Fixed Point Theory, 23(2022), No. 1, 219-238
DOI: 10.24193/fpt-ro.2022.1.14
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SPLIT EQUALITY FIXED POINT PROBLEMS AND COMMON NULL POINT PROBLEMS IN HILBERT SPACES

MOHAMMAD ESLAMIAN*, S. AZARMI** AND G. ZAMANI ESKANDANI**

*Department of Mathematics,
University of Science and Technology of Mazandaran,
P.O.Box: 48518-78413, Behshahr, Iran
E-mail: eslamian@mazust.ac.ir, mhmdeslamian@gmail.com

**Department of Pure Mathematics, Faculty of Mathematical Sciences,
University of Tabriz, Tabriz, Iran
E-mail: s.azarmi@tabrizu.ac.ir, zamani@tabrizu.ac.ir

Abstract. In this paper we introduce and study a new algorithm for finding a solution of the split equality fixed point problem for quasi-nonexpansive mappings and a common zero of a finite family of inverse strongly monotone mappings in Hilbert spaces. A numerical example to support our main theorem will be exhibited. Our results improve and generalize some recent results in the literature.
Key Words and Phrases: Split equality fixed point problems, inverse strongly monotone mappings, null point problem.
2020 Mathematics Subject Classification: 47J25, 47N10, 47H10, 65J15.

1. Introduction

The variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \qquad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $F : C \to H$ is a mapping. We denote the solutions set of (1.1) by $VI(C, F)$. Variational inequality problem has a great impact and influence in the classes of mathematical problems and it is widely studied in many fields of pure and applied sciences. Several iterative schemes have been proposed for solving variational inequality (see [2, 12, 13, 16, 20, 24, 26]). Among all the iterative methods for VIP, the simplest one is gradient projection method as follows:

$$x_{n+1} = P_C(I - \lambda F)x_n,$$

for each $n \in \mathbb{N}$, where P_C is the metric projection of H into C and λ is a positive real number. The convergence of this method can be proved under a strong condition that the mapping F is strongly monotone and Lipschitz continuous. It requires repetitive use of P_C, that it works only when the explicit form of P_C is known (e.g., C is a closed ball or a closed cone). In 2001, Yamada [29] introduced the following so-called hybrid steepest descent method:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n, \tag{1.2}$$

for each $n \in \mathbb{N}$. Under certain conditions, the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique point in $VI(Fix(T), F)$. This method does not require the closed form expression of P_C but instead requires a closed form expression of a nonexpansive mapping T, whose fixed point set is C.

Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. Given mappings $F : H_1 \to H_1$ and $G : H_2 \to H_2$. The split variational inequality problem (SVIP) introduced first by Censor et al. [8] can be formulated as follows:
find

$$x^* \in C : \langle F(x^*), x - x^* \rangle \geq 0 \qquad \forall x \in C$$

such that

$$y^* = Ax^* \in Q : \langle G(y^*), y - y^* \rangle \geq 0 \qquad \forall y \in Q.$$

So SVIP is the problem of finding $x^* \in VI(C, F)$ such that $Ax^* \in VI(Q, G)$. A special case of the SVIP, when $F = G = 0$, is the split feasibility problem (SFP) which has been studied by many authors (see [3, 4, 6, 7, 11, 18, 21, 22, 23]).
In [19], Moudafi introduced the following split equality fixed point problem (SEFP). Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators and let C and Q be two nonempty closed convex subsets of H_1 and H_2, respectively, let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators such that $Fix(S) \neq \emptyset$ and $Fix(T) \neq \emptyset$. The split equality fixed point problem (SEFP) is to find

$$x \in Fix(S), \quad y \in Fix(T) \qquad \text{such that} \quad Ax = By. \tag{1.3}$$

In addition, let $F : H_1 \to H_1$ be a monotone and L- Lipschitz continuous operator on C and $G : H_2 \to H_2$ be a monotone and K- Lipschitz continuous operator on Q such that $Fix(S) \cap VI(C, F) \neq \emptyset$ and $Fix(T) \cap VI(Q, G) \neq \emptyset$. The split equality variational inequality and fixed point problem (SEVIP) introduced by Eslamian [9] is to find points

$$x \in Fix(S) \cap VI(C, F), \quad y \in Fix(T) \cap VI(Q, G) \qquad \text{such that} \quad Ax = By. \tag{1.4}$$

If $F = G = 0$, then the SEVIP reduces to the split equality fixed point problem.
Motivated by the above works, the purpose of this paper is to introduce a new algorithm for finding a solution of split equality variational inequality problem for inverse strongly monotone operators and a common fixed points of a finite family of quasi-nonexpansive mappings which does not require any knowledge of the operator norms. A numerical example to support our main theorem will be exhibited.

2. Preliminaries and lemmas

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. Let C be a nonempty closed convex subset of H. we denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in H by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. By P_C, we denote the metric projection from H onto C. Namely, for each $x \in H$, $P_C(x)$ is the unique element in C such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

Let T be a mapping of C into H. We use $Fix(T)$ to stand for the fixed point set of T. A mapping $T : C \longrightarrow H$ is said to be

• Firmly nonexpansive if

$$\|T(x) - T(y)\|^2 + \|(I - T)(x) - (I - T)(y)\|^2 \leq \|x - y\|, \qquad \forall x, y \in C;$$

• Nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \qquad \forall x, y \in C;$$

• Quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$\|T(x) - p\| \leq \|x - p\|, \qquad \forall x \in C, \ \ p \in Fix(T);$$

• Lipschitz continuous with constant $L > 0$ if

$$\|T(x) - T(y)\| \leq L\|x - y\|, \qquad \forall x, y \in C;$$

• monotone if

$$\langle T(x) - T(y), x - y \rangle \geq 0, \qquad \forall x, y \in C;$$

• Inverse-strongly monotone with constant $\alpha > 0$, $(\alpha - ism)$ if

$$\langle T(x) - T(y), x - y \rangle \geq \alpha \|T(x) - T(y)\|^2, \qquad \forall x, y \in C.$$

It is not hard to see that α-inverse-strongly monotone mappings are Lipschitz continuous. A mapping $T : H \to H$ is called α-averaged if there exists $\alpha \in (0, 1)$ such that

$$T = (1 - \alpha)I + \alpha S,$$

where $S : H \to H$ is nonexpansive mapping. More information on metric projections, firmly nonexpansive mappings and averaged mappings can be found in the book by Goebel and Reich [14].

Definition 2.1. Let $T : H \to H$ be a mapping, then $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in H, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, imply $x = Tx$.

We will use the following lemmas.

Lemma 2.2. ([1]) *Let C be a nonempty closed convex subset of H and $T : C \to H$ be a quasi-nonexpansive mapping. Then $Fix(T)$ is closed and convex.*

Lemma 2.3. ([1]) *Let $T : H \to 2^H$. The resolvent of T is $J_T = (I + T)^{-1}$. Then the following hold:*

(i) *J_T is firmly nonexpansive if and only if T is monotone,*
(ii) *$Fix(J_T) = T^{-1}(0)$.*

Lemma 2.4. ([27]) *Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.5. ([28]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\vartheta_n)a_n + \vartheta_n\delta_n, \qquad n \geq 0,$$

where $\{\vartheta_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty}\vartheta_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\vartheta_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. ([17]) *Let* $\{t_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $t_{n_i} < t_{n_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{\tau(n)\} \subset \mathbb{N}$ *such that* $\tau(n) \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $n \in \mathbb{N}$*:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.7. ([10]) *Let* H *be a Hilbert space and* $x_i \in H$, $(1 \leq i \leq m)$. *Then for any given* $\{\lambda_i\}_{i=1}^m \subset (0,1)$ *with* $\sum_{i=1}^m \lambda_i = 1$ *and for any positive integer* k, j *with* $1 \leq k < j \leq m$, *we have*

$$\|\sum_{i=1}^m \lambda_i x_i\|^2 \leq \sum_{i=1}^m \lambda_i\|x_i\|^2 - \lambda_k\lambda_j\|x_k - x_j\|^2.$$

Lemma 2.8. ([5]) *Let* $T : H \to H$ *be a mapping.*

(i) T *is nonexpansive if and only if the complement* $I - T$ *is* $\frac{1}{2}$*-inverse strongly monotone.*

(ii) *If* T *is* ν*-inverse strongly monotone, then for* $\gamma > 0$, γT *is* $\frac{\nu}{\gamma}$*-inverse strongly monotone.*

(iii) *For* $\alpha \in (0,1)$, T *is* α*-averaged if and only if* $I - T$ *is* $\frac{1}{2\alpha}$*-inverse strongly monotone.*

3. Main results

Now we state and prove our main results of this paper.

Theorem 3.1. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces,* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be bounded linear operators. Let for* $i = 1, 2, ..., m$, $F_i : H_1 \to H_1$ *be a finite family of* κ_i*-inverse strongly monotone mappings and* $G_i : H_2 \to H_2$ *be a finite family of* ι_i*-inverse strongly monotone mappings. Let* $\{T_i\}_{i=1}^m : H_1 \to H_1$ *and* $\{S_i\}_{i=1}^m : H_2 \to H_2$ *be two finite families of quasi-nonexpansive mappings such that* $S_i - I$ *and* $T_i - I$ *are demiclosed at* 0. *Suppose*

$$\Omega = \{(x,y) : x \in \bigcap_{i=1}^m (Fix(T_i) \cap F_i^{-1}(0)),\;\; y \in \bigcap_{i=1}^m (Fix(S_i) \cap G_i^{-1}(0)),\;\; Ax = By\} \neq \emptyset.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in H_1, \quad y_0,\, \upsilon \in H_2$ and by

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n) \\ t_n = \alpha_n^{(0)} z_n + \sum_{i=1}^m \alpha_n^{(i)} T_i z_n \\ u_n = (I - \mu^{(m)}\theta_n^{(m)} F_m) \circ \ldots \circ (I - \mu^{(2)}\theta_n^{(2)} F_2) \circ (I - \mu^{(1)}\theta_n^{(1)} F_1) t_n, \\ x_{n+1} = \beta_n \vartheta + (1-\beta_n) u_n \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n) \\ s_n = \alpha_n^{(0)} w_n + \sum_{i=1}^m \alpha_n^{(i)} S_i w_n \\ v_n = (I - \rho^{(m)}\delta_n^{(m)} G_m) \circ \ldots \circ (I - \rho^{(2)}\delta_n^{(2)} G_2) \circ (I - \rho^{(1)}\delta_n^{(1)} G_1) s_n, \\ y_{n+1} = \beta_n \upsilon + (1-\beta_n) v_n, \end{cases} \tag{3.1}$$

for all $n \geq 0$, where the step-size γ_n is chosen in such a way that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon\right), n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set

$$\Pi = \{n : Ax_n - By_n \neq 0\}.$$

Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n\}$, $\{\delta_n^{(i)}\}$ and $\{\theta_n^{(i)}\}$ satisfy the following conditions:

(i) $\sum_{i=0}^m \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(0)} \alpha_n^{(i)} > 0$, *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^\infty \beta_n = \infty$,

(iii) $\{\mu^{(i)}\theta_n^{(i)}\} \subset [a_i, b_i] \subset (0, 2\kappa_i)$ *and* $\{\rho^{(i)}\delta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\iota_i)$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequences $\{(x_n, y_n)\}$ converge strongly to $(x^*, y^*) \in \Omega$.

Proof. Put

$$\begin{cases} z_n^{(1)} = (I - \mu^{(1)}\theta_n^{(1)} F_1) t_n \\ z_n^{(2)} = (I - \mu^{(2)}\theta_n^{(2)} F_2) z_n^{(1)} \\ \vdots \\ z_n^{(m)} = u_n = (I - \mu^{(m)}\theta_n^{(m)} F_m) z_n^{(m-1)} \end{cases}$$

and

$$\begin{cases} y_n^{(1)} = (I - \rho^{(1)}\delta_n^{(1)} G_1) s_n \\ y_n^{(2)} = (I - \rho^{(2)}\delta_n^{(2)} G_2) y_n^{(1)} \\ \vdots \\ y_n^{(m)} = v_n = (I - \rho^{(m)}\delta_n^{(m)} G_m) y_n^{(m-1)} \end{cases}$$

By using Lemma 2.8, since F_1 is κ_1-inverse strongly monotone and G_1 is ι_1-inverse strongly monotone, we can rewrite $z_n^{(1)}$ and $y_n^{(1)}$ as

$$z_n^{(1)} = (1 - \lambda_n^{(1)}) t_n + \lambda_n^{(1)} V_n^{(1)} t_n, \tag{3.2}$$

and

$$y_n^{(1)} = (1 - \xi_n^{(1)}) s_n + \xi_n^{(1)} W_n^{(1)} s_n.$$

where $V_n^{(1)}$ is a nonexpansive mapping of H_1 into H_1, $W_n^{(1)}$ is a nonexpansive mapping of H_2 into H_2,

$$\lambda_n^{(1)} = \frac{\mu^{(1)}\theta_n^{(1)}}{2\kappa_1} \text{ and } \xi_n^{(1)} = \frac{\rho^{(1)}\delta_n^{(1)}}{2\iota_1}$$

for all $n \in \mathbb{N}$. Take $(x^*, y^*) \in \Omega$. We have

$$\begin{aligned}\|z_n^{(1)} - x^*\|^2 &= \|(1-\lambda_n^{(1)})t_n + \lambda_n^{(1)}V_n^{(1)}t_n - x^*\|^2\\ &\leq (1-\lambda_n^{(1)})\|t_n - x^*\|^2 + \lambda_n^{(1)}\|V_n^{(1)}t_n - x^*\|^2\\ &\quad -\lambda_n^{(1)}(1-\lambda_n^{(1)})\|V_n^{(1)}t_n - t_n\|^2\\ &\leq \|t_n - x^*\|^2 - \lambda_n^{(1)}(1-\lambda_n^{(1)})\|V_n^1 t_n - t_n\|^2.\end{aligned}$$

By a similar argument for $i \in \{2, 3, 4, ..., m\}$, we get

$$\|z_n^{(i)} - x^*\|^2 \leq \|z_n^{(i-1)} - x^*\|^2 - \lambda_n^{(i)}(1-\lambda_n^{(i)})\|V_n^{(i)}z_n^{(i-1)} - z_n^{(i-1)}\|^2, \tag{3.3}$$

and

$$\|y_n^{(i)} - y^*\|^2 \leq \|y_n^{(i-1)} - y^*\|^2 - \xi_n^{(i)}(1-\xi_n^{(i)})\|W_n^{(i)}y_n^{(i-1)} - y_n^{(i-1)}\|^2, \tag{3.4}$$

where $V_n^{(i)}$ are nonexpansive mappings of H_1 into H_1 and $W_n^{(i)}$ are nonexpansive mappings of H_2 into H_2 for all $n \in \mathbb{N}$. From (3.1) we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|x_n - \gamma_n A^*(Ax_n - By_n) - x^*\|^2\\ &= \|x_n - x^*\|^2 + \gamma_n^2\|A^*(Ax_n - By_n)\|^2\\ &\quad - 2\gamma_n\langle x_n - x^*, A^*(Ax_n - By_n)\rangle\\ &= \|x_n - x^*\|^2 + \gamma_n^2\|A^*(Ax_n - By_n)\|^2\\ &\quad - 2\gamma_n\langle Ax_n - Ax^*, (Ax_n - By_n)\rangle\\ &= \|x_n - x^*\|^2 + \gamma_n^2\|A^*(Ax_n - By_n)\|^2 - \gamma_n\|Ax_n - Ax^*\|^2\\ &\quad - \gamma_n\|Ax_n - By_n\|^2 + \gamma_n\|By_n - Ax^*\|^2.\end{aligned} \tag{3.5}$$

Similarly, we also have

$$\begin{aligned}\|w_n - y^*\|^2 &= \|y_n + \gamma_n B^*(Ax_n - By_n) - y^*\|^2\\ &= \|y_n - y^*\|^2 + \gamma_n^2\|B^*(Ax_n - By_n)\|^2 - \gamma_n\|By_n - By^*\|^2\\ &\quad - \gamma_n\|Ax_n - By_n\|^2 + \gamma_n\|Ax_n - By^*\|^2.\end{aligned} \tag{3.6}$$

By adding equalities (3.5), (3.6) and by taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned}\|z_n - x^*\|^2 + \|w_n - y^*\|^2 &= \|x_n - x^*\|^2 + \|y_n - y^*\|^2 \\ &\quad + \gamma_n^2(\|A^*(Bx_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2) \\ &\quad - 2\gamma_n\|Ax_n - By_n\|^2.\end{aligned} \tag{3.7}$$

Using Lemma 2.7 for each $i \in \{1, 2, ..., m\}$, we have

$$\begin{aligned}\|t_n - x^*\|^2 &= \|\alpha_n^{(0)} z_n + \textstyle\sum_{i=1}^m \alpha_n^{(i)} T_i z_n - x^*\|^2 \\ &\leq \alpha_n^{(0)}\|z_n - x^*\|^2 + \textstyle\sum_{i=1}^m \alpha_n^{(i)}\|T_i z_n - x^*\|^2 \\ &\quad - \alpha_n^{(0)}\alpha_n^{(i)}\|T_i z_n - z_n\|^2 \\ &\leq \|z_n - x^*\|^2 - \alpha_n^{(0)}\alpha_n^{(i)}\|T_i z_n - z_n\|^2\end{aligned} \tag{3.8}$$

Similarly, we can obtain

$$\begin{aligned}\|s_n - y^*\|^2 &= \|\alpha_n^{(0)} w_n + \textstyle\sum_{i=1}^m \alpha_n^{(i)} S_i w_n - y^*\|^2 \\ &\leq \alpha_n^{(0)}\|w_n - y^*\|^2 + \textstyle\sum_{i=1}^m \alpha_n^{(i)}\|S_i w_n - y^*\|^2 \\ &\quad - \alpha_n^{(0)}\alpha_n^{(i)}\|S_i w_n - w_n\|^2 \\ &\leq \|w_n - y^*\|^2 - \alpha_n^{(0)}\alpha_n^{(i)}\|S_i w_n - w_n\|^2\end{aligned} \tag{3.9}$$

From (3.3) and (3.4), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\beta_n\vartheta + (1-\beta_n)u_n - x^*\|^2 \\ &= \beta_n\|\vartheta - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 - \beta_n(1-\beta_n)\|u_n - \vartheta\|^2 \\ &\leq \beta_n\|\vartheta - x^*\|^2 + (1-\beta_n)\|t_n - x^*\|^2 - \beta_n(1-\beta_n)\|u_n - \vartheta\|^2 \\ &\quad - (1-\beta_n)\lambda_n^{(1)}(1-\lambda_n^{(1)})\|V_n^{(1)} t_n - t_n\|^2 - ... \\ &\quad - (1-\beta_n)\lambda_n^{(m)}(1-\lambda_n^{(m)})\|V_n^{(m)} z_n^{(m-1)} - z_n^{(m-1)}\|^2,\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1} - y^*\|^2 &= \|\beta_n\upsilon + (1-\beta_n)v_n - y^*\|^2 \\ &= \beta_n\|\upsilon - y^*\|^2 + (1-\beta_n)\|v_n - y^*\|^2 - \beta_n(1-\beta_n)\|v_n - \upsilon\|^2 \\ &\leq \beta_n\|\upsilon - y^*\|^2 + (1-\beta_n)\|s_n - y^*\|^2 - \beta_n(1-\beta_n)\|v_n - \upsilon\|^2 \\ &\quad - (1-\beta_n)\xi_n^{(1)}(1-\xi_n^{(1)})\|W_n^{(1)} s_n - s_n\|^2 - ... \\ &\quad - (1-\beta_n)\xi_n^{(m)}(1-\xi_n^{(m)})\|W_n^{(m)} y_n^{(m-1)} - y_n^{(m-1)}\|^2.\end{aligned}$$

By adding the two last inequalities and using (3.7), (3.8) and (3.9), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &+ \|y_{n+1}-y^*\|^2 \\
&\leq \beta_n(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2) \\
&+ (1-\beta_n)(\|t_n-x^*\|^2+\|s_n-y^*\|^2) \\
&\leq\ \beta_n(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2) \\
&+ (1-\beta_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2) \\
&\leq\ \max\{(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2),(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\} \\
&\vdots \\
&\leq\ \max\{(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2),(\|x_0-x^*\|^2+\|y_0-y^*\|^2)\}.
\end{aligned}$$

Thus $\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{z_n\},\{w_n\},\{s_n\}$ and $\{t_n\}$ are all bounded.

$$\begin{aligned}
\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\leq (1-\beta_n)(\|t_n-x^*\|^2+\|s_n-y^*\|^2) \\
&+ \beta_n(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2) \\
&- \beta_n(1-\beta_n)\|u_n-\vartheta\|^2 \\
&- (1-\beta_n)\lambda_n^{(1)}(1-\lambda_n^{(1)})\|V_n^{(1)}t_n-t_n\|^2 \\
&\vdots \\
&- (1-\beta_n)\lambda_n^{(m)}(1-\lambda_n^{(m)})\|V_n^{(m)}z_n^{(m-1)}-z_n^{(m-1)}\|^2 \\
&- \beta_n(1-\beta_n)\|v_n-\upsilon\|^2 \\
&- (1-\beta_n)\xi_n^{(1)}(1-\xi_n^{(1)})\|W_n^{(1)}s_n-s_n\|^2 \\
&\vdots \\
&- (1-\beta_n)\xi_n^{(m)}(1-\xi_n^{(m)})\|W_n^{(m)}y_n^{m-1}-y_n^{m-1}\|^2 \\
&\leq (1-\beta_n)(\|x_n-x^*\|^2+\|y_n-y^*\|^2) \\
&+ \beta_n(\|\vartheta-x^*\|^2+\|\upsilon-y^*\|^2) \\
&- (1-\beta_n)\gamma_n[2\|Ax_n-By_n\|^2 \\
&- \gamma_n(\|B^*(Ax_n-By_n)\|^2+\|A^*(Ax_n-By_n)\|^2)] \\
&- (1-\beta_n)\alpha_n^{(0)}\alpha_n^{(i)}\|T_iz_n-z_n\|^2 \\
&- (1-\beta_n)\alpha_n^{(0)}\alpha_n^{(i)}\|S_iw_n-w_n\|^2 \\
&+ \beta_n(1-\beta_n)\|u_n-\vartheta\|^2
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&-(1-\beta_n)\lambda_n^{(1)}(1-\lambda_n^{(1)})\|V_n^{(1)}t_n - t_n\|^2\\
&\vdots\\
&-(1-\beta_n)\lambda_n^{(m)}(1-\lambda_n^{(m)})\|V_n^{(m)}z_n^{(m-1)} - z_n^{(m-1)}\|^2\\
&-\beta_n(1-\beta_n)\|v_n - \upsilon\|^2\\
&-(1-\beta_n)\xi_n^{(1)}(1-\xi_n^{(1)})\|W_n^{(1)}s_n - s_n\|^2\\
&\vdots\\
&-(1-\beta_n)\xi_n^{(m)}(1-\xi_n^{(m)})\|W_n^{(m)}y_n^{m-1} - y_n^{m-1}\|^2.
\end{aligned}
$$

By our assumption that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon\right),$$

we have

$$(\gamma_n + \epsilon)(\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2) \leq 2\|Ax_n - By_n\|^2.$$

From above inequality and inequality (3.10), we can obtain

$$
\begin{aligned}
(1-\beta_n)\gamma_n\epsilon(\|B^* \quad &(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2)\\
&\leq (1-\beta_n)\gamma_n[2\|Ax_n - By_n\|^2\\
&-\gamma_n(\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2)]\\
&\leq (1-\beta_n)(\|x_n - x^*\|^2 + \|y_n - y^*\|^2)\\
&-\|x_{n+1} - x^*\|^2 - \|y_{n+1} - y^*\|^2\\
&+\beta_n(\|\vartheta - x^*\|^2 + \|\upsilon - y^*\|^2).
\end{aligned}
\tag{3.11}
$$

Set$(\vartheta^*, \upsilon^*) = P_\Omega(\vartheta, \upsilon)$. Put $\Gamma_n = \|x_n - \vartheta^*\|^2 + \|y_n - \upsilon^*\|^2$ for all $n \in \mathbb{N}$. We finally analyze the inequality (3.11) by considering the following two cases.

Case A. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq n_0$ (for n_0 large enough). In this case, since Γ_n is bounded, the $\lim_{n\to\infty}\Gamma_n$ exists. Since $\lim_{n\to\infty}\beta_n = 0$, from (3.11) and by our assumption on $\{\gamma_n\}$, we have

$$\lim_{n\to\infty}(\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2) = 0.$$

So we obtain that $\lim_{n\to\infty}\|B^*(Ax_n - By_n)\| = 0$ and $\lim_{n\to\infty}\|A^*(Ax_n - By_n)\| = 0$. This implies that $\lim_{n\to\infty}\|Ax_n - By_n\| = 0$. Since $\{\gamma_n\}$ is bounded, we deduce

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\gamma_n\|A^*(Ax_n - By_n)\| = 0, \tag{3.12}$$

and

$$\lim_{n\to\infty}\|w_n - y_n\| = \lim_{n\to\infty}\gamma_n\|B^*(Ax_n - By_n)\| = 0. \tag{3.13}$$

By assumption (i) and (3.10), we get

$$\lim_{n\to\infty}\|S_i w_n - w_n\| = \lim_{n\to\infty}\|T_i z_n - z_n\| = 0, \quad i \in \{1, 2, ..., m\}.$$

Hence

$$\|t_n - z_n\| = \|\alpha_n^{(0)} z_n + \sum_{i=1}^{m} \alpha_n^{(i)} T_i z_n - z_n\| \le \sum_{i=1}^{m} \alpha_n^{(i)} \|T_i z_n - z_n\| \to 0$$

$$\|x_n - t_n\| \le \|x_n - z_n\| + \|z_n - t_n\| \to 0. \tag{3.14}$$

Now we claim that $(\omega_w(x_n), \omega_w(y_n)) \subset \Omega$, where

$$\omega_w(x_n) = \{x \in H_1 : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\}\, of\ \{x_n\}\}.$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, we have $\omega_w(x_n)$ and $\omega_w(y_n)$ are nonempty. Now, take $\widehat{x} \in \omega_w(x_n)$ and $\widehat{y} \in \omega_w(y_n)$. Thus, there exists a subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{y_{n_i}\}$ of $\{y_n\}$ which $x_{n_i} \rightharpoonup \widehat{x}$ and $y_{n_i} \rightharpoonup \widehat{y}$. Without loss of generality, we can assume that $x_n \rightharpoonup \widehat{x}$ and $y_n \rightharpoonup \widehat{y}$. From (3.12) and (3.13), we have $z_n \rightharpoonup \widehat{x}$ and $w_n \rightharpoonup \widehat{y}$. On the other hand, demiclosedness of $T_i - I$ in 0, for each $i \in \{1, 2, ..., m\}$ implies that $\widehat{x} \in \bigcap_{i=1}^{m} Fix(T_i)$. By similar argument, we obtain that $\widehat{y} \in \bigcap_{i=1}^{m} Fix(S_i)$. Since $\{\theta_{n_i}^{(1)}\}$ is bounded, we can fined a subsequence $\{\theta_{n_{i_j}}^{(1)}\}$ converging to $\theta^{(1)}$ such that $\mu^{(1)}\theta^{(1)} \in [a_1, b_1]$. From (3.14) we have $t_{n_{i_j}} \rightharpoonup \widehat{x}$. Since $\{t_{n_{i_j}}\}$ is bounded and F_1 is inverse strongly monotone, we know that $\{F_1 t_{n_{i_j}}\}$ is bounded.

$$\|(I - \mu^{(1)}\theta_{n_{i_j}}^{(1)} F_1) t_{n_{i_j}} - (I - \mu^{(1)}\theta^{(1)} F_1) t_{n_{i_j}}\| \le |\mu^{(1)}\theta_{n_{i_j}}^{(1)} - \mu^{(1)}\theta^{(1)}| \|F_1 t_{n_{i_j}}\|.$$

From $\theta_{n_{i_j}}^{(1)} \to \theta^{(1)}$, we have

$$\|(I - \mu^{(1)}\theta_{n_{i_j}}^{(1)} F_1) t_{n_{i_j}} - (I - \mu^{(1)}\theta^{(1)} F_1) t_{n_{i_j}}\| \to 0.$$

From (3.10) and (3.2) we have

$$\|z_n^{(1)} - t_n\| \to 0,$$

hence

$$\|(I - \mu^{(1)}\theta_{n_{i_j}}^{(1)} F_1) t_{n_{i_j}} - t_{n_{i_j}}\| \to 0.$$

Since

$$\begin{aligned}\|(I - \mu^{(1)}\theta^{(1)} F_1) t_{n_{i_j}} - t_{n_{i_j}}\| \quad &\le \|(I - \mu^{(1)}\theta_{n_{i_j}}^{(1)} F_1) t_{n_{i_j}} - t_{n_{i_j}}\| \\ &\quad + \|(I - \mu^{(1)}\theta^{(1)} F_1) t_{n_{i_j}} - (I - \mu^{(1)}\theta_{n_{i_j}}^{(1)} F_1) t_{n_{i_j}}\|,\end{aligned}$$

we get

$$\|(I - \mu^{(1)}\theta^{(1)} F_1) t_{n_{i_j}} - t_{n_{i_j}}\| \to 0, \qquad n \to \infty.$$

From Lemmas 2.4 and 2.8, we obtain that

$$\widehat{x} \in Fix(I - \mu^{(1)}\theta^{(1)} F_1) = F_1^{-1}(0).$$

By similar argument for $i \in \{2, 3, ..., m\}$, we get

$$\widehat{x} \in \bigcap_{i=1}^{m} F_i^{-1}(0) \text{ and } \widehat{y} \in \bigcap_{i=1}^{m} G_i^{-1}(0).$$

On the other hand, $A\widehat{x} - B\widehat{y} \in \omega_w(Ax_n - By_n)$ and weakly lower semi continuity of the norm imply that

$$\|A\widehat{x} - B\widehat{y}\| \leq \liminf_{n\to\infty} \|Ax_n - By_n\| = 0.$$

Thus $(\widehat{x}, \widehat{y}) \in \Omega$. We also have the uniqueness of the weak cluster point of $\{x_n\}$ are $\{y_n\}$, (see [30] for details) which implies that the whole sequences $\{(x_n, y_n)\}$ weakly convergence to a point $(\widehat{x}, \widehat{y}) \in \Omega$. Next we prove that the sequences $\{(x_n, y_n)\}$ converges strongly to $(\vartheta^*, \upsilon^*)$. Now, we show that

$$\limsup_{n\to\infty}(\langle \vartheta - \vartheta^*, x_n - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_n - \upsilon^*\rangle) \leq 0.$$

Choose a subsequence $\{x_{n_k}\}$, $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that

$$\begin{aligned}&\limsup_{n\to\infty}(\langle \vartheta - \vartheta^*, x_n - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_n - \upsilon^*\rangle)\\ =&\lim_{n\to\infty}(\langle \vartheta - \vartheta^*, x_{n_k} - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_{n_k} - \upsilon^*\rangle).\end{aligned}$$

Since the sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are bounded, there exist subsequences $\{x_{n_{k_j}}\}$, $\{y_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_k}\}$, respectively such that converges $x_{n_{k_j}} \rightharpoonup \widehat{x}$, $y_{n_{k_j}} \rightharpoonup \widehat{y}$, and $(\widehat{x}, \widehat{y}) \in \Omega$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \widehat{x}$, $y_{n_k} \rightharpoonup \widehat{y}$.
It follows from prperties of projection that

$$\begin{aligned}\limsup\nolimits_{n\to\infty}&(\langle \vartheta - \vartheta^*, x_n - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_n - \upsilon^*\rangle)\\ &= \lim\nolimits_{n\to\infty}(\langle \vartheta - \vartheta^*, x_{n_k} - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_{n_k} - \upsilon^*\rangle)\\ &= \langle \vartheta - \vartheta^*, \widehat{x} - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, \widehat{y} - \upsilon^*\rangle\\ &= \langle (\vartheta - \vartheta^*, \upsilon - \upsilon^*), (\widehat{x} - \vartheta^*, \widehat{y} - \upsilon^*)\rangle\\ &= \langle (\vartheta, \upsilon) - (\vartheta^*, \upsilon^*), (\widehat{x}, \widehat{y}) - (\vartheta^*, \upsilon^*)\rangle \leq 0.\end{aligned}$$

From the inequality, $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, we find that

$$\begin{aligned}\|x_{n+1} - \vartheta^*\|^2 &= \|\beta_n\vartheta + (1 - \beta_n)u_n - \vartheta^*\|^2\\ &\leq (1 - \beta_n)^2\|u_n - \vartheta^*\| + 2\beta_n\langle \vartheta - \vartheta^*, x_{n+1} - \vartheta^*\rangle.\end{aligned}$$

Similarly we obtain that

$$\|y_{n+1} - \upsilon^*\|^2 \leq (1 - \beta_n)^2\|v_n - \upsilon^*\|^2 + 2\beta_n\langle \upsilon - \upsilon^*, y_{n+1} - \upsilon^*\rangle.$$

By adding the two last inequalities, we have that

$$\begin{aligned}\|x_{n+1} - \vartheta^*\|^2 &+ \|y_{n+1} - \upsilon^*\|^2\\ &\leq (1 - \beta_n)^2(\|u_n - \vartheta^*\|^2 + \|v_n - \upsilon^*\|^2)\\ &+ 2\beta_n(\langle \vartheta - \vartheta^*, x_{n+1} - \vartheta^*\rangle + \langle \upsilon - \upsilon^*, y_{n+1} - \upsilon^*\rangle).\end{aligned}$$

It immediately follows that

$$\begin{aligned}\Gamma_{n+1} &\leq (1-\beta_n)^2\Gamma_n + 2\beta_n\eta_n \\ &= (1-2\beta_n)\Gamma_n + \beta_n^2\Gamma_n + 2\beta_n\eta_n \\ &\leq \quad (1-2\beta_n)\Gamma_n + 2\beta_n\{\tfrac{\beta_n N}{2} + \eta_n) \\ &\leq \quad (1-\rho_n)\Gamma_n + \rho_n\delta_n,\end{aligned}$$

where

$$\eta_n = \langle \vartheta - \vartheta^*, x_{n+1} - \vartheta^* \rangle + \langle v - v^*, y_{n+1} - v^* \rangle,$$
$$N = sup\{\|x_n - \vartheta^*\|^2 + \|y_n - v^*\|^2 : n \geq 0\},$$
$$\rho_n = 2\beta_n \text{ and } \delta_n = \frac{\beta_n N}{2} + \eta_n.$$

It is easy to see that $\rho_n \to 0$,

$$\sum_{n=1}^{\infty} \rho_n = \infty \text{ and } \limsup_{n\to\infty} \delta_n \leq 0.$$

Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty} \Gamma_n = 0$. Consequently $\lim_{n\to\infty} \|x_n - \vartheta^*\| = \lim_{n\to\infty} \|y_n - v^*\| = 0$, that is $(x_n, y_n) \to (\vartheta^*, v^*)$.

Case B. Assume that $\{\Gamma_n\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Now, it follows from (3.10) that

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \leq \beta_n(\|\vartheta - \vartheta^*\|^2 + \|v - v^*\|^2) - \beta_n\Gamma_{\tau(n)}.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\{x_n\}$ and $\{y_n\}$ are bounded, we derive that

$$\lim_{n\to\infty} (\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)}) = 0. \tag{3.15}$$

Following an argument similar to that in Case A, we have

$$\Gamma_{\tau(n)+1} \leq (1-\rho_{\tau(n)})\Gamma_{\tau(n)} + \rho_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty} \delta_{\tau(n)} \leq 0$. Since $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have

$$\rho_{\tau(n)}\Gamma_{\tau(n)} \leq \rho_{\tau(n)}\delta_{\tau(n)}.$$

Since $\rho_{\tau(n)} > 0$ we deduce that

$$\Gamma_{\tau(n)} \leq \delta_{\tau(n)}.$$

From $\limsup_{n\to\infty} \delta_{\tau(n)} \leq 0$, we get $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. This together with (3.15), implies that $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Thus by Lemma 2.6, we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}.$$

Therefore $(x_n, y_n) \to (\vartheta^*, v^*)$. This completes the proof.

4. Corollaries

Putting $\vartheta = \upsilon = 0$ in 3.1 we obtain the following result.

Theorem 4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $F : H_1 \to H_1$ be a κ-inverse strongly monotone mapping and $G : H_2 \to H_2$ be a ι-inverse strongly monotone mapping. Let $\{T_i\}_{i=1}^m : H_1 \to H_1$ and $\{S_i\}_{i=1}^m : H_2 \to H_2$ be two finite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Suppose*

$$\Omega = \{(x,y) : x \in (\cap_{i=1}^m (Fix(T_i)) \cap F^{-1}(0)),$$
$$y \in (\cap_{i=1}^m (Fix(S_i)) \cap G^{-1}(0)), \quad Ax = By\} \neq \emptyset.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 \in H_1, y_0 \in H_2$ and by:

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n) \\ t_n = \alpha_n^{(0)} z_n + \sum_{i=1}^m \alpha_n^{(i)} T_i z_n \\ x_{n+1} = (1-\beta_n)(I - \mu\theta_n F_1) t_n \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n) \\ s_n = \alpha_n^{(0)} w_n + \sum_{i=1}^m \alpha_n^{(i)} S_i w_n \\ y_{n+1} = (1-\beta_n)(I - \rho\delta_n G_1) s_n, \end{cases} \tag{4.1}$$

for all $n \geq 0$, where the step-size γ_n is chosen in such a way that

$$\gamma_n \in \left(\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon\right), n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set

$$\Pi = \{n : Ax_n - By_n \neq 0\}.$$

Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n\}$, $\{\delta_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(i) $\sum_{i=0}^m \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(0)} \alpha_n^{(i)} > 0$, *for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^\infty \beta_n = \infty$,

(iii) $\{\mu\theta_n\} \subset [a,b] \subset (0, 2\kappa)$ *and* $\{\rho\delta_n\} \subset [c,d] \subset (0, 2\iota)$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequences $\{(x_n, y_n)\}$ converge strongly to $(x^*, y^*) \in \Omega$, where (x^*, y^*) is also a point in

$$\{(x,y) : \ x \in VI(\cap_{i=1}^m (Fix(T_i), F), \ \ y \in VI(\cap_{i=1}^m (Fix(S_i), G)), \ \ Ax = By\}.$$

Let f be a continuously Fréchet differentiable and convex functional on H and let ∇f be the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse strongly monotone, (see [15]).

Theorem 4.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let for $i = 1, 2, ..., m$, f_i be continuously Fréchet differentiable convex functionals on H_1 such that ∇f_i be $\frac{1}{\kappa_i}$-Lipschitz continuous and g_i be continuously Fréchet differentiable convex functionals on H_2 such that ∇g_i be $\frac{1}{\iota_i}$-Lipschitz continuous. Let $\{T_i\}_{i=1}^m : H_1 \to H_1$ and $\{S_i\}_{i=1}^m : H_2 \to H_2$ be two finite* *families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at* 0*. Suppose*

$$\Omega = \{(x, y) : x \in \bigcap_{i=1}^{m} (Fix(T_i) \cap (\nabla f_i)^{-1}(0)),$$

$$y \in \bigcap_{i=1}^{m} (Fix(S_i) \cap (\nabla g_i)^{-1}(0)), \quad Ax = By\} \neq \emptyset.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in H_1, y_0,\ \upsilon \in H_2$ and by:

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n) \\ t_n = \alpha_n^{(0)} z_n + \sum_{i=1}^{m} \alpha_n^{(i)} T_i z_n \\ u_n = (I - \mu^{(m)}\theta_n^{(m)}\nabla f_m) \circ ... \circ (I - \mu^{(2)}\theta_n^{(2)}\nabla f_2) \circ (I - \mu^{(1)}\theta_n^{(1)}\nabla f_1) t_n, \\ x_{n+1} = \beta_n \vartheta + (1 - \beta_n) u_n \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n) \\ s_n = \alpha_n^{(0)} w_n + \sum_{i=1}^{m} \alpha_n^{(i)} S_i w_n \\ v_n = (I - \rho^{(m)}\delta_n^{(m)}\nabla g_m) \circ ... \circ (I - \rho^{(2)}\delta_n^{(2)}\nabla g_2) \circ (I - \rho^{(1)}\delta_n^{(1)}\nabla g_1) s_n, \\ y_{n+1} = \beta_n \upsilon + (1 - \beta_n) v_n, \end{cases}$$

for all $n \geq 0$, where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon), n \in \Pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set

$$\Pi = \{n : Ax_n - By_n \neq 0\}.$$

Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n\}$, $\{\delta_n^{(i)}\}$ and $\{\theta_n^{(i)}\}$ satisfy the following conditions:

(i) $\sum_{i=0}^{m} \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(0)}\alpha_n^{(i)} > 0$*, for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\beta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\{\mu^{(i)}\theta_n^{(i)}\} \subset [a_i, b_i] \subset (0, 2\kappa_i)$ *and* $\{\rho^{(i)}\delta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\iota_i)$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequences $\{(x_n, y_n)\}$ converge strongly to $(x^*, y^*) \in \Omega$.

Let C be a closed convex subset of a real Hilbert space H. Then a mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

Put $A = I - T$. Then A is $(1 - k)/2$-inverse-strongly-monotone (see [25]).

Theorem 4.3. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let for $i = 1, 2, ..., m$, $F_i : H_1 \to H_1$ be a finite family of κ_i-strictly pseudocontractive mappings and $G_i : H_2 \to H_2$ be a finite family of ι_i-strictly pseudocontractive mappings. Let $\{T_i\}_{i=1}^{m} : H_1 \to H_1$ and $\{S_i\}_{i=1}^{m} : H_2 \to H_2$*

be two finite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0*. Suppose*

$$\Omega = \{(x,y) : x \in \bigcap_{i=1}^{m} (Fix(T_i) \cap Fix(F_i)),$$

$$y \in \bigcap_{i=1}^{m} (Fix(S_i) \cap Fix(G_i)), \quad Ax = By\} \neq \emptyset.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in H_1, \quad y_0, \upsilon \in \mathcal{H}_2$ and by:

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n) \\ t_n = \alpha_n^{(0)} z_n + \sum_{i=1}^m \alpha_n^{(i)} T_i z_n \\ u_n = ((1-\mu^{(m)}\theta_n^{(m)})I + \mu^{(m)}\theta_n^{(m)} F_m) \circ ... \circ ((1-\mu^{(1)}\theta_n^{(1)})I + \mu^{(1)}\theta_n^{(1)} F_1) t_n, \\ x_{n+1} = \beta_n \vartheta + (1-\beta_n) u_n \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n) \\ s_n = \alpha_n^{(0)} w_n + \sum_{i=1}^m \alpha_n^{(i)} S_i w_n \\ v_n = ((1-\rho^{(m)}\delta_n^{(m)})I + \rho^{(m)}\delta_n^{(m)} G_m) \circ ... \circ ((1-\rho^{(1)}\delta_n^{(1)})I + \rho^{(1)}\delta_n^{(1)} G_1) s_n, \\ y_{n+1} = \beta_n \upsilon + (1-\beta_n) v_n, \end{cases}$$

for all $n \geq 0$, where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon), n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set

$$\Pi = \{n : Ax_n - By_n \neq 0\}.$$

Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n\}$, $\{\delta_n^{(i)}\}$ and $\{\theta_n^{(i)}\}$ satisfy the following conditions:

(i) $\sum_{i=0}^m \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(0)} \alpha_n^{(i)} > 0$*, for each* $i \in \{1, 2, ..., m\}$,

(ii) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^\infty \beta_n = \infty$,

(iii) $\{\mu^{(i)}\theta_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1-\kappa_i)$ *and* $\{\rho^{(i)}\delta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 1-\iota_i)$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequences $\{(x_n, y_n)\}$ converge strongly to $(x^*, y^*) \in \Omega$.

By setting $T_i = S_i = I$ $(i = 1, 2, ..., m)$, in Theorem 4.3, we have the following result.

Theorem 4.4. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let for $i = 1, 2, ..., m$, $F_i : H_1 \to H_1$ be a finite family of κ_i-inverse strongly monotone mappings and $G_i : H_2 \to H_2$ be a finite family of ι_i-inverse strongly monotone mappings. Suppose*

$$\Omega = \{(x,y) : x \in \bigcap_{i=1}^{m} F_i^{-1}(0),$$

$$y \in \bigcap_{i=1}^{m} G_i^{-1}(0), \quad Ax = By\} \neq \emptyset.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in H_1$, y_0, $\upsilon \in H_2$ and by:

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n) \\ u_n = (I - \mu^{(m)}\theta_n^{(m)} F_m) \circ ... \circ (I - \mu^{(2)}\theta_n^{(2)} F_2) \circ (I - \mu^{(1)}\theta_n^{(1)} F_1) z_n, \\ x_{n+1} = \beta_n \vartheta + (1-\beta_n) u_n \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n) \\ v_n = (I - \rho^{(m)}\delta_n^{(m)} G_m) \circ ... \circ (I - \rho^{(2)}\delta_n^{(2)} G_2) \circ (I - \rho^{(1)}\delta_n^{(1)} G_1) w_n, \\ y_{n+1} = \beta_n \upsilon + (1-\beta_n) v_n, \end{cases}$$

for all $n \geq 0$, where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|Ax_n - By_n\|^2}{\|B^*(Ax_n - By_n)\|^2 + \|A^*(Ax_n - By_n)\|^2} - \epsilon), n \in \Pi$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set

$$\Pi = \{n : Ax_n - By_n \neq 0\}.$$

Let the sequences $\{\beta_n\}$, $\{\delta_n^{(i)}\}$ and $\{\theta_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\{\mu^{(i)}\theta_n^{(i)}\} \subset [a_i, b_i] \subset (0, 2\kappa_i)$ *and* $\{\rho^{(i)}\delta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\iota_i)$ *for each* $i \in \{1, 2, ..., m\}$.

Then, the sequences $\{(x_n, y_n)\}$ converge strongly to $(x^*, y^*) \in \Omega$.

5. Numerical example

In this section, we will present a numerical example in the two-dimensional space of real numbers to show that our algorithm is efficient.

Example 5.1. Let $H_1 = H_2 = H_3 = \mathbb{R}^2$ with the Euclidean norm. We take $m = 2$ in Theorem 4.1. Assume that

$$T_i(x_1, x_2) = \left(\frac{\sin(x_1)}{i}, \frac{\sin(x_2)}{i}\right),$$

$$S_i(x_1, x_2) = \left(\frac{\arctan(x_1)}{i}, \frac{\arctan(x_2)}{i}\right),$$

$$F_i(x) = \frac{ix}{2} \text{ and } G_i(x) = \frac{ix}{3}.$$

Also we define

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

hence

$$A^* = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad B^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

Put

$$\gamma_n = \frac{1}{2},\ \beta_n = \frac{1}{n\ln(4n)},\ \mu^{(i)} = (i+1),$$
$$\theta_n^{(i)} = \frac{1}{i(1+i)},\ \rho^{(i)} = \frac{i+1}{2},\ \delta_n^{(i)} = \frac{2}{i(1+i)}$$

and $\alpha_n^0 = \alpha_n^i = \frac{1}{3}$. It can be observed that all the assumptions of Theorem 4.1 are satisfied and $\Omega = \{(0,0)\}$.
Taking $(x_{0,1}, x_{0,2}) = (y_{0,1}, y_{0,2}) = (0.5, 0.5)$ and $\vartheta = \upsilon = (0.01, 0.01)$, we obtain the following algorithm:

$$\begin{bmatrix} x_{(n+1),1} \\ x_{(n+1),2} \end{bmatrix} = \begin{bmatrix} \frac{1}{100n\ln 4n} + (1-\frac{1}{n\ln 4n})(\frac{1}{24}x_{n,1} + \frac{1}{8}\sin(\frac{1}{2}x_{n,1})) \\ \frac{1}{100n\ln 4n} + (1-\frac{1}{n\ln 4n})(\frac{1}{24}x_{n,2} + \frac{1}{24}y_{n,2} + \frac{1}{8}\sin(\frac{1}{2}x_{n,2} + \frac{1}{2}y_{n,2})) \end{bmatrix},$$
$$\begin{bmatrix} y_{(n+1),1} \\ y_{(n+1),2} \end{bmatrix} = \begin{bmatrix} \frac{1}{100n\ln 4n} + (1-\frac{1}{n\ln 4n})(\frac{4}{27}y_{n,1} + \frac{2}{9}\arctan(y_{n,1})) \\ \frac{1}{100n\ln 4n} + (1-\frac{1}{n\ln 4n})(\frac{2}{27}x_{n,2} + \frac{2}{27}y_{n,2} + \frac{2}{9}\arctan(\frac{1}{2}x_{n,2} + \frac{1}{2}y_{n,2})) \end{bmatrix}.$$

So we have the numerical results in Tables 1 and 2 and Figures 1 and 2.

Table 1. Numerical results of Example 5.1

n	x_n	$\|x_n\|_2$
1	$(0.50000000, 0.50000000)$	0.70710678
2	$(0.02163620, 0.03552313)$	0.04159348
3	$(0.00411632, 0.00968892)$	0.01052708
4	$(0.00171269, 0.00359991)$	0.00398656
5	$(0.00106400, 0.00175051)$	0.00204851
$\vdots$	$\vdots$	$\vdots$
98	$(1.9329e-5, 2.2440e-5)$	$2.9617e-05$
99	$(1.9099e-05, 2.2172e-05)$	$2.9263e-05$
100	$(1.8873e-05, 2.1910e-05)$	$2.8918e-05$

Table 2. Numerical results of Example 5.1

n	y_n	$\|y_n\|_2$
1	$(0.50000000, 0.50000000)$	0.70710678
2	$(0.05656474, 0.05656474)$	0.07999463
3	$(0.01830684, 0.01535186)$	0.02389184
4	$(0.00721181, 0.00535644)$	0.00898341
5	$(0.00333185, 0.00241107)$	0.00411251
$\vdots$	$\vdots$	$\vdots$
98	$(2.7638e-05, 2.6441e-05)$	$3.8249e-05$
99	$(2.7308e-05, 2.6125e-05)$	$3.7792e-05$
100	$(2.6984e-05, 2.5816e-05)$	$3.7345e-05$

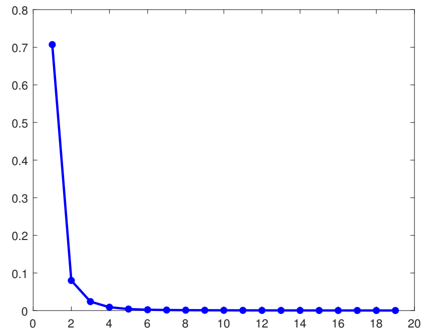

FIGURE 1. Plotting of $\|x_n\|_2$ in Table 1

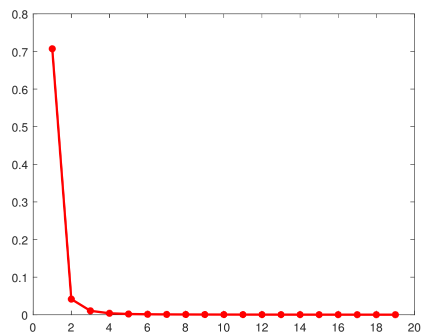

FIGURE 2. Plotting of $\|y_n\|_2$ in Table 2

REFERENCES

[1] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer-Verlag, 2011.

[2] D.P. Bertsekas, E.M. Gafni, *Projection methods for variational inequalities with applications to the traffic assignment problem*, Math. Progr. Study, **17**(1982), 139-159.

[3] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, **18**(2002), 441-453.

[4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, **20**(2004), 103-120.

[5] L.C. Ceng, Q.H. Ansari, J.C. Yao, *Some iterative method for finding fixed points and solving constrained convex minimization problems*, Number Algebra, Control Optim, **1**(3)(2011), 341-359.

[6] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol., **51**(2006), 2353-2365.

[7] Y. Censor, T. Elfving, *A multiprojection algorithms using Bragman projection in a product space*, Numerical Algorithms, **8**(1994), 221-239.

[8] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numerical Algorithms, **59**(2012), 301-323.
[9] M. Eslamian, *Strong convergence of split equality variational inequality and fixed point problem*, Riv. Mat. Univ. Parma, **8**(2)(2017), 225-246.
[10] M. Eslamian, *Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A. Matematicas, **107**(2013), 299-307.
[11] M. Eslamian, G.Z. Eskandani, M. Raeisi, *Split common null point and common fixed point problems between Banach spaces and Hilbert spaces*, Mediterr. J. Math., **14**(3)(2017), 119.
[12] A. Gibali, S. Reich, R. Zalas, *Iterative methods for solving variational inequalities in Euclidean space*, J. Fixed Point Theory Appl., **17**(2015), 775-811.
[13] A. Gibali, S. Reich, R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optimization, **66**(2017), 417-437.
[14] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
[15] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive nonself-mappings and inverse-strongly-monotone mappings*, J. Convex Anal., **11**(2004), 69-79.
[16] D. Kinderlehrer, G. Stampaccia, *An Iteration to Variational Inequalities and their Applications*, Academic Press, New York, 1990.
[17] P.E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization*, Set-Valued Analysis, **16**(2008), 899-912.
[18] E. Masad, S. Reich, *A note on the multiple-set split convex feasibility problem in Hilbert space*, J. Nonlinear Convex Anal., **8**(2007), 367-371.
[19] A. Moudafi, *Alternating CQ-algorithm for convex feasibility and split fixed-point problems*, J. Nonlinear Convex Anal., **15**(2014), 809-818.
[20] P.M. Pardalos, Th.M. Rassias, A.A. Khan, *Nonlinear Analysis and Variational Problems*, Springer, Berlin, 2010.
[21] M. Raeisi, G.Z. Eskandani, M. Eslamian, *A general algorithm for multiple-sets split feasibility problem involving resolvents and Bregman mappings*, Optimization, **67**(2)(2018), 309-327.
[22] M. Raeisi, G.Z. Eskandani, *A hybrid extragradient method for a general split equality problem involving resolvents and pseudomonotone bifunctions in Banach spaces*, Calcolo 56, **43** (2019), https://doi.org/10.1007/s10092-019-0341-4.
[23] G.Z. Eskandani, M. Raeisi, *Solving a general split equality problem without prior knowledge of operator norms in Banach spaces*, Results Math., **76**(2021), article no. 4, https://doi.org/10.1007/s00025-020-01312-2.
[24] R.T. Rockafellar, R.J.-B. Wets, *Variational Analysis*, 2nd edition, Springer, New York, 2004.
[25] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118**(2003), 417-428.
[26] M. Tian, B.N. Jiang, *Weak convergence theorem for zero points of inverse strongly monotone mapping and fixed points of nonexpansive mapping in Hilbert space*, Optimization, **66**(2017), 1689-1698.
[27] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math Anal Appl., **298**(2004), 279-291.
[28] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.
[29] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, In: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, (D. Butnariu, Y. Censor, and S. Reich - Eds.) Amsterdam, Elsevier, 2001, 473-504.
[30] J. Zhao, *Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms*, Optimization, **64**(2015), 2619-2630.

Received: April 18, 2019; Accepted: March 27, 2021.