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A NOTE ON THE RATE OF CONVERGENCE OF VISCOSITY ITERATIONS

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Abstract. In [6], Moudafi introduced the so-called viscosity iterative method to approximate a fixed point of a nonexpansive mapping and proved the strong convergence of the generated sequence. Since then, several authors extended the convergence result in different settings and for mappings satisfying general metric conditions. Anyway, to the best of our knowledge and beside numerical simulations, little is known about the speed of convergence of the method itself. In this paper, we propose a step in this direction by giving an estimate for the rate of convergence of viscosity sequences generated by quasi-nonexpansive mappings in the setting of q-uniformly smooth Banach spaces.
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1. Introduction

Iterative methods for approximating fixed points have a long story and their study had been one of the most prolific field in fixed point theory in the last decades. For a comprehensive introduction on the topic, we refer to books [2] and [4]. Despite the intensive literature on the subject, very little had been written about the rate of convergence of the generated sequences. One of the first results in this direction had been given by Baillon ad Bruck in [1] (see also [3] for a simpler proof) and it can be expressed as follows.

Theorem 1.1. *Let X be a Banach space with a closed, convex and nonempty set C, let $\lambda \in (0,1)$ be a fixed parameter and let $T : C \to C$ be a nonexpansive map with nonempty fixed point set $Fix(T)$. Fix $x_0 \in C$ and let $\{x_n\}$ be defined by the Krasnoselskii iteration*

$$x_{n+1} = \lambda x_n + (1-\lambda)Tx_n. \tag{1.1}$$

Then the rate of asymptotic stability is given by

$$\|x_n - Tx_n\| \leq \frac{diam(C)}{\sqrt{\pi\lambda(1-\lambda)n}}.$$

In [1], the authors conjectured that a similar results holds for the Krasnoselskii-Mann (or segmenting Mann) iteration, generated by the inductive step

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n,$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence. The conjecture was then positively solved in 2014 by Cominetti, Soto and Vaisman (see [5]). Their proof exploits some properties of special functions together with an identity for Catalan numbers.

Focusing on other classical results, Halpern and viscosity iterations had been widely studied in literature for several classes of mappings and in different settings and their importance resides on the strong convergence to a fixed point of the produced sequence.

The aim of this article is to provide a convergence rate of the iterative viscosity approximation for quasi-nonexpansive mappings in a q-uniformly smooth Banach space X. Our result shows that the convergence rate depends not only on the smoothness of X (i.e., on q), but also on the coefficient of the contraction used in the viscosity approximation.

2. Preliminaries

2.1. **Uniformly smooth spaces.** We briefly recall some facts and notations regarding Banach spaces and, in particular, q-uniformly smooth spaces which will represent the natural setting for our results .

Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{X^*}$ be the normalized duality mapping given by

$$J(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\|^* = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality pairing. In the sequel, we shall denote single-valued duality mappings by j. Given $q > 1$, we shall use J_q to denote the generalized duality mapping given by

$$J_q(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^q \text{ and } \|f\|^* = \|x\|^{q-1}\}.$$

We recall the following relation

$$J_q(x) := \|x\|^{q-2} J(x).$$

The modulus of smoothness $\rho : [0, +\infty) \to [0, +\infty)$ is defined by

$$\rho(t) := \sup \left\{ \frac{\|x + y\| - \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t \right\}.$$

X is said to be uniformly smooth if $\lim \rho(t)/t = 0$ and q-uniformly smooth (with $q > 1$) if there exists a constant c with the property that $\rho(t) \leq ct^q$.

We cite Hilbert spaces, Sobolev spaces $W^{m,q}(\Omega)$, as well as Lebesgue spaces $L^q(\Omega)$ and l^q, with $q \in (1, \infty)$, as examples of q-uniformly smooth Banach spaces. Also every superreflexive Banach space admits an equivalent renorming for which it is q-uniformly smooth (see [7]). For a proof of previous facts and for more examples, we refer to [9].

The following proposition summarizes some useful properties of q-uniformly smooth spaces, we refer to [13] and the book [4] for a proof and discussions.

Proposition 2.1. *Let $q > 1$ and let $(X, \|\cdot\|)$ be q-uniformly smooth. Then*

(1) *X is a reflexive Banach space,*
(2) *The generalized duality mapping is single-valued and norm to norm uniformly continuous. Moreover $j_q(\lambda x) = \lambda^{q-1} j_q(x)$ for any $\lambda > 0$.*
(3) *There exists a constant $K_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + K_q\|y\|^q \tag{2.1}$$

(4) *If $q > 2$ then X is $2-$uniformly smooth.*

2.2. **Nonexpansive maps and iterations.** Let C be a nonempty, convex and closed subset of the Banach space X. Recall that a self-mapping $T : C \to C$ is said to be non-expansive if

$$\|Tx - Ty\| \le \|x - y\| \qquad \forall x, y \in C$$

and quasi non-expansive, whenever the above inequality holds when $y = Ty$ is a fixed point of T. We shall use $Fix(T)$ to denote the fixed point set of T. Throughout the paper we assume that $Fix(T)$ is not empty.

The viscosity approximation method of selecting a point in $Fix(T)$ was first introduced by Moudafi [6], who proved the following.

Theorem 2.2. *In a Hilbert space H, let $C \subset H$ be closed, convex and nonempty and let $T, f : C \to C$ be a nonexpansive mapping with a fixed point and a contraction respectively. Define $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n.$$

Suppose that $\{\alpha_n\}$ satisfies the conditions

(M1) $\lim_{n\to\infty} \alpha_n = 0$;
(M2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(M3) $\lim_{n\to\infty} |1/\alpha_n - 1/\alpha_{n-1}| = 0$.

Then $\{x_n\}$ converges strongly to the unique solution $\tilde{p} \in Fix(T)$ of the variational inequality

$$\langle (I - f)\tilde{p}, p - \tilde{p}\rangle \ge 0 \qquad \forall p \in Fix(T). \tag{2.2}$$

In other words, $\tilde{p}$ is the unique fixed point of the map $P_{Fix(T)}f$.

Later, Xu improved the above result by generalizing it to the setting of uniformly smooth Banach spaces:

Theorem 2.3. [16] *Let X be a uniformly smooth Banach space and $C \subset X$ be closed, convex and nonempty, let T, f and $\{x_n\}$ as in the previous theorem and suppose that $\{\alpha_n\}$ satisfies conditions $(M1)$,$(M2)$ and the following*

(M3*) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$ *or* $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$.

Then $\{x_n\}$ converges strongly to the unique solution $\tilde{p} \in Fix(T)$ of the variational inequality

$$\langle (I - f)\tilde{p}, j_q(p - \tilde{p})\rangle \ge 0 \qquad \forall p \in Fix(T).$$

Remark 2.4. We stress that conditions $(M1)$,$(M2)$ and $(M3^*)$ allow the natural choice $\alpha_n = \frac{1}{n+1}$, which is excluded by $(M1)$-$(M3)$. Further remarks on the necessity of conditions $(M1)$-$(M3)$ and $(M3^*)$ and their role in optimization can be found in [14], [15],[10] and [11]. See [12] also for a recent on finding zeros of nonexpansive, accretive mappings in Banach spaces.

The following lemma is used in the original proof of Theorem 2.3.

Lemma 2.5. [14] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \qquad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence in $\mathbb{R}$ such that

(1) $\lim_{n\to\infty} \gamma_n = 0$;

(2) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(3) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For the convergence rate we also need a lemma due to Chung which can be found in [8].

Lemma 2.6. (Chung, cf. [8, Lemma 4, p. 45]) *Assume a nonnegative sequence (u_n) satisfies the condition*

$$u_{n+1} \leq \left(1 - \frac{c}{n}\right) u_n + \frac{d}{n^{p+1}}, \quad n > 0, \tag{2.3}$$

where $d > 0$, $p > 0$, $c > 0$ are constants. Then

$$\begin{array}{ll} u_n \leq d(c-p)^{-1}n^{-p} + o\left(n^{-p}\right), & \text{if } c > p, \\ u_n = O\left(n^{-c}\log n\right), & \text{if } c = p, \\ u_n = O\left(n^{-c}\right), & \text{if } c < p. \end{array}$$

3. Main results

Theorem 3.1. *Let $q \in (1,2]$ and $\kappa \in [0,1)$ be given. Let $(X, \|\cdot\|)$ be a q-uniformly smooth Banach space, C a nonempty, closed and convex subset of X, $T : C \to C$ a quasi-nonexpansive map with $Fix(T) \neq \emptyset$, and $f : C \to C$ a κ-contraction. Suppose that there exists $\tilde{p} \in Fix(T)$ such that*

$$\langle (I-f)\tilde{p}, j_q(y-\tilde{p})\rangle \geq 0 \tag{3.1}$$

for $y \in T(C)$. Then the viscosity iteration with initial guess $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n = 1, 2, \cdots, \tag{3.2}$$

converges strongly to $\tilde{p}$, which also solves (2.2), whenever $\{\alpha_n\} \subset (0,1)$ satisfies the conditions $(M1)$ and $(M2)$ of Theorem 2.2. Moreover, if $\alpha_n = \frac{1}{n}$ for $n > 0$, we have the convergence rate:

$$\|x_n - \tilde{p}\| \leq \begin{cases} O\left(\frac{1}{n^{1-\frac{1}{q}}}\right) & \text{if } \kappa < \frac{1}{q}, \\ O\left(\frac{(\log n)^{1/q}}{n^{1-\frac{1}{q}}}\right) & \text{if } \kappa = \frac{1}{q}, \\ O\left(\frac{1}{n^{1-\kappa}}\right) & \text{if } \kappa > \frac{1}{q}. \end{cases} \tag{3.3}$$

Proof. By following a standard argument, it holds that

$$\|x_n - \tilde{p}\| \le \max\{\|x_1 - p\|, (1-\kappa)^{-1}\|f(\tilde{p}) - \tilde{p}\|\}.$$

Consequently, the sequences $\{x_n\}, \{Tx_n\}, \{f(x_n)\}$ are all bounded.
Applying (2.1) to (3.2) arrives at

$$\begin{aligned}
\|x_{n+1} - \tilde{p}\|^q &= \|(1-\alpha_n)(Tx_n - \tilde{p}) + \alpha_n(f(x_n) - \tilde{p})\|^q \\
&\le \|(1-\alpha_n)(Tx_n - \tilde{p})\|^q + q\alpha_n\langle f(x_n) - \tilde{p}, j_q\left((1-\alpha_n)(Tx_n - \tilde{p})\right)\rangle \\
&\quad + K_q\|\alpha_n(f(x_n) - \tilde{p})\|^q \\
&\le (1-\alpha_n)^q\|Tx_n - \tilde{p}\|^q - q\alpha_n(1-\alpha_n)^{q-1}\langle (I-f)\tilde{p}, j_q(Tx_n - \tilde{p})\rangle \\
&\quad + q\alpha_n(1-\alpha_n)^{q-1}\|f(x_n) - f(\tilde{p})\|\|Tx_n - \tilde{p}\|^{q-1} + \alpha_n^q K_q\|f(x_n) - \tilde{p}\|^q \\
&\le (1-\alpha_n)^q\|x_n - \tilde{p}\|^q + q\kappa\alpha_n(1-\alpha_n)^{q-1}\|x_n - \tilde{p}\|^q \\
&\quad + \alpha_n^q K_q \sup_{k\ge 1}\|f(x_k) - \tilde{p}\|^q.
\end{aligned}$$

Setting $u_n := \|x_n - \tilde{p}\|^q$ and $\tilde{K}_q := K_q \sup_{k\ge 1}\|f(x_k) - \tilde{p}\|^q$, we get

$$u_{n+1} \le (1-\alpha_n)^{q-1}(1-\alpha_n(1-\kappa q))u_n + \tilde{K}_q\alpha_n^q. \tag{3.4}$$

Since $(1-\alpha_n)^{q-1} \le 1-(q-1)\alpha_n$ and since

$$(1-(q-1)\alpha_n)(1-\alpha_n(1-\kappa q)) = 1 - q(1-\kappa)\alpha_n + (q-1)(1-\kappa q)\alpha_n^2,$$

(3.4) is reduced to

$$u_{n+1} \le (1-q(1-\kappa)\alpha_n)u_n + (q-1)(1-\kappa q)\alpha_n^2 u_n + \tilde{K}_q\alpha_n^q. \tag{3.5}$$

An easy application of Lemma 2.5 to (3.5) immediately implies that $u_n \to 0$, namely, $x_n \to \tilde{p}$ in norm.
Now if $\kappa < \frac{1}{q}$ (i.e., $\kappa q < 1$), then we infer from (3.5) that (noting $\alpha_n^2 \le \alpha_n^q$)

$$u_{n+1} \le (1-q(1-\kappa)\alpha_n)u_n + d_q\alpha_n^q. \tag{3.6}$$

where $d_q := (q-1)(1-\kappa q)\sup_{n\ge 1} u_n + \tilde{K}_q$. In particular, if $\alpha_n = \frac{1}{n}$, we may rewrite (3.6) as

$$u_{n+1} \le \left(1 - \frac{q(1-\kappa)}{n}\right)u_n + \frac{d_q}{n^q}. \tag{3.7}$$

Since $\kappa q < 1$, which is equivalent to $q(1-\kappa) > q-1$, we can apply Lemma 2.6 to the case where $c := q(1-\kappa)$, $p := q-1$ and $d := d_p$ to get

$$u_n \le d_q(1-\kappa q)n^{-(q-1)} + o(n^{-(q-1)}).$$

This proves the first case of (3.3).
Next assume $\kappa q \ge 1$; then (3.5) is reduced to

$$u_{n+1} \le (1-q(1-\kappa)\alpha_n)u_n + \tilde{K}_q\alpha_n^q. \tag{3.8}$$

In particular, if $\alpha_n = \frac{1}{n}$, we obtain from (3.8)

$$u_{n+1} \le \left(1 - \frac{q(1-\kappa)}{n}\right)u_n + \frac{\tilde{K}_q}{n^q}. \tag{3.9}$$

Further, if $\kappa q = 1$, then $q(1-\kappa) = q-1$. Applying Lemma 2.2 yields

$$u_n = O(\log n/n^{q-1}).$$

This proves the second case of (3.3).
Finally, if $\kappa q > 1$, then $q(1-\kappa) < q-1$, and we can again apply Lemma 2.2 to get

$$u_n = O(1/n^{q(1-\kappa)}).$$

This proves the third case of (3.3).

Corollary 3.2. *Let $X, C, \{\alpha_n\}$ and T be as in the previous theorem. Fix $u \in C$ and let $\{z_n\}$ be the Halpern iteration defined by $z_1 \in C$ and*

$$z_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n.$$

Suppose that the following inequality is satisfied for some $\tilde{p} \in Fix(T)$,

$$\langle \tilde{p} - u, j_q(y-\tilde{p})\rangle \geq 0 \qquad \forall y \in T(C).$$

Then $\{z_n\}$ converges strongly to $\tilde{p}$. Moreover, if $\alpha_n = \frac{1}{n}$, then we have the convergence rate:

$$\|z_n - \tilde{p}\| \leq \frac{[(q-1)\sup_{n\geq 1}\|z_n-\tilde{p}\|^q + K_q\|u-\tilde{p}\|^q]^{1/q}}{n^{(q-1)/q}} + o\left(\frac{1}{n^{(q-1)/q}}\right) \tag{3.10}$$

for $n \geq 1$.

Proof. We take the contraction f in Theorem 3.1 to be $f(x) \equiv u$ and thus $\kappa = 0$. It turns out that (3.7) is reduced to (recalling $\alpha_n = 1/n$)

$$u_{n+1} \leq \left(1 - \frac{q}{n}\right) u_n + \frac{d_q}{n^q}, \tag{3.11}$$

where

$$d_q = (q-1)\sup_{n\geq 1}\|z_n - \tilde{p}\|^q + K_q\|u-\tilde{p}\|^q.$$

Applying Lemma 2.2 (the case of $c > p$ with $c = q$ and $p = q-1$) to (3.11) immediately yields

$$u_n \leq \frac{d_p}{n^{q-1}} + o\left(\frac{1}{n^{q-1}}\right).$$

This is equivalent to (3.10).

Remark 3.3. We remark that condition 3.1 can be lowered by assuming existence of $\tilde{p} \in Fix(T)$ such that

$$\langle (I-f)\tilde{p}, j_q(x_n - \tilde{p})\rangle \geq 0, \qquad n \in \mathbb{N}.$$

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