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# $\Delta$-CONVERGENCE OF CONVEX COMBINATIONS OF TWO MAPS ON $p$-UNIFORMLY CONVEX METRIC SPACES

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**Abstract.** In this paper, we first study some properties of the convex combination metric and the convex combination of two maps on a $p$-uniformly convex metric space. Also, we study the $\Delta$-convergence of an iterative sequence for a convex combination of two maps on $p$-uniformly convex metric spaces.
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## 1. Introduction

The problem to finding a point in the intersection of two closed convex sets has been studied by many mathematicians, e.g, [29, 8, 2, 6, 15, 7, 18, 5, 1, 10, 12, 13] and references cited therein. Actually, it is called the convex feasibility problem which arises in many mathematical objects, e.g, optimization theory and image reconstruction problems etc., (see [6]). One of famous methods to solving convex feasibility problems in Hilbert spaces is the averaged projection method. More precisely, for two closed convex subsets $A, B$ of a Hilbert space $H$ with $A \cap B \neq \emptyset$, we can define the iterative sequence $\{x_n\}$ by

$$x_n := \frac{1}{2}\left(P_A x_{n-1} + P_B x_{n-1}\right), \quad x_0 \in H, \tag{1.1}$$

where $P_A$ and $P_B$ are corresponding metric projections for $A$ and $B$, respectively, which is called the averaged projection sequence. In [2] the author proved that the sequence $\{x_n\}$ in (1.1) weakly converges (in Hilbert space sense) to some point in $A \cap B$. In [7], the authors proved that there exists an averaged projection sequence which weakly converges, but does not converge in the norm-sense.

The convex feasibility problem has been extended to general geodesic metric spaces and general maps by many mathematicians. For example, CAT($\kappa$)-space with $\kappa \geq 0$ (see [3, 10]) and $p$-uniformly convex metric space (see [13, 11]). In metric spaces, we can not define the averaged projection sequence since there are no linear structures.

But, fortunately, in a geodesic metric space $X$, using the notion of geodesic, we can define the averaged projection sequence as follows:

$$x_n := P_A x_{n-1} \#_{1/2} P_B x_{n-1}, \tag{1.2}$$

where $x_0$ is a given starting point and $x \#_{1/2} y$ is the point $\gamma(1/2)$ on a geodesic $\gamma : [0,1] \to X$ joining $\gamma(0) = x$ and $\gamma(1) = y$.

The main purpose of this paper is to prove $\Delta$-convergence of an iterative sequence for convex combinations of two maps $T$ and $S$ on a $p$-uniformly convex metric space $X$ as follows:

$$Kx := (T \#_\lambda S)x := Tx \#_\lambda Sx$$

for $\lambda \in (0,1)$. Indeed, the averaged projection sequence (1.2) can be rewritten as

$$x_n = K^n x_0, \tag{1.3}$$

where $K = P_A \#_{1/2} P_B$. Thus the convergence of (1.2) can be proved by the convergence of the sequence $\{x_n\}$ in (1.3). For the proof of the $\Delta$-convergence, we first study some properties of the convex combination metric and convex combinations of two maps on a $p$-uniformly convex metric space. Using the properties we also prove that the sequence given as in (1.3) is asymptotically regular.

This paper is organized as follows. In Section 2, we recall the basic notion of a $p$-uniformly convex metric space and some properties, and we study several properties of the convex combination of two maps on a $p$-uniformly convex metric space for our study. In Section 3, we first recall the notion of $\Delta$-convergence and their properties, and then we study the $\Delta$-convergence of an iterative sequence for a convex combination of two maps on $p$-uniformly convex metric spaces.

## 2. The convex combination metric on $p$-uniformly convex metric spaces

Let $(X, d)$ be a metric space. A metric space $X$ is called a *geodesic metric space* if for any two point $x, y \in X$, there exists a geodesic $\gamma$ joining them. For $2 \leq p < \infty$, a geodesic metric space $(X, d)$ is said to be a *$p$-uniformly convex metric space with parameter $c_X > 0$* [21, 23, 24] if there exists a constant $c_X := c(p) \in (0,1]$ such that for any $z \in X$ and geodesic $\gamma : [0,1] \to X$,

$$d(z, \gamma(t))^p \leq (1-t) d(z, \gamma(0))^p + t d(z, \gamma(1))^p - c_X t(1-t) d(\gamma(0), \gamma(1))^p, \; t \in [0,1]. \tag{2.1}$$

Now, we give important examples of $p$-uniformly convex metric spaces.

**Example 2.1.** A Banach space $B$ with a norm $\|\cdot\|_B$ is called *uniformly convex* if $\delta_B(\epsilon) > 0$ for all $\epsilon \in (0,2]$, where

$$\delta_B(\epsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\|_B \,\middle|\, \|x\|_B = \|y\|_B = 1, \|x-y\|_B \geq \epsilon, x, y \in B \right\}$$

which is called the modulus of convexity of $B$. A Banach space $B$ is called *$p$-uniformly convex* with $p > 1$ if there exists a constant $c_B > 0$ such that $\delta_B(\epsilon) \geq c_B \cdot \epsilon^p$ for all $0 < \epsilon \leq 2$. In particular, $L^p$-spaces ($p \geq 2$) are $p$-uniformly convex Banach spaces, in fact, $\delta_{L^p}(\epsilon) \geq \epsilon^p/(p2^p)$ (see [14, 25]). If $B$ is a $p$-uniformly convex Banach space for $p \geq 2$, then $B$ is a $p$-uniformly convex metric space with parameter $4c/2^p$, where $c = c(c_B, p) > 0$ (see [21] for more details).

**Example 2.2.** Let $(X,d)$ be a Hadamard space (or complete CAT(0)-space). Then $(X,d)$ is a 2-uniformly convex metric space with parameter $c_X = 1$. Indeed, for any $x, y, z \in X$ and any geodesic $\gamma$ joining $x$ and $y$,

$$d(z,\gamma(t))^2 \leq (1-t)d(z,x)^2 + td(z,y)^2 - t(1-t)d(x,y)^2,$$

holds (see [28]). If $(X,d)$ is a complete $p$-uniformly convex metric space with $c_X = 1$, using Proposition 2.5 in [21], then we have $p = 2$. Thus, $(X,d)$ becomes a Hadamard space.

For a non-empty subset $F \subseteq X$, the *diameter* of $F$ is defined by

$$\operatorname{diam}(F) := \sup_{x,y\in F}\{d(x,y)\}.$$

**Example 2.3.** For $\kappa > 0$, let $(M,d)$ be a CAT($\kappa$) space with $\operatorname{diam}(\mathrm{M}) < \frac{\pi}{2\sqrt{\kappa}}$. Then for any $x, y, z \in M$ and any geodesic $\gamma$ joining $x$ and $y$, there exists a constant $c_M \in (0,1)$ such that

$$d(z,\gamma(t))^2 \leq (1-t)d(z,x)^2 + td(z,y)^2 - c_M t(1-t)d(x,y)^2,\ t \in [0,1] \tag{2.2}$$

holds. (see [21, 24]). Therefore, any CAT($\kappa$) space $M$ with $\operatorname{diam}(\mathrm{M}) < \frac{\pi}{2\sqrt{\kappa}}$ is a 2-uniformly convex metric space with parameter $c_M \in (0,1)$.

Let $(X,d)$ be a $p$-uniformly convex metric space with parameter $c_X \in (0,1]$ and $\lambda \in (0,1)$ be given. We now define the metric function $d_\lambda : X^2 \times X^2 \to \mathbb{R}_+$ by

$$d_\lambda\left((a,b),(x,y)\right) := \sqrt[p]{(1-\lambda)d(a,x)^p + \lambda d(b,y)^p}, \quad (a,b),(x,y) \in X \times X.$$

Note that $(X^2, d_\lambda)$ is also a $p$-uniformly convex metric space with same parameter $c_{X^2} \in (0,1)$. Since it is well known that $(X^2, d_\lambda)$ is a geodesic metric space if $(X,d)$ is a geodesic metric space (see [9, 27]), we only check that (2.1) for $d_\lambda$. But it is immediate using the definition of $d_\lambda$ and (2.1).

**Remark 2.4.** If $X$ is a complete CAT($\kappa$) space with $\kappa \geq 0$ (we assume that $X$ has a diameter $\pi/2\sqrt{\kappa}$ if $\kappa > 0$, and $\infty$ if $\kappa = 0$), then by using Exercise 1.9. (1c) in [9], $(X^2, d_\lambda)$ is also CAT($\kappa$) space.

A map $T : C \subseteq X \to X$ is said to satisfy property $(P1)$ ([1]) if $\operatorname{Fix}(T) \neq \emptyset$ and there exists $\beta > 0$ such that for $x \in C$ and $z \in \operatorname{Fix}(T)$

$$d(Tx,z)^p \leq d(x,z)^p - \beta d(Tx,x)^p.$$

**Example 2.5.** Let $(X,d)$ be a complete $p$-uniformly convex metric space. Every firmly nonexpansive map $T : X \to X$ satisfies (P1) with $\beta = c_X/2$ (see [1]).

**Proposition 2.6.** *Let $(X,d)$ be a $p$-uniformly convex metric space with parameter $c_X \in (0,1]$ and $\lambda \in (0,1)$ be given. Let $T, S : X \to X$ be two maps which satisfy the property $(P1)$. Define a map $V : X^2 \to X^2$ by for any $(x,y) \in X \times X$*

$$V(x,y) := (Tx, Sy).$$

*Then $V$ also satisfies the property $(P1)$ with respect to $d_\lambda$.*

*Proof.* Using the definition of $d_\lambda$ and the property $(P1)$ of $T$ and $S$, it is immediate. □

Now, we recall the notion of the metric projection operator on a $p$-uniformly convex metric space $X$. Let $F$ be a closed convex subset of $X$. Then for any $x \in X$, there exists a unique point $P_F x \in F$ such that

$$d(x, F) := \inf_{y \in F} d(x, y) = d(x, P_F x)$$

hold. (see [21]). We call $P_F : X \to F$ the metric projection operator onto $F$.

**Proposition 2.7.** *Let $\lambda \in (0, 1)$ be given and $(X, d)$ be a $p$-uniformly convex metric space with parameter $c_X \geq \lambda^{p-1} + (1-\lambda)^{p-1}$. Consider the closed convex subset of $X \times X$*

$$\Delta_X := \{(x, x) \mid x \in X\}.$$

*and define the map $P : X^2 \to X^2$ by for any $(x, y) \in X \times X$*

$$P(x, y) := (x\#_\lambda y, x\#_\lambda y),$$

*where $x\#_\lambda y$ is the point $\gamma(\lambda)$ of geodesic $\gamma$ connecting $x$ and $y$. Then $P$ is the metric projection operator to $\Delta_X \subseteq X \times X$, i.e. $P = P_{\Delta_X}$.*

*Proof.* The proof is a simple modification of the proof in [27]. To prove the result, we only show that for any $x, y, u \in X$

$$d_\lambda ((x, y), (x\#_\lambda y, x\#_\lambda y)) \leq d_\lambda ((x, y), (u, u)).$$

Indeed, by (2.1), we have

$$c_X (1-\lambda)\lambda d(x, y)^p \leq (1-\lambda) d(u, x)^p + \lambda d(u, y)^p. \tag{2.3}$$

We also obtain that by definition of $d_\lambda$,

$$\begin{aligned} d_\lambda ((x, y), (x\#_\lambda y, x\#_\lambda y))^p &= (1-\lambda) d(x, x\#_\lambda y)^p + \lambda d(y, x\#_\lambda y)^p \\ &= (1-\lambda)\lambda^p d(x, y)^p + \lambda(1-\lambda)^p d(y, x)^p \\ &= \lambda(1-\lambda)(\lambda^{p-1} + (1-\lambda)^{p-1}) d(x, y)^p. \end{aligned}$$

Thus by (2.3) and the assumption of $c_X$, we have

$$\begin{aligned} d_\lambda ((x, y), (x\#_\lambda y, x\#_\lambda y))^p &\leq (1-\lambda) d(u, x)^p + \lambda d(u, y)^p \\ &= d_\lambda ((x, y), (u, u))^p. \end{aligned}$$

The proof is complete. □

**Proposition 2.8.** *Let $(X, d)$ be a $p$-uniformly convex metric space with parameter $c_X \in (0, 1)$ and $\lambda \in (0, 1)$ be given. Let $T, S : X \to X$ be two maps. Define the map $K : X \to X$ by*

$$Kx = (T\#_\lambda S)x := Tx\#_\lambda Sx,$$

*$V : X^2 \to X^2$ and $P : X^2 \to X^2$ are given by in Proposition 2.6 and Proposition 2.7, respectively. Then we have the following properties:*

(i) $\mathrm{Fix}(P \circ V) = \{(z, z) \mid z \in \mathrm{Fix}(K)\}$;
(ii) *for any $x, y \in X$, $d_\lambda((x, x), (y, y)) = d(x, y)$;*
(iii) *for any $x \in X$ and any $n \in \mathbb{N}$, $(P \circ V)^n = (K^n x, K^n x)$.*

*Proof.* It is immediate, using the definitions of $d_\lambda$, $K$, $V$ and $P$. □

**Proposition 2.9.** *Let $\lambda \in (0,1)$ be given and $(X,d)$ be a $p$-uniformly convex metric space with parameter $c_X = \lambda^{p-1} + (1-\lambda)^{p-1} < 1$. Let $\epsilon \geq 0$ be given and $A, B$ be two subsets of $X$ and $z \in X$. Suppose that $a \in A$ and $b \in B$ are elements in $X$ such that for any $x \in X$, $y \in A$ and $w \in B$*

$$d_\lambda((z,z),(a,b))^p \leq d_\lambda((x,x),(y,w))^p + \frac{\epsilon^p}{2^p}\lambda(1-\lambda)c_X. \tag{2.4}$$

*Then for any $y \in A$ and $w \in B$, we have*

$$d(a,b) \leq d(y,w) + \epsilon.$$

*Proof.* We take $x := a\#_\lambda b$, $y := a$ and $w := b$ in the assumption (2.4), then we have

$$(1-\lambda)d(z,a)^p + \lambda d(z,b)^p \leq \lambda(1-\lambda)c_X d(a,b)^p + \frac{\epsilon^p}{2^p}\lambda(1-\lambda)c_X,$$

where $c_X = \lambda^{p-1} + (1-\lambda)^{p-1}$. By using (2.1), we obtain that

$$d(z, a\#_\lambda b)^p \leq \frac{\epsilon^p}{2^p}\lambda(1-\lambda)c_X. \tag{2.5}$$

On the other hand, we have

$$d_\lambda((z,z),(a,b)) \leq d_\lambda((y\#_\lambda w, y\#_\lambda w),(y,w)) + \frac{\epsilon}{2}\sqrt[p]{\lambda(1-\lambda)c_X}.$$

Thus we obtain that

$$\begin{aligned}
&d_\lambda((a\#_\lambda b, a\#_\lambda b),(a,b)) \leq d_\lambda((a\#_\lambda b, a\#_\lambda b),(z,z)) + d_\lambda((z,z)(a,b))\\
&\leq \frac{\epsilon}{2}\sqrt[p]{\lambda(1-\lambda)c_X} + d_\lambda((y\#_\lambda w, y\#_\lambda w),(y,w)) + \frac{\epsilon}{2}\sqrt[p]{\lambda(1-\lambda)c_X}\\
&= \frac{\epsilon}{2}\sqrt[p]{\lambda(1-\lambda)c_X} + \sqrt[p]{\lambda(1-\lambda)c_X}\,d(y,w) + \frac{\epsilon}{2}\sqrt[p]{\lambda(1-\lambda)c_X}.
\end{aligned}$$

Therefore, we have

$$\sqrt[p]{\lambda(1-\lambda)c_X}\,d(a,b) \leq \sqrt[p]{\lambda(1-\lambda)c_X}\,d(y,w) + \epsilon\sqrt[p]{\lambda(1-\lambda)c_X}.$$

The proof is complete. □

Let $A$ and $B$ be two subsets of $X$. Let $S_{A,B}$ be the set of all pairs $(x,y)$ such that

$$d(x,y) = d(A,B) := \inf_{(a,b)\in A\times B} d(a,b).$$

**Proposition 2.10.** *Let $\lambda \in (0,1)$ be given and $(X,d)$ be a $p$-uniformly convex metric space with parameter $c_X \geq \lambda^{p-1} + (1-\lambda)^{p-1}$. Let $A$ and $B$ be two subsets of $X$. Then*

$$S_{\Delta_X, A\times B} \supseteq \{((a\#_\lambda b, a\#_\lambda b),(a,b)) \mid (a,b) \in S_{A,B}\}. \tag{2.6}$$

*In particular, if $c_X = \lambda^{p-1} + (1-\lambda)^{p-1}$, then we have*

$$S_{\Delta_X, A\times B} = \{((a\#_\lambda b, a\#_\lambda b),(a,b)) \mid (a,b) \in S_{A,B}\}. \tag{2.7}$$

*Proof.* We first prove (2.6). For $(a,b) \in S_{A,B}$, put $z := a\#_\lambda b$. Then we only show that

$$d_\lambda((z,z),(a,b)) \leq d_\lambda((x,x),(y,w)), \quad x \in X, y \in A, w \in B$$

i.e.,

$$\lambda(1-\lambda)(\lambda^{p-1} + (1-\lambda)^{p-1})d(a,b)^p \leq (1-\lambda)d(x,y)^p + \lambda d(x,w)^p.$$

By using (2.1) for $\gamma(t) := y\#_\lambda w$, we have

$$c_X\lambda(1-\lambda)d(y,w)^p \leq (1-\lambda)d(x,y)^p + \lambda d(x,w)^p,$$

where $c_X \geq \lambda^{p-1} + (1-\lambda)^{p-1}$. Since $(a,b) \in S_{A,B}$, we have for any $y \in A$ and $w \in B$, $d(a,b) \leq d(y,w)$. It gives the proof of (2.6). By Proposition 2.9 with $\epsilon = 0$ in (2.5), the converse inclusion of (2.7) is immediate. □

## 3. Convergence results for convex combinations of two maps

Now, we recall the notion of a weak type convergence in general geodesic metric spaces for our study. A notion of weak type convergence of (geodesic) metric spaces was first introduced by T. Lim in [22]. It is called the $\Delta$-convergence. Many authors studied the $\Delta$-convergence of several sequences of maps in geodesic metric spaces, e.g., CAT($\kappa$) spaces with $\kappa \geq 0$ and $p$-unifomly convex metric spaces, etc, see [20, 16, 5, 17, 4, 3, 12, 13, 11, 10]. In Hilbert spaces, it is well known the $\Delta$-convergence coincides with the weak convergence of the Hilbert space sense.

Let $(X,d)$ be a geodesic metric space and $\{x_n\} \subseteq X$ be a bounded sequence. For a given point $x \in X$, put

$$r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) := \left\{ x \in X \,|\, r(x,\{x_n\}) = \inf_{x\in X} r(x,\{x_n\}) \right\}.$$

It is clear that $z \in A(\{x_n\})$ if and only if $\limsup_{n\to\infty} d(z,x_n) \leq \limsup_{n\to\infty} d(x,x_n)$ for any $x \in X$.

We now recall the notion of $\Delta$-convergence in $X$. A sequence $\{x_n\} \subseteq X$ is said to $\Delta$-*converge* (or weakly converge) to $x \in X$ if for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, the point $x$ is the unique asymptotic center of $\{x_{n_k}\}$, and in this case, $x$ is called the $\Delta$-*limit* of $\{x_n\}$. A point $x \in X$ is called a $\Delta$-*cluster point* of $\{x_n\} \subseteq X$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ $\Delta$-converges to $x$.

The following proposition is one of the important results to obtain the $\Delta$-convergence results in $p$-uniformly convex metric spaces.

**Proposition 3.1** ([1]). *Let $(X,d)$ be a complete $p$-uniformly convex metric space. Let $\{x_n\}$ be a bounded sequence in $X$. Then we have the following properties:*

(i) *$A(\{x_n\})$ has only one point.*

(ii) *$\{x_n\}$ has a $\Delta$-convergent subsequence i.e., $\{x_n\}$ has a $\Delta$-cluster point $x \in X$.*

We recall the notion of Fejér monotone sequences in (geodesic) metric spaces. Let $F$ be a non-empty subset of a metric space $(X,d)$ and let $\{x_n\}$ be a sequence in $X$. A sequence $\{x_n\}$ is said to be *Fejér monotone with respect to $F$* if

$$d(x_{n+1}, z) \leq d(x_n, z), \quad z \in F \quad n \in \mathbb{N}.$$

It is clear that if $\{x_n\}$ is Fejér monotone with respect to (w.r.t) $F$, then $\{x_n\}$ is a bounded sequence.

**Lemma 3.2.** *Let $(X, d)$ be a complete $p$-uniformly convex metric space with parameter $c_X \in (0, 1]$ and let $F$ be a nonempty subset of $X$. Suppose that a sequence $\{x_n\} \subseteq X$ is Fejér monotone w.r.t $F$, and any $\Delta$-cluster point of $\{x_n\}$ belongs to $F$, then $\{x_n\}$ $\Delta$-converges to a point in $F$.*

*Proof.* The proof is exactly same as the proof of [17, Lemma 3.2], using Proposition 3.1. □

Now, we recall the notion of $\Delta$-demiclosedness of functions. Let $(X, d)$ be a geodesic metric space. A function $T : X \to X$ is said to be $\Delta$-*demiclosed* if for any $\Delta$-convergent sequence $\{x_n\}$, its $\Delta$-limit belongs to $\mathrm{Fix}(T)$ whenever $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. It is clear that the identity function $I$ on $X$ is $\Delta$-demiclosed.

**Example 3.3.**

(i) Let $(X, d)$ be a complete $p$-uniformly convex metric space. Every firmly nonexpansive map $T : X \to X$, (that is,

$$d(Tx, Ty) \leq d(x\#_t Tx, y\#_t Ty), \quad x, y \in X, \quad t \in [0, 1)$$

is $\Delta$-demiclosed (see [13]).

(ii) Let $(M, d)$ be a complete $\mathrm{CAT}(\kappa)$ space with $\kappa \geq 0$ and $F$ be a non-empty closed convex subset of $M$. Then the metric projection map $P_F$ is $\Delta$-demiclosed (see [19]).

**Lemma 3.4** ([11])**.** *Let $(M, d)$ be a complete $\mathrm{CAT}(\kappa)$ space with $\kappa \geq 0$ ($\mathrm{diam(M)} < \frac{\pi}{2\sqrt{\kappa}}$ if $\kappa > 0$). Let $A_1$ and $A_2$ be two closed convex subsets of $M$ with $A_1 \cap A_2 \neq \emptyset$ and $P_{A_1}$ and $P_{A_2}$ be corresponding (metric) projections, respectively. Then $P_{A_1}\#_t P_{A_2}$ is also $\Delta$-demiclosed for all $t \in [0, 1]$.*

*Proof.* The proof can be found in [11]. But for the convenience and completeness, we give the proof. Put $P = P_{A_1}\#_t P_{A_2}$ for $t \in (0, 1)$. Let $\{x_n\}$ be a (bounded) sequence in $M$ and $z$ an element of $X$ such that $d(Px_n, x_n) \to 0$ as $n \to \infty$ and suppose that $\{x_n\}$ $\Delta$-converges to $z$. Note that since for any $q \in A_1 \cap A_2$,

$$0 \leq d(x_n, q) - d(Px_n, q) \leq d(Px_n, x_n)$$

we have

$$d(x_n, q) - d(Px_n, q) \to 0$$

as $n \to \infty$ which implies that

$$\lim_{n\to\infty} d(x_n, q)^2 - d(Px_n, q)^2 = 0.$$

Since for any $q \in A_1 \cap A_2$,

$$\begin{aligned} d(Px_n, q)^2 &\leq (1-t)d(P_{A_1}x_n, q)^2 + td(P_{A_2}x_n, q)^2 - c_M t(1-t)d(P_{A_1}x_n, P_{A_2}x_n)^2 \\ &\leq d(x_n, q)^2 - c_M t(1-t)d(P_{A_1}x_n, P_{A_2}x_n)^2, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(P_{A_1}x_n, P_{A_2}x_n) = 0.$$

Thus we have

$$d(P_{A_1}x_n, Px_n) = td(P_{A_1}x_n, P_{A_2}x_n) \to 0$$

as $n \to \infty$ which implies that $P_{A_1} z = z$ since $P_{A_1}$ is $\Delta$-demiclosed. By similar method, we have $P_{A_2} z = z$. Since

$$\begin{aligned}\limsup_{n\to\infty} d(Pz, x_n)^2 &\le \limsup_{n\to\infty} \left[(1-t)d(P_{A_1}z, x_n) + td(P_{A_1}z, x_n)^2\right] \\ &= \limsup_{n\to\infty} d(z, x_n)^2,\end{aligned}$$

by uniqueness of $\Delta$-limit, we conclude that $P(z) = z$. The proof is complete. □

For our study, we now introduce some assumption in $p$-uniformly convex metric spaces. Let $(X, d)$ be a complete $p$-uniformly convex metric space with parameter $c_X \in (0, 1]$. We consider the following condition:

**(A)** For any closed convex set $F$ of $(X, d)$, the metric projection map $P_F : X \to F$ satisfies the property $(P1)$.

**Remark 3.5.** For $\kappa \ge 0$, every complete CAT$(\kappa)$ space, Hilbert space and Banach space satisfies the property **(A)** (see, e.g, [1]). Therefore, every metric projection map on a complete CAT$(\kappa)$ space satisfies the property $(P1)$.

From now on, we always assume that every complete $p$-uniformly convex metric space satisfies the property **(A)** (i.e., every metric projection map on complete $p$-uniformly convex metric spaces satisfies the property $(P1)$).

Now, we recall the notion of asymptotic regularity of a sequence in metric spaces. A sequence $\{x_n\}$ in a metric space $(X, d)$ is called *asymptotic regular* if $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

**Theorem 3.6.** *Let $\lambda \in (0, 1)$ be given and $(X, d)$ be a complete $p$-uniformly convex metric space with parameter $c_X \ge \lambda^{p-1} + (1-\lambda)^{p-1}$. Let $T, S : X \to X$ be maps satisfying the property $(P1)$. Define the map $K : X \to X$ by*

$$Kx = (T \#_\lambda S)x := Tx \#_\lambda Sx.$$

*If $\mathrm{Fix}(K) \neq \emptyset$ then we have that $K$ is asymptotically regular. (i.e., $\{x_n := K^n x\}$ for $x \in X$ is asymptotic regular).*

*Proof.* Let $x \in X$ and define

$$x_n := K^n x, \quad n \in \mathbb{N}.$$

We only show that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Define $V : X^2 \to X^2$ and $P : X^2 \to X^2$ by in Proposition 2.6 and Proposition 2.7, respectively. Then by (i) in Proposition 2.8, we have $\mathrm{Fix}(P \circ V) \neq \emptyset$. On the other hands, we define

$$\hat{x}_n := (P \circ V)^n(x, x).$$

Since $P$ is a metric projection map to $\Delta_X \subseteq X \times X$ (see Proposition 2.7), by assumption, $P$ satisfies the condition **(A)** which implies that $P$ also satisfies the property $(P1)$. So by Corollary 3.1 in [1], we have that

$$\lim_{n\to\infty} d_\lambda(\hat{x}_n, \hat{x}_{n+1}) = 0.$$

Since, by (iii) in Proposition 2.8,

$$\hat{x}_n = (x_n, x_n),$$

by (ii) in Proposition 2.8, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

The proof is complete. □

**Theorem 3.7.** *Let $\lambda \in (0,1)$ be given and $(X,d)$ be a complete $p$-uniformly convex metric space with parameter $c_X \geq \lambda^{p-1} + (1-\lambda)^{p-1}$. Let $T, S : X \to X$ be two maps satisfying the property $(P1)$. Suppose that $T\#_\lambda S$ is $\Delta$-demiclosed and $\mathrm{Fix}(T\#_\lambda S) = \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$. Put $x_n := (T\#_\lambda S)^n x_0$, for an initial point $x_0 \in X$. Then $\{x_n\}$ $\Delta$-converges to a point in $\mathrm{Fix}(T\#_\lambda S)$.*

*Proof.* Let $x_0 \in X$ be given. Define the sequence $\{x_n\}$ by

$$x_n := K^n x_0 = (T\#_\lambda S)^n x_0$$

for all $n \in \mathbb{N}$. Since, $T$ and $S$ satisfy the property $(P1)$, $T$ and $S$ are quasi-nonexpansive maps i.e.,

$$\begin{aligned} d(Tx, z) &\leq d(x, z), \;\; z \; \in \mathrm{Fix}(T) \\ d(Sx, z) &\leq d(x, w), \;\; w \; \in \mathrm{Fix}(S). \end{aligned}$$

So, for any $z \in \mathrm{Fix}(T\#_\lambda S) = \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$ we have

$$\begin{aligned} d((T\#_\lambda S)x, z)^p &\leq (1-\lambda) d(Tx, z)^p + \lambda d(Sx, z)^p \\ &\leq d(x, z)^p. \end{aligned}$$

Therefore, we have

$$d(x_{n+1}, z) = d((T\#_\lambda S)((T\#_\lambda S)^n x_0), z) \leq d((T\#_\lambda S)^n x_0) = d(x_n, z).$$

Thus $\{x_n\}$ is is Fejér monotone w.r.t $\mathrm{Fix}(T\#_\lambda S)$. Also, by Theorem 3.6, we have

$$\lim_{n\to\infty} d(x_n, Kx_n) = 0. \tag{3.1}$$

Since $\{x_n\}$ is bounded, by Proposition 3.1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ $\Delta$-converges to a point $z \in M$. Since $K$ is a $\Delta$-demiclosed (by assumption), by combining (3.1), we have that $z \in \mathrm{Fix}(T\#_\lambda S)$. Therefore, by Lemma 3.2, we conclude that $\{x_n\}$ $\Delta$-converges to some point $z \in \mathrm{Fix}(T\#_\lambda S)$ as $n \longrightarrow \infty$. □

Note that in the case of a complete 2-uniformly convex metric spaces $X$, it is clear that for any $\lambda \in (0,1)$, $1 = c_X = \lambda^{2-1} + (1-\lambda)^{2-1}$ (i.e., $X$ is a CAT(0) space) and that $\mathrm{Fix}(P_A\#_t P_B) = \mathrm{Fix}(P_A) \cap \mathrm{Fix}(P_B) = A \cap B$ for all $t \in (0,1)$ (see [11]). The following result is clear by Remark 3.5, Lemma 3.4 and the above note.

**Corollary 3.8.** *Let $\lambda \in (0,1)$ be given. Let $(X,d)$ be a complete* CAT(0) *space (or complete 2-uniformly convex metric space with parameter $c_X = 1$). Let $A$ and $B$ be two closed convex subsets of $X$ with $A \cap B \neq \emptyset$ and $P_A$ and $P_B$ be corresponding (metric) projection maps, respectively. Put $x_n := (P_A\#_\lambda P_B)^n x_0$ for an initial point $x_0 \in X$. Then $\{x_n\}$ $\Delta$-converges to a point in $A \cap B$.*

**Remark 3.9.** In CAT(0) spaces, every firmly nonexpansive map having fixed points is strongly nonexpansive (see [26, Lemma 3.2]) and $T_1 \#_t T_2$ for $t \in (0,1)$ also strongly nonexpansive for two firmly nonexpansive map $T_1, T_2$(see [26, Lemma 3.5]) and so $T_1 \#_t T_2$ is $\Delta$-demiclosed by nonexpansiveness of $T_1 \#_t T_2$ (see [13]).

The following result is clear by Example 2.5, Remark 3.9 and [26, Lemma 3.5].

**Corollary 3.10.** *Let $\lambda \in (0,1)$ be given. Let $(X,d)$ be a complete* CAT(0) *space. Let $T_1$ and $T_2$ be two firmly nonexpansive maps on $X$. Put $x_n := (T_1 \#_\lambda T_2)^n x_0$ for an initial point $x_0 \in X$.*

*Then $\{x_n\}$ $\Delta$-converges to a point in* $\mathrm{Fix}(T_1 \#_\lambda T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_1)$.

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