mathcal{X}$. It is to be assumed that a pair (P, Q) satisfies a property, if both P and Q individually satisfy that property. For example, we say a pair (P, Q) is convex if and only if P and Q are convex. If $\|a - b\| = \text{dist}(P, Q)$ for some $(a, b) \in P \times Q$, then (a, b) is called a *proximal point*. The proximal pair of (P, Q) is denoted by (P_0, Q_0) and defined as

$$P_0 = \{a \in P : \exists\, b' \in Q \mid \|a - b'\| = \text{dist}(P, Q)\},$$
$$Q_0 = \{b \in Q : \exists\, a' \in P \mid \|a' - b\| = \text{dist}(P, Q)\}.$$

Notice that (P_0, Q_0) maybe empty, but in particular, if (P, Q) is a nonempty, bounded, closed and convex pair in a reflexive Banach space $\mathcal{X}$, then (P_0, Q_0) is a nonempty pair and it is easy to see that it also closed and convex.

The pair (P, Q) is said to be *proximinal* provided that $P_0 = P$ and $Q_0 = Q$.

A mapping $T : P \cup Q \to P \cup Q$ is called *cyclic* if $T(P) \subseteq Q$ and $T(Q) \subseteq P$. By extension, T will be said *noncyclic* if $T(P) \subseteq P$ and $T(Q) \subseteq Q$. The mapping T is called *relatively nonexpansive* if it satisfies $\|Ta - Tb\| \leq \|a - b\|$ whenever $a \in P$ and $b \in Q$. In especial case, if $P = Q$, then T is said to be a nonexpansive mapping. Recall that $T : P \cup Q \to P \cup Q$ is compact means that $(\overline{T(P)}, \overline{T(Q)})$ is a compact pair.

Definition 2.1. Let (P, Q) be a nonempty pair in a normed linear space $\mathcal{X}$ and T ba a cyclic mapping on $P \cup Q$. A point $w^* \in P \cup Q$ is called a best proximity point for the mapping T provided that $\|w^* - Tw^*\| = \text{dist}(P, Q)$. In the case that T is noncyclic, then a point $(u^*, v^*) \in P \times Q$ is a best proximity pair for T if

$$u^* = Tu^*, \quad v^* = Tv^*, \quad \|u^* - v^*\| = \text{dist}(P, Q).$$

The first existence result of best proximity points (pairs) for cyclic (noncyclic) relatively nonexpansive mappings was established in [8] (see Theorem 2.1 and 2.2 of [8]). Their main conclusions is based on a geometric property, called *proximal normal structure* defined on a nonempty and convex pair of subsets of a Banach space $\mathcal{X}$.

It was announced in [9] that every nonempty, compact and convex pair in a Banach space $\mathcal{X}$ has the proximal normal structure. Using these facts, the following existence result was proved.

Theorem 2.1. (Theorem 3.2 of [11]) *Let (P, Q) be a nonempty, bounded, closed and convex pair in a Banach space $\mathcal{X}$ such that P_0 is nonempty. Assume that $T : P \cup Q \to P \cup Q$ is a cyclic relatively nonexpansive mapping. If T is compact, then T has a best proximity point.*

Before stating the same result of Theorem 2.1 for noncyclic mappings, let us recall a Banach space $\mathcal{X}$ is strictly convex if for $a, b, x \in \mathcal{X}$ and $\Lambda > 0$,

$$\big[\|a - x\| \leq \Lambda,\ \|b - x\| \leq \Lambda,\ a \neq b\big] \Rightarrow \Big\| \frac{a+b}{2} - x \Big\| < \Lambda,$$

holds. The L^p space $(1 < p < \infty)$ and Hilbert spaces are examples of strictly convex Banach spaces.

Theorem 2.2. (Theorem 4.1 of [11]) *Let (P, Q) be a nonempty, bounded, closed and convex pair in a strictly convex Banach space $\mathcal{X}$ such that P_0 is nonempty. Assume that $T : P \cup Q \to P \cup Q$ is a noncyclic relatively nonexpansive mapping. If T is compact, then T has a best proximity pair.*

We refer to [1, 19] for some existence results of best proximity points for various classes of cyclic mappings in the setting of reflexive Banach spaces.

2.2. **Simulation Functions.**

Definition 2.2. ([15]) Let $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. Then Ξ is called a simulation function if it satisfies the following conditions:

(KSR-1) $\Xi(0, 0) = 0$,
(KSR-2) $\Xi(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,
(KSR-3) if $\{s_j\}$ and $\{t_j\}$ are sequences in $(0, +\infty)$ such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j > 0$, then $\limsup_{j\to\infty} \Xi(t_j, s_j) < 0$.

However de-Hierro and Samet [14] modified the above defined notion slightly and enlarged the simulation functions family by replacing condition (KSR-3) with

(DS-3) if $\{s_j\}$ and $\{t_j\}$ are sequences in $(0, +\infty)$ such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j > 0$ and $t_j < s_j$ then $\limsup_{j\to\infty} \Xi(t_j, s_j) < 0$.

In a parallel development, Argoubi et al. [3] found that the condition (KSR-1) is redundant and can be deduced from (KSR-2) and (KSR-3) or (DS-3). They redefined the simulation function by removing the condition (KSR-1) as below.

Definition 2.3. ([3]) Let $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a mapping. Then Ξ is called a simulation function if it satisfies the following conditions:

(ASV-1) $\Xi(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,
(ASV-2) if $\{s_j\}$ and $\{t_j\}$ are sequences in $(0, +\infty)$ such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j > 0$ and $t_j < s_j$ then $\limsup_{j\to\infty} \Xi(t_j, s_j) < 0$.

The family of all simulation functions in the sense of Definition 2.3 will be denoted by $\mathcal{Z}_{ASV}$ in the sequel. Moreover, the family of all real functions $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ which only satisfy the condition (ASV-1) will be denoted by $\mathcal{Z}_{(ASV-1)}$.
Here, we present some examples to illustrate the simulation functions.
Example 2.1. ([3]) Suppose a function $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\Xi(q,p) = \delta p - q, \ \forall p, q \in \mathbb{R}^+,$$

where $0 \leq \delta < 1$. Then $\Xi \in \mathcal{Z}_{ASV}$.
Example 2.2. ([3]) Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping which satisfies $\limsup_{q \to r^+} \varphi(q) < 1$ for any $r \in \mathbb{R}^+$. If a function $\Xi_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\Xi_1(q,p) = \varphi(p)p - q, \ \forall p, q \in \mathbb{R}^+,$$

then $\Xi_1 \in \mathcal{Z}_{ASV}$.
Example 2.3. ([3]) If $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is an upper semi-continuous mapping satisfying $\kappa(q) < q$, and a function $\Xi_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is defined with

$$\Xi_2(q,p) = \kappa(p) - q, \ \forall p, q \in \mathbb{R}^+,$$

then $\Xi_2 \in \mathcal{Z}_{ASV}$.

For furthermore examples of simulation functions, we refer to [6].
Definition 2.4.A self-mapping T on a metric space $(\mathcal{X}, d)$ is called a Z-contraction if there exists $\Xi \in \mathcal{Z}_{ASV}$ such that

$$\Xi(d(Tx,Ty), d(x,y)) \geq 0, \ x, y \in \mathcal{X}. \tag{2.1}$$

It was announced in [15] that every Z-contraction defined on a complete metric space, admits a unique fixed point (see also [17, 18] for more information).

In [6] the authors generalized the class of Z-contractions by using the notion of MNC as follows.

Definition 2.5. Let K be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{X}$ and $T : K \to K$ be a continuous operator. We say that T is a $Z_{\mathcal{N}}$-contraction if there exists $\Xi \in \mathcal{Z}_{ASV}$ for which

$$\Xi(\mathcal{N}(T(C)), \mathcal{N}(C)) \geq 0,$$

for any nonempty subset C of K.

In this way, the extended version of Darbo's fixed point theorem was presented in [6] as follows.
Theorem 2.3. *Let K be a nonempty, bounded, closed and convex subset of a Banach space $\mathcal{X}$ and $T : K \to K$ be a continuous operator. If T is a $Z_{\mathcal{N}}$-contraction in the sense of Definition 2.5, then T has a fixed point.*

3. Existence results

Throughout this section, we assume that $\mathcal{N}$ is an MNC on a Banach space $\mathcal{X}$ and (P, Q) is a nonempty pair $\mathcal{X}$.
Also, if $T : P \cup Q \to P \cup Q$ is a cyclic (noncyclic) mapping, the set of all nonempty, bounded, closed, convex, proximinal and T-invariant pair $(\mathcal{M}_1, \mathcal{M}_2) \subseteq (P, Q)$ with

$\operatorname{dist}(\mathcal{M}_1, \mathcal{M}_2) = \operatorname{dist}(P, Q)$ will be denoted by $\mathcal{M}_T(P, Q)$. Notice that $\mathcal{M}_T(P, Q)$ maybe empty, but in particular if (P, Q) is a nonempty weakly compact and convex pair in a Banach space X and T is cyclic (noncyclic) relatively nonexpansive, then $(P_0, Q_0) \in \mathcal{M}_T(P, Q)$ (see [10] for more details).

We now introduce the first class of cyclic (noncyclic) mappings.

Definition 3.1. A mapping $T : P \cup Q \to P \cup Q$ is said to be a cyclic (noncyclic) Ξ_g-condensing operator if T is cyclic (noncyclic) and there exists $\Xi \in \mathcal{Z}_{ASV}$ such that

$$\Xi\bigg(\mathcal{N}\big(T(\mathcal{M}_1)\cup T(\mathcal{M}_2)\big)+g\big(\mathcal{N}(T(\mathcal{M}_1)\cup T(\mathcal{M}_2))\big), \mathcal{N}\big(\mathcal{M}_1\cup\mathcal{M}_2\big)+g\big(\mathcal{N}(\mathcal{M}_1\cup\mathcal{M}_2)\big)\bigg)\geq 0,$$

for any $(\mathcal{M}_1, \mathcal{M}_2) \in \mathcal{M}_T(P, Q)$ where $g : [0, \infty) \to [0, \infty)$ is a continuous function.

We are now ready to establish the first existence theorem of best proximity points.

Theorem 3.1. *Let (P, Q) be a nonempty, weakly compact and convex pair in a Banach space $\mathcal{X}$ and $T : P \cup Q \to P \cup Q$ be a cyclic relatively nonexpansive mapping which is a Ξ_g-condensing operator. Then T has a best proximity point.*

Proof. As we mentioned, $(P_0, Q_0) \in \mathcal{M}_T(P, Q) \neq \emptyset$.
Let us define a pair (G_n, H_n) as $G_n = \overline{con}(T(G_{n-1}))$ and $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$, where $G_0 = P_0$ and $H_0 = Q_0$. We claim that $G_{n+1} \subseteq H_n$ and $H_n \subseteq G_{n-1}$ for all $n \in \mathbb{N}$. We have $H_1 = \overline{con}(T(H_0)) = \overline{con}(TQ_0)) = \overline{con}(P_0) \subseteq P_0 = G_0$. Therefore, $T(H_1) \subseteq T(G_0)$. So $H_2 = \overline{con}(T(H_1)) \subseteq \overline{con}(T(G_0)) = G_1$. Continuing this pattern, we get $H_n \subseteq G_{n-1}$ by using induction. Similarly, we can see that $G_{n+1} \subseteq H_n$ for all $n \in \mathbb{N}$. Thus $G_{n+2} \subseteq H_{n+1} \subseteq G_n \subseteq H_{n-1}$ for all $n \in \mathbb{N}$. Hence, we get a decreasing sequence $\{(G_{2n}, H_{2n})\}$ of nonempty, closed and convex pairs in $P_0 \times Q_0$. Moreover, $T(H_{2n}) \subseteq T(G_{2n-1}) \subseteq \overline{con}(T(G_{2n-1})) = G_{2n}$ and $T(G_{2n}) \subseteq T(H_{2n-1}) \subseteq \overline{con}(T(H_{2n-1})) = H_{2n}$. Therefore for all $n \in \mathbb{N}$, the pair (G_{2n}, H_{2n}) is T-invariant.
Now if $(u, v) \in P_0 \times Q_0$ is a proximinal point, then

$$\operatorname{dist}(G_{2n}, H_{2n}) \leq \|T^{2n}u - T^{2n}v\| \leq \|u - v\| = \operatorname{dist}(P, Q).$$

Next, we show that the pair (G_n, H_n) is proximinal using mathematical induction. Obviously for $n = 0$, the pair (G_0, H_0) is proximinal. Suppose (G_k, H_k) is proximinal and x is an arbitrary member of $G_{k+1} = \overline{con}(T(G_k))$. Then it is represented as

$$x = \sum_{l=1}^{m} \lambda_l T(x_l)$$

with $x_l \in G_k$, $m \in \mathbb{N}$, $\lambda_l \geq 0$ and

$$\sum_{l=1}^{m} \lambda_l = 1.$$

Due to proximinality of the pair (G_k, H_k), there exists $y_l \in H_k$ for $1 \leq l \leq m$ such that $\|x_l - y_l\| = \operatorname{dist}(G_k, H_k) = \operatorname{dist}(P, Q)$. Take

$$y = \sum_{l=1}^{m} \lambda_l T(y_l).$$

Then $y \in \overline{con}(T(H_k)) = H_{k+1}$ and

$$\|x - y\| = \| \sum_{l=1}^{m} \lambda_l T(x_l) - \sum_{l=1}^{m} \lambda_l T(y_l)\| \leq \sum_{l=1}^{m} \lambda_l \|x_l - y_l\| = \operatorname{dist}(P, Q).$$

This means that the pair (G_{k+1}, H_{k+1}) is proximinal. It is worth noticing that if

$$\mathcal{N}(G_{2j} \cup H_{2j}) + g(\mathcal{N}(G_{2j} \cup H_{2j})) = 0$$

for some $j \in \mathbb{N}$, then $\mathcal{N}(G_{2j}) = 0$ and $\mathcal{N}(H_{2j}) = 0$, that is, $T : G_{2j} \cup H_{2j} \to G_{2j} \cup H_{2j}$ is a compact operator and the result can be concluded from Theorem 2.1, immediately. So we assume that $\mathcal{N}(G_{2n} \cup H_{2n}) + g(\mathcal{N}(G_{2n} \cup H_{2n})) > 0$ for all $n \in \mathbb{N}$. In view of the fact that T is a Ξ_g-condensing operator, there exists $\Xi \in \mathcal{Z}_{ASV}$ so that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\Xi\Big(\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})), \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\Big) \\
=&\Xi\Big(\mathcal{N}(\overline{con}(T(G_n)) \cup \overline{con}(T(H_n)) + g(\mathcal{N}(\overline{con}(T(G_n) \cup \overline{con}(T(H_n))),\\
&\mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\Big)\\
=&\Xi\big(\mathcal{N}(T(G_n) \cup T(H_n)) + g(\mathcal{N}(T(G_n) \cup T(H_n))), \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\big)\\
\geq& \ 0.
\end{aligned}$$

By (ASV-1), we get

$$\begin{aligned}
0 \leq&\Xi\big(\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})), \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\big)\\
<&\mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)) - \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})).
\end{aligned}$$

This implies that

$$\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})) < \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)), \quad \forall n \in \mathbb{N}.$$

Thus $\{\mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\}$ is a strictly decreasing sequence of positive real numbers and so, there exists $\gamma \geq 0$ such that

$$\lim_{n\to\infty} \big[\mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\big] = \gamma.$$

Let $\gamma > 0$ and set

$$t_n := \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))$$

and

$$s_n := \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)).$$

Then $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that $t_n, s_n \to \gamma > 0$ and $t_n < s_n$.
It now follows from (ASV-2) that

$$\begin{aligned}
\limsup_{n\to\infty} &\,\Xi\big(\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})), \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\big)\\
&= \limsup_{n\to\infty} \Xi(t_n, s_n) < 0,
\end{aligned}$$

which is a contradiction since $\Xi(t_n, s_n) \geq 0$ for all $n \in \mathbb{N}$. So $\gamma = 0$ and this turns that

$$\mathcal{N}(G_{2n} \cup H_{2n}) + g(\mathcal{N}(G_{2n} \cup H_{2n})) \to 0.$$

Since $g \geq 0$ is continuous, we have

$$\lim_{n\to\infty} \mathcal{N}(G_{2n} \cup H_{2n}) = \max\left\{\lim_{n\to\infty} \mathcal{N}(G_{2n}), \lim_{n\to\infty} \mathcal{N}(H_{2n})\right\} = 0$$

and

$$\lim_{n\to\infty} g(\mathcal{N}(G_{2n} \cup H_{2n})) = 0.$$

Now let

$$G_\infty = \bigcap_{n=0}^{\infty} G_{2n} \text{ and } H_\infty = \bigcap_{n=0}^{\infty} H_{2n}.$$

By property (d) of MNC, the pair (G_∞, H_∞) is nonempty, convex, compact and T-invariant with $\operatorname{dist}(G_\infty, H_\infty) = \operatorname{dist}(P, Q)$ and this ensures that T admits a best proximity point. □

The next theorem is a noncyclic version of Theorem 3.1 in the setting of strictly convex Banach spaces.

Theorem 3.2. *Let (P, Q) be a nonempty, weakly compact and convex pair in a strictly convex Banach space $\mathcal{X}$ and $T : P \cup Q \to P \cup Q$ be a noncyclic relatively nonexpansive mapping which is a Ξ_g-condensing operator. Then T has a best proximity pair.*

Proof. By a similar argument of Theorem 3.1, let us define the pair (G_n, H_n) as $G_n = \overline{con}(T(G_{n-1}))$ and $H_n = \overline{con}(T(H_{n-1}))$, $n \geq 1$, where $G_0 = P_0$ and $H_0 = Q_0$. Since T is noncyclic, $H_1 = \overline{con}(T(H_0)) = \overline{con}(T(Q_0)) \subseteq Q_0 = H_0$. Therefore, $T(H_1) \subseteq T(H_0)$. Thus $H_2 = \overline{con}(T(H_1)) \subseteq \overline{con}(T(H_0)) = H_1$. Continuing this pattern, we get $H_n \subseteq H_{n-1}$ by using induction. Equivalently, $G_{n+1} \subseteq G_n$ for all $n \in \mathbb{N}$. Hence we get a decreasing sequence $\{(G_n, H_n)\}$ consist of nonempty, closed and convex pairs in (P_0, Q_0) such that $T(H_n) \subseteq T(H_{n-1}) \subseteq \overline{con}(T(H_{n-1})) = H_n$ and $T(G_n) \subseteq T(G_{n-1}) \subseteq \overline{con}(T(G_{n-1})) = G_n$ which implies that (G_n, H_n) is T-invariant for all $n \in \mathbb{N}$. From the proof of Theorem 3.1, we can see that (G_n, H_n) is a proximinal pair with $\operatorname{dist}(G_n, H_n) = \operatorname{dist}(P, Q)$ for all $n \in \mathbb{N} \cup \{0\}$.
Notice that if $\mathcal{N}(G_j \cup H_j) + g(\mathcal{N}(G_j \cup H_j)) = 0$ for some $j \in \mathbb{N}$, then $\mathcal{N}(G_j \cup H_j) = 0$ as $g \geq 0$, which ensures that $\max\{\mathcal{N}(G_j), \mathcal{N}(H_j)\} = 0$. Thus $T : G_j \cup H_j \to G_j \cup H_j$ is a compact operator and the result follows from Theorem 2.2.
Suppose $\mathcal{N}(G_n \cup H_n) + G(\mathcal{N}(G_n \cup H_n)) > 0$ for all $n \in \mathbb{N}$. By a similar argument of Theorem 3.1 $\{\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))\}$ is a strictly decreasing sequence of positive real numbers, so there exists $\gamma \geq 0$ for which

$$\lim_{n\to\infty} \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)) = \gamma.$$

Suppose $\gamma > 0$. If we set

$$t_n := \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))$$

and

$$s_n := \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)),$$

we have $t_n, s_n \to \gamma$ and $t_n < s_n$, for all $n \in \mathbb{N}$ and so by (ASV-2),

$$\limsup_{n\to\infty} \Xi\big(\mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})), \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))\big)$$

$$= \limsup_{n\to\infty} \varXi(t_n, s_n) < 0. \tag{3.1}$$

On the other hand by an equivalent discussion of Theorem 3.1, since T is a $\varXi_g$-condensing operator,

$$\varXi(t_n, s_n) = \varXi\Big(\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1})), \mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\Big)$$

$$\geq 0, \quad \forall n \in \mathbb{N},$$

which is a contradiction by (3.1). Thus $\gamma = 0$ and $\mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)) \to 0$ as $n \to \infty$. This gives, $\lim_{n\to\infty} \mathcal{N}(G_n \cup H_n) = 0$ and $\lim_{n\to\infty} g(\mathcal{N}(G_n \cup H_n)) = 0$ as $g \geq 0$. Thereby,

$$\max\Big\{\lim_{n\to\infty} \mathcal{N}(G_n), \lim_{n\to\infty} \mathcal{N}(H_n)\Big\} = 0.$$

Now if

$$G_\infty := \bigcap_{n=0}^{\infty} G_n \text{ and } H_\infty := \bigcap_{n=0}^{\infty} H_n,$$

then by property (d) of MNC, (G_∞, H_∞) is a nonempty, convex, compact and T-invariant pair with $\operatorname{dist}(G_\infty, H_\infty) = \operatorname{dist}(P, Q)$. It now follows from Theorem 2.2 that T admits a best proximity pair. □

In what follows we need the following classes of functions which will be used in the sequel.

Definition 3.2. ([2]) Let $F([0,\infty))$ be the class of all functions $f : [0,\infty) \to [0,\infty)$. Then by Θ we denote the class of all operators

$$O(\bullet;\cdot) : F([0,\infty)) \to F([0,\infty)), \quad \text{by} \quad f \mapsto O(f;\cdot)$$

satisfying the following conditions:

(i) $O(f;t) > 0$ for $t > 0$ and $O(f;0) = 0$;
(ii) $\lim_{n\to\infty} O(f;t_n) = O(f; \lim_{n\to\infty} t_n)$;
(iii) $O(f;t) \leq O(f,s)$ for $t \leq s$.

We mention that in the original definition of the class of Θ which was appeared in [2], another additional assumption must satisfy, which is as below:

(iv) $O(f; \max\{t,s\}) = \max\{O(f;t), O(f;s)\}$ for some $f \in F([0,\infty))$.

It is worth noticing that we do not need the assumption (iv) in our main results.

Definition 3.3. Let Ψ denote the class of all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(i) ψ is continuous;
(ii) $\psi(t) < t$ for $t > 0$.

By Ψ' we mean the subclass of Ψ consists of $\psi : [0,\infty) \to [0,\infty)$ which satisfies the following additional condition

(iii) ψ is non-decreasing.

Here, we present the second class of cyclic (noncyclic) mappings as below.

Definition 3.4. A mapping $T : P \cup Q \to P \cup Q$ is said to be a cyclic (noncyclic) $O{-}\psi{-}\Xi_g$-condensing operator if T is cyclic (noncyclic) and there exists $\Xi \in \mathcal{Z}_{(ASV-1)}$ such that for any $(\mathcal{M}_1, \mathcal{M}_2) \in \mathcal{M}_T(P, Q)$,

$$\Xi\Big(O\big(f; \mathcal{N}(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) + g(\mathcal{N}(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)))\big),$$

$$\psi\big(O(f; \mathcal{N}(\mathcal{M}_1 \cup \mathcal{M}_2) + g(\mathcal{N}(\mathcal{M}_1 \cup \mathcal{M}_2)))\big)\Big) \geq 0,$$

where $f \in F([0, \infty)), O(\bullet; \cdot) \in \Theta, \psi \in \Psi$ and $g : [0, \infty) \to [0, \infty)$ is a continuous function.

We are now in a position to prove the other existence theorems for best proximity points (pairs).

Theorem 3.3. *Let (P, Q) be a nonempty, weakly compact and convex pair in a Banach space $\mathcal{X}$ and $T : P \cup Q \to P \cup Q$ be a cyclic relatively nonexpansive mapping which is an $O - \psi - \Xi_g$-condensing operator. Then T has a best proximity point.*

Proof. Consider the sequence $\{(G_n, H_n)\}$ as in the proof of Theorem 3.1 such that

$$\mathcal{N}(G_{2n} \cup H_{2n}) + g(\mathcal{N}(G_{2n} \cup H_{2n})) > 0, \quad \forall n \in \mathbb{N}.$$

Since T is an $O - \psi - \Xi_g$-condensing operator for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\Xi\Big(O\big(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))\big),\\
&\quad \psi\big(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)))\big)\Big)\\
=&\Xi\Big(O\big(f; \mathcal{N}(\overline{con}(T(G_n)) \cup \overline{con}(T(H_n))) + g(\overline{con}(T(G_n)) \cup \overline{con}(T(H_n)))\big)\big),\\
&\quad \psi\big(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)))\big)\Big)\\
=&\Xi\Big(O\big(f; \mathcal{N}(T(G_n) \cup T(H_n)) + g(T(G_n) \cup T(H_n)))\big),\\
&\quad \psi\big(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)))\big)\Big)\\
\geq&0.
\end{aligned}$$

By (ASV-1), we get

$$\begin{aligned}
0 \leq&\Xi\Big(O\big(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))\big),\\
&\psi\big(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)))\big)\Big)\\
<& \ \psi\big(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n)))\big)\\
&- O\big(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1}))\big).
\end{aligned}$$

This implies

$$\begin{aligned}
&O\big(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))\big)\\
&<\psi\big(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)))\big)\\
&<O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)), \quad \forall n\in\mathbb{N}. \qquad (3.2)
\end{aligned}$$

It follows from the condition (iii) of $O(\bullet;\cdot)$ that

$$\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))<\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))$$

for any $n\in\mathbb{N}$. Therefore, $\{\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\}$ is a decreasing sequence of positive real numbers. Thus there exists $\gamma\geq 0$ such that

$$\lim_{n\to\infty}\big[\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\big]=\gamma.$$

Suppose $\gamma>0$. By using the properties (i) of ψ and the property (ii) of $O(\bullet;\cdot)$, we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\psi(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\\
&\quad=\psi(O(f;\lim_{n\to\infty}\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\\
&\quad=\psi(O(f;\gamma)).
\end{aligned}$$

By (3.2), we get

$$\begin{aligned}
O(f;\gamma)&=\lim_{n\to\infty}O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\\
&=\lim_{n\to\infty}\psi(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)))\\
&=\psi(O(f;\gamma)).
\end{aligned}$$

From the condition (ii) of ψ, we must have $O(f;\gamma)=0$, but since $\gamma>0$, from the condition (i) of $O(\bullet;\cdot)$ we have $O(f;\gamma)>0$ which is impossible. Therefore, $\gamma=0$ and so $\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n}))\to 0$ as $n\to\infty$. Now the result follows by a similar discussion of Theorem 3.1. □

By using a same method of the proof of Theorem 3.3, we obtain the next best proximity pair result. We omit the proof since it follows similar patterns to those given for the proofs of Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *Let (P,Q) be a nonempty, weakly compact and convex pair in a strictly convex Banach space $\mathcal{X}$ and $T:P\cup Q\to P\cup Q$ be a noncyclic relatively nonexpansive mapping which is an $O-\psi-\Xi_g$-condensing operator. Then T admits a best proximity pair.*

Let $\triangle$ denote the class of all functions $\beta:[0,\infty)\to[0,1)$ which satisfy the condition

$$t_n\to 0 \quad \text{whenever} \quad \beta(t_n)\to 1.$$

This family of functions was introduced by Geraghty in [12].

The third family of cyclic (noncyclic) mappings will be presented by using the class of Geraghty's functions as follows.

Definition 3.5. A mapping $T : P \cup Q \to P \cup Q$ is said to be a cyclic (noncyclic) $O - \psi - \beta - \Xi_g$-condensing operator if T is cyclic (noncyclic) and there exists $\Xi \in \mathcal{Z}_{(ASV-1)}$ such that for any $(\mathcal{M}_1, \mathcal{M}_2) \in \mathcal{M}_T(P, Q)$,

$$\begin{aligned}\Xi\Big(&\psi\big(O(f; \mathcal{N}(T(\mathcal{M}_1) \cup T(\mathcal{M}_2)) + g(\mathcal{N}(T(\mathcal{M}_1) \cup T(\mathcal{M}_2))))\big),\\ &\beta(O(f; \psi(\mathcal{N}(\mathcal{M}_1 \cup \mathcal{M}_2))))\psi(O(f; \mathcal{N}(\mathcal{M}_1 \cup \mathcal{M}_2) + g(\mathcal{N}(\mathcal{M}_1 \cup \mathcal{M}_2))))\Big) \geq 0,\end{aligned} \tag{3.3}$$

where $f \in F([0, \infty)), O(\bullet; \cdot) \in \Theta, \psi \in \Psi', \beta \in \triangle$ and $g : [0, \infty) \to [0, \infty)$ is a continuous function.

We now state the following existence theorem.

Theorem 3.5. *Let (P, Q) be a nonempty, weakly compact and convex pair in a Banach space $\mathcal{X}$ and $T : P \cup Q \to P \cup Q$ be a cyclic relatively nonexpansive mapping which is an $O - \psi - \beta - \Xi_g$-condensing operator. Then T has a best proximity point.*

Proof. Consider the sequence $\{(G_n, H_n)\}$ as in the proof of Theorem 3.1 such that

$$\mathcal{N}(G_{2n} \cup H_{2n}) + g(\mathcal{N}(G_{2n} \cup H_{2n})) > 0, \quad \forall n \in \mathbb{N}.$$

Since T is an $O - \psi - \beta - \Xi_g$-condensing operator, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}&\Xi\Big(\psi\big(O(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})))\big),\\ &\quad\beta(O(f; \psi(\mathcal{N}(G_n \cup H_n))))\psi(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))))\Big)\\ =&\Xi\Big(\psi(O(f; \mathcal{N}(\overline{con}(T(G_n)) \cup \overline{con}(T(H_n))) + g(\mathcal{N}(\overline{con}(T(G_n)) \cup \overline{con}(T(H_n)))))),\\ &\quad\beta(O(f; \psi(\mathcal{N}(G_n \cup H_n))))\psi(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))))\Big)\\ =&\Xi\Big(\psi(O(f; \mathcal{N}(T(G_n) \cup T(H_n) + g(\mathcal{N}(T(G_n) \cup T(H_n))))),\\ &\quad\beta(O(f; \psi(\mathcal{N}(G_n \cup H_n))))\psi(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))))\Big)\\ \geq&0.\end{aligned}$$

By (ASV-1), we get

$$\begin{aligned}0 \leq&\Xi\Big(\psi\big(O(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})))\big),\\ &\quad\beta(O(f; \psi(\mathcal{N}(G_n \cup H_n))))\psi(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))))\Big)\\ <&\beta(O(f; \psi(\mathcal{N}(G_n \cup H_n))))\psi(O(f; \mathcal{N}(G_n \cup H_n) + g(\mathcal{N}(G_n \cup H_n))))\\ &\quad - \psi\big(O(f; \mathcal{N}(G_{n+1} \cup H_{n+1}) + g(\mathcal{N}(G_{n+1} \cup H_{n+1})))\big).\end{aligned}$$

This implies

$$
\begin{aligned}
0 \leq& \psi\big(O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1})))\big)\\
<&\beta(O(f;\psi(\mathcal{N}(G_n\cup H_n))))\psi(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\\
<&\psi(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)))).
\end{aligned} \tag{3.4}
$$

That means

$$
\begin{aligned}
&O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1})))\\
&\quad< O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))).
\end{aligned}
$$

Therefore,

$$
\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))<\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)).
$$

Let

$$
\lim_{n\to\infty}\big[\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\big]=\gamma
$$

for some $\gamma\geq 0$.
Notice that by (3.4)

$$
\begin{aligned}
0<&\frac{\psi\big(O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1})))\big)}{\psi(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))}\\
\leq&\ \beta(O(f;\psi(\mathcal{N}(G_n\cup H_n))))<1.
\end{aligned}
$$

This gives us

$$
\lim_{n\to\infty}\beta(O(f;\psi(\mathcal{N}(G_n\cup H_n))))=1.
$$

Then by the property of $\beta\in\triangle$, we get

$$
\lim_{n\to\infty}O(f;\psi(\mathcal{N}(G_n\cup H_n)))=0.
$$

Thus we have

$$
\lim_{n\to\infty}\mathcal{N}(G_n\cup H_n)=0
$$

and so, $\gamma=0$. Therefore,

$$
\lim_{n\to\infty}[\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n}))]=0.
$$

Now if we define

$$
G_\infty:=\bigcap_{n=0}^{\infty}G_{2n}\text{ and }H_\infty:=\bigcap_{n=0}^{\infty}H_{2n},
$$

then (G_∞,H_∞) is a nonempty, compact and convex pair in a Banach space $\mathcal{X}$ which is T-invariant with

$$
\operatorname{dist}(G_\infty,H_\infty)=\operatorname{dist}(P,Q).
$$

Hence by Theorem 2.1 T has a best proximity point. □

Next we present the noncyclic version of Theorem 3.5.

Theorem 3.6. *Let (P,Q) be a nonempty, weakly compact and convex pair in a strictly convex Banach space $\mathcal{X}$ and $T : P \cup Q \to P \cup Q$ be a noncyclic relatively nonexpansive mapping which is an $O-\psi-\beta-\Xi_g$-condensing operator. Then T has a best proximity pair.*

Proof. The proof follows from the proof of Theorems 3.3 and 3.5. □

We finish this section by introducing the fourth family of cyclic (noncyclic) mappings and proving the existence theorems for them. To this end, we recall the following classes of functions.

Definition 3.6. ([16]) Let Υ denote the class of all MT-functions $\chi : [0,\infty) \to [0,1)$ satisfying the condition

$$\limsup_{s\to t^+} \chi(s) < 1 \quad \text{for all } t \in [0,\infty).$$

We note that if $\chi : [0,1) \to [0,1)$ is a non-decreasing function or a non-increasing function, then χ is an MT-function.

Definition 3.7. Let Ω denote the set of all functions $\omega : [0,\infty) \to [0,\infty)$ satisfying:

(i) ω is non-decreasing;

(ii) $\omega(t) = 0$ if and only if $t = 0$.

Definition 3.8. A mapping $T : P \cup Q \to P \cup Q$ is said to be a cyclic (noncyclic) $O-\omega-\chi-\Xi_g$-condensing operator if T is cyclic (noncyclic) and there exists $\Xi \in \mathcal{Z}_{(ASV-1)}$ such that for any $(\mathcal{M}_1,\mathcal{M}_2) \in \mathcal{M}_T(P,Q)$,

$$\begin{aligned}\Xi\Big(&\omega\big(O(f;\mathcal{N}(T(\mathcal{M}_1)\cup T(\mathcal{M}_2)) + g(\mathcal{N}(T(\mathcal{M}_1)\cup T(\mathcal{M}_2))))\big),\\ &\chi(O(f;\omega(\mathcal{N}(\mathcal{M}_1\cup\mathcal{M}_2))))\omega\big(O(f;\mathcal{N}(\mathcal{M}_1\cup\mathcal{M}_2) + g(\mathcal{N}(\mathcal{M}_1\cup\mathcal{M}_2)))\big)\Big) \geq 0,\end{aligned} \tag{3.5}$$

where $f \in F([0,\infty))$, $O(\bullet;\cdot) \in \Theta$, $g : [0,\infty) \to [0,\infty)$ is a continuous function, $\chi \in \Upsilon$ and $\omega \in \Omega$.

We now state the following best proximity point theorem for cyclic $O-\omega-\chi-\Xi_g$-condensing operators.

Theorem 3.7. *Let (P,Q) be a nonempty, weakly compact and convex pair in a Banach space $\mathcal{X}$ and $T : P\cup Q \to P\cup Q$ be a cyclic relatively nonexpansive mapping which is an $O-\omega-\chi-\Xi_g$-condensing operator. Then T has a best proximity point.*

Proof. Again consider the sequence $\{(G_n,H_n)\}$ as in the proof of Theorem 3.1 such that

$$\mathcal{N}(G_{2n}\cup H_{2n}) + g(\mathcal{N}(G_{2n}\cup H_{2n})) > 0, \quad \forall n \in \mathbb{N}.$$

Since T is an $O-\omega-\chi-\Xi_g$-condensing operator, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\Xi\bigg(\omega\big(O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))\big),\\
&\quad \chi(O(f;\omega(\mathcal{N}(G_n)\cup H_n))))\omega\big(O(f;\mathcal{N}(\mathcal{N}(G_n)\cup H_n)+g(\mathcal{N}(\mathcal{N}(G_n\cup H_n)))))\big)\bigg)\\
&\quad = \Xi\bigg(\omega\big(O(f;\mathcal{N}(\overline{con}(T(G_n))\cup\overline{con}(T(H_n)))+g(\mathcal{N}(\overline{con}(T(G_n))\cup\overline{con}(T(H_n)))))\big),\\
&\quad \chi(O(f;\omega(\mathcal{N}(G_n)\cup H_n))))\omega\big(O(f;\mathcal{N}(\mathcal{N}(G_n)\cup H_n)+g(\mathcal{N}(\mathcal{N}(G_n\cup H_n)))))\big)\bigg)\\
&\quad = \Xi\bigg(\omega\big(O(f;\mathcal{N}(T(G_n)\cup T(H_n))+g(\mathcal{N}(T(G_n)\cup T(H_n))))\big),\\
&\quad \chi(O(f;\omega(\mathcal{N}(G_n)\cup H_n)))\omega\big(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)))))\big)\bigg)\\
&\quad \geq 0.
\end{aligned}$$

By (ASV-1) we obtain

$$\begin{aligned}
&\omega\big(O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))\big)\\
&\quad \leq \chi(O(f;\omega(\mathcal{N}(G_n\cup H_n))))\omega\big(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\\
&\quad < \omega\big(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\big). \qquad (3.6)
\end{aligned}$$

Assume that

$$\lim_{n\to\infty}\omega\big(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\big)=\eta, \qquad (3.7)$$

for some $\eta \geq 0$. If $\eta > 0$, since $\chi \in \Upsilon$, we have

$$\limsup_{t\to\eta^+}\chi(O(f;t))<1 \text{ and } \chi(O(f;v))<1,$$

then there exists $\delta \in [0,1)$, $\varepsilon > 0$ such that $\chi(O(f;t)) < \delta$ for all $t \in [\eta, \eta+\varepsilon)$. By (3.7), let $N \in \mathbb{N}$ be such that

$$\eta \leq \omega(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))<\eta+\varepsilon, \quad \forall n \geq N.$$

It now follows from the relation (3.6) that

$$\begin{aligned}
&\omega(O(f;\mathcal{N}(G_{n+1}\cup H_{n+1})+g(\mathcal{N}(G_{n+1}\cup H_{n+1}))))\\
&\leq \chi(O(f;\omega(\mathcal{N}(G_n\cup H_n)))\omega(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))\\
&\leq \delta\omega(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n)))), \quad \forall n \geq N. \qquad (3.8)
\end{aligned}$$

By taking the limit of the relation (3.8), we obtain $\eta \leq \delta\eta$, which means taht $\eta = 0$ and so

$$\lim_{n\to\infty}\omega(O(f;\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))))=0.$$

In view of the fact that ω is non-decreasing function and by the property (ii) of $O(\bullet;\cdot)$, we conclude that the sequence $\{\mathcal{N}(G_n\cup H_n)+g(\mathcal{N}(G_n\cup H_n))\}$ and so it's even subsequence, that is, $\{\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n}))\}$ is a non-increasing sequence of positive numbers. Now if $\lim_{n\to\infty}\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n}))=\gamma$ for some $\gamma \geq 0$, then by the fact that ω is non-decreasing and by considering the property (iii) of $O(\bullet;\cdot)$, we deduce that

$$\omega(O(f;\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n})))) \geq \omega(O(f;\gamma)),$$

and hence

$$0=\lim_{n\to\infty}\omega(O(f;\mathcal{N}(G_{2n}\cup H_{2n})+g(\mathcal{N}(G_{2n}\cup H_{2n})))) \geq \omega(O(f;\gamma)), \quad \forall n\in\mathbb{N},$$

which ensures that $\gamma = 0$. By a similar argument of the proof of Theorem 3.1 the result follows. □

Here is the noncyclic version of Theorem 3.7.

Theorem 3.8. *Let (P,Q) be a nonempty, weakly compact and convex pair in a strictly convex Banach space $\mathcal{X}$ and $T: P\cup Q \to P\cup Q$ be a noncyclic relatively nonexpansive mapping which is an $O-\omega-\chi-\Xi_g$-condensing operator. Then T has a best proximity pair.*

4. Application to system of integro-differential equations

This section is dedicated to prove a result which shows the existence of optimum solutions for a system of functional integro-differential equations.

Let $\alpha,\beta,\rho\in\mathbb{R}^+$ with $\rho<\alpha$. Let $t_0\in\mathbb{R}$ and $\mathcal{X}$ be Banach space. We denote by $C(I,\mathcal{X})$, the Banach space of all continuous mappings from $I=[t_0-\alpha,t_0+\alpha]$ into $\mathcal{X}$, endowed with the supremum norm. Also, let $B_1=B(u_1;\beta)$ and $B_2=B(u_2;\beta)$ be closed balls in $\mathcal{X}$, where $u_1,u_2\in\mathcal{X}$. Assume that $k_i: I\times I\times B_i\to\mathcal{X}$ and $F_i: I\times B_i\times B_i\to\mathcal{X}$, with $i=1,2$, are continuous mappings, and k_i is k_i-invariant. Let us consider the following system:

$$\begin{cases} u'(t)=F_1(t,u(t),\displaystyle\int_{t_0}^{t}k_1(t,s,u(s))ds), & u(t_0)=u_1,\\ v'(t)=F_2(t,v(t),\displaystyle\int_{t_0}^{t}k_2(t,s,v(s))ds), & v(t_0)=u_2,\end{cases} \tag{4.1}$$

where the integral is the Bochner integral. Let $J=[t_0-\rho,t_0+\rho]$ and define

$$\mathcal{M}_1=\{x: J\to B_1 : x\in C(J,\mathcal{X}), x(t_0)=u_1\}$$

and

$$\mathcal{M}_2=\{y: J\to B_2 : y\in C(J,\mathcal{X}), y(t_0)=u_2\}.$$

Clearly, $(\mathcal{M}_1,\mathcal{M}_2)$ is a bounded, closed and convex pair in $C(J,\mathcal{X})$. Also, for any $(u,v)\in\mathcal{M}_1\times\mathcal{M}_2$, we have

$$\|u_1-u_2\|\leq\sup_{t\in J}\|u(t)-v(t)\|=\|u-v\|,$$

and so, $\operatorname{dist}(\mathcal{M}_1,\mathcal{M}_2)=\|u_1-u_2\|$.

Now, let $T : \mathcal{M}_1 \cup \mathcal{M}_2 \to C(J, \mathcal{X})$ be the operator defined as

$$Tu(t) = \begin{cases} u_2 + \displaystyle\int_{t_0}^{t} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau, & u \in \mathcal{M}_1, \\ u_1 + \displaystyle\int_{t_0}^{t} F_2(\tau, u(\tau), \int_{t_0}^{\tau} k_2(\tau, s, u(s))ds)d\tau, & u \in \mathcal{M}_2. \end{cases} \tag{4.2}$$

We show that T is a cyclic operator. Indeed, for $u \in \mathcal{M}_1$ we have

$$\begin{aligned} \|Tu(t) - u_2\| &= \| \int_{t_0}^{t} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau \| \\ &\leq \left| \int_{t_0}^{t} \|F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)\| d\tau \right| \\ &\leq K_1 \rho, \end{aligned}$$

where

$$K_i = \sup\{\|F_i(t, u(t), \int_{t_0}^{t} k_i(t, s, u(s))ds)\| : (t, u) \in I \times B_i\}, \quad i = 1, 2.$$

Now, if we assume

$$\rho < \frac{\beta}{\max_{i \in \{1,2\}} K_i},$$

we get $\|Tu(t) - u_2\| \leq \beta$ for all $t \in J$ and $Tu \in \mathcal{M}_2$. The same argument shows that $u \in \mathcal{M}_2$ implies $Tu \in \mathcal{M}_1$.

It should be clear that $w \in \mathcal{M}_1 \cup \mathcal{M}_2$ is an optimum solution of the system (4.1) if $\|w - Tw\| = \operatorname{dist}(\mathcal{M}_1 \cup \mathcal{M}_2)$ is satisfied. Equivalently, w is the best proximity point of the operator T. Before proving the actuality of optimum solution of system (4.1), we recall an extension of the *Mean-Value Theorem*.

Theorem 4.1. ([11]) *Let I, J, B_i, F_i and k_i with $i \in \{1, 2\}$ be given as above discussion, where $t_0, t \in J$ such that $t_0 < t$. Then*

$$\begin{aligned} u_j + \int_{t_0}^{t} F_i(\tau, u(\tau), \int_{t_0}^{\tau} k_i(\tau, s, u(s))ds)d\tau &\in u_j \\ &+ (t - t_0)\overline{con}(\{F_i(\tau, u(\tau), \int_{t_0}^{\tau} k_i(\tau, s, u(s))ds) : \tau \in [t_0, t]\}), \end{aligned}$$

for $(i, j) \in \{(1, 2), (2, 1)\}$.

We give the following result which guarantees the existence of an optimal solution for the system of equations (4.1).

Theorem 4.2. *Under the aforesaid assumptions let*

$$\rho < \frac{\beta}{\max\{K_1, K_2\}}$$

and $\mathcal{N}$ be an MNC on $C(J, \mathcal{X})$, for which the following conditions hold:

(1) *For any bounded pair* $(N_1, N_2) \subseteq (B_1, B_2)$, *there is a mapping* $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ *which is upper semi-continuous and satisfies* $\kappa(t) < t$ *such that*

$$\mathcal{N}\big(F_1(\mathcal{J} \times N_1 \times N_1) \cup F_2(\mathcal{J} \times N_2 \times N_2)\big) < \frac{\kappa\big(\mathcal{N}(N_1 \cup N_2)\big)}{\rho},$$

(2) $\|F_1(t, u(t), \int_{t_0}^t k_1(t, s, u(s))ds) - F_2(t, v(t), \int_{t_0}^t k_2(t, s, v(s))ds)\|$
$\leq \frac{1}{\rho}(\|u(t) - v(t)\| - \|u_2 - u_1\|), \quad \forall (u, v) \in \mathcal{M}_1 \times \mathcal{M}_2.$

Then the problem (4.1) *has an optimal solution.*

Proof. Since T is a cyclic operator, it follows trivially that $T(\mathcal{M}_1)$ is a bounded subset of $\mathcal{M}_2$. We show that $T(\mathcal{M}_1)$ is also an equicontinuous subset of $\mathcal{M}_2$. Suppose $t, t' \in J$ and $u \in \mathcal{M}_1$. We observe that

$$\begin{aligned}
&\|Tu(t) - Tu(t')\| \\
&= \| \int_{t_0}^{t} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau - \int_{t_0}^{t'} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, x(s))ds)d\tau \| \\
&\leq \left| \int_{t}^{t'} \|F_1(\tau, x(\tau), \int_{t_0}^{\tau} k_1(\tau, s, x(s))ds)\| d\tau \right| \\
&\leq K_1|t - t'|,
\end{aligned}$$

that is, $T(\mathcal{M}_1)$ is equicontinuous too. With the same argument, one can show that $T(\mathcal{M}_2)$ is equicontinuous. Now, by applying Arzela-Ascoli theorem, it follows that the pair $(\mathcal{M}_1, \mathcal{M}_2)$ is relatively compact. Our aim is to prove that T is a relatively nonexpansive and Ξ_g-condensing operator. For each $(u, v) \in \mathcal{M}_1 \times \mathcal{M}_2$ with the help of assumption (2), we have

$$\begin{aligned}
\|Tu(t) - Tv(t)\| =& \Big\| u_1 + \int_{t_0}^{t} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau \\
& - \big(u_2 + \int_{t_0}^{t} F_2(\tau, v(\tau), \int_{t_0}^{\tau} k_2(\tau, s, v(s))ds)d\tau\big) \Big\| \\
\leq& \|u_2 - u_1\| + \Big| \int_{t_0}^{t} \|F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau \\
& - F_2(\tau, v(\tau), \int_{t_0}^{\tau} k_2(\tau, s, v(s))ds)\| d\tau \Big| \\
\leq& \|u_2 - u_1\| + \frac{1}{\rho}\Big| \int_{t_0}^{t} (\|u(\tau) - v(\tau)\| - \|u_2 - u_1\|)d\tau \Big| \\
\leq& \|u_2 - u_1\| + (\|u - v\| - \|u_2 - u_1\|) = \|u - v\|.
\end{aligned}$$

Thereby,

$$\|Tu - Tv\| = \sup_{t \in J} \|Tu(t) - Tv(t)\| \leq \|u - v\|,$$

that is, T is relatively nonexpansive.
Now suppose that the pair $(N_1, N_2) \subseteq \big(\mathcal{M}_1, \mathcal{M}_2\big)$ is a nonempty, bounded, closed,

convex and proximinal pair which is T-invariant and

$$\mathrm{dist}(N_1, N_2) = \mathrm{dist}(\mathcal{M}_1, \mathcal{M}_2)\ (= \|u_2 - u_1\|).$$

It now follows from Theorem 4.1 and assumption (1) that

$$\begin{aligned}
&\mathcal{N}(T(N_1) \cup T(N_2))\\
&= \max\{\mathcal{N}(T(N_1)), \mathcal{N}(T(N_2))\}\\
&= \max\left\{\sup_{t\in J}\{\mathcal{N}(\{Tu(t) : u \in N_1\})\}, \sup_{t\in J}\{\mathcal{N}(\{Tv(t) : v \in N_2\})\}\right\}\\
&= \max\left\{\sup_{t\in J}\{\mathcal{N}(\{u_2 + \int_{t_0}^{t} F_1(\tau, u(\tau), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds)d\tau : u \in N_1\})\},\right.\\
&\left.\sup_{t\in J}\{\mathcal{N}(\{u_1 + \int_{t_0}^{t} F_2(\tau, v(\tau), \int_{t_0}^{\tau} k_2(\tau, s, v(s))ds)d\tau : v \in N_2\})\}\right\},
\end{aligned}$$

and that

$$\begin{aligned}
&\mathcal{N}(T(N_1) \cup T(N_2))\\
&\leq \max\left\{\sup_{t\in J}\{\mathcal{N}(\{u_2 + (t - t_0)\overline{con}(\{F_1(\tau, u(t), \int_{t_0}^{\tau} k_1(\tau, s, u(s))ds) : \tau \in [t_0, t]\})\})\},\right.\\
&\left.\sup_{t\in J}\{\mathcal{N}(\{u_1 + (t - t_0)\overline{con}(\{F_2(t, v(t), \int_{t_0}^{\tau} k_2(\tau, s, v(s))ds) : \tau \in [t_0, t]\})\})\}\right\}\\
&\leq \max\left\{\sup_{0\leq\lambda\leq\rho}\{\mu(\{u_2 + \lambda\overline{con}(\{F_1(J \times N_1 \times N_1)\})\})\},\right.\\
&\left.\sup_{0\leq\lambda\leq\rho}\{\mathcal{N}(\{u_1 + \lambda\overline{con}(\{F_2(J \times N_2 \times N_2)\})\})\}\right\}\\
&= \max\left\{\rho\mathcal{N}(F_1(J \times N_1 \times N_1)), \rho\mathcal{N}(F_2(J \times N_2 \times N_2))\right\}\\
&= \rho\mathcal{N}(\{F_1(J \times N_1 \times N_1) \cup F_2(J \times N_2 \times N_2)\})\\
&< \rho\frac{\kappa(\mathcal{N}(N_1 \cup N_2))}{\rho} = \kappa(\mathcal{N}(N_1 \cup N_2)).
\end{aligned}$$

Thus we get

$$\kappa(\mathcal{N}(N_1 \cup N_2)) - \mathcal{N}(T(N_1 \cup T(N_2))) \geq 0.$$

If we take $\Xi(t, s) = \kappa(s) - t$ and $g : [0, \infty) \to [0, \infty)$ given as $g(s) = 0$ for all $s \in [0, \infty)$, the necessary requirements of Theorem 3.1 are satisfied. So the operator T has a best proximity point which is an optimal solution of the system of (4.1). □

Remark 4.1. It is worth noticing that in Theorem 3.1 the considered pair (P, Q) is weakly compact, whereas the pair (B_1, B_2) and so, $(\mathcal{M}_1, \mathcal{M}_2)$ in Theorem 4.2 is not weakly compact. We mention that the weakly compactness condition of (P, Q) in Theorem 3.1 is used to establish nonemptiness of the proximal pair (P_0, Q_0). However, in Theorem 4.2, $\big((\mathcal{M}_1)_0, (\mathcal{M}_2)_0\big)$ is nonempty, because of the fact $(u_1, u_2) \in (\mathcal{M}_1, \mathcal{M}_2)$.

As an application of Theorem 4.2, we obtain the next existence result of a solution for a system of integro-differential equations.

Corollary 4.1. *Under the above notations and the assumptions of Theorem 4.2, if $u_1 = u_2$, then the system*

$$\begin{cases} u'(t) = F_1(t, u(t), \int_{t_0}^{t} k_1(t, s, u(s))ds), & u(t_0) = u_1, \\ v'(t) = F_2(t, v(t), \int_{t_0}^{t} k_2(t, s, v(s))ds), & v(t_0) = u_1, \end{cases} \tag{4.3}$$

has a solution.

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