Fixed Point Theory, 23(2022), No. 1, 179-198
DOI: 10.24193/fpt-ro.2022.1.11
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A STOCHASTIC PRODUCTION PLANNING PROBLEM

E.C. CANEPA*, D.-P. COVEI** AND T.A. PIRVU***

*University Politehnica of Bucharest,
Department of Mathematical Methods and Models,
313 Splaiul Independenţei, Bucharest 060042, Romania
E-mail: cristinacanepa@yahoo.com

**The Bucharest University of Economic Studies,
Department of Applied Mathematics,
Piaţa Romană, 1st district, Postal Code: 010374, Postal Office: 22, Romania
E-mail: dragos.covei@csie.ase.ro

***McMaster University, Department of Mathematics and Statistics
1280 Main Street West, Hamilton, ON, L8S 4K1, Canada
E-mail: tpirvu@math.mcmaster.ca

Abstract. Stochastic production planning problems were studied in several works; the model with one production good was discussed in [3]. The extension to several economic goods is not a trivial issue as one can see from the recent works [8], [9] and [13]. The following qualitative aspects of the problem are analyzed in [9]: the existence of a solution and its characterization through dynamic programming/Hamilton Jacobi Bellman (HJB) equation, as well as the verification (i.e., the solution of the HJB equation yields the optimal production of the goods). In this paper, we stylize the model of [8] and [9] in order to provide some quantitative answers to the problem. This is possible especially because we manage to solve the HJB equation in closed form. We point to a fixed point characterization of the optimal production rates. Among other results, we find that the optimal production rates adjusted for demand are the same across all the goods and they also turn to be independent of some model parameters. Moreover we show that production rates (adjusted for demand) are increasing in the aggregate number of goods produced, and they are also uniformly bounded. Numerical experiments show some patterns of the output.
Key Words and Phrases: Stochastic production problem, stochastic control, fixed point.
2020 Mathematics Subject Classification: 35J57, 35J47, 47H10, 60J70.

1. Introduction

Production planning problems were studied for quite some time. [21] considered a stochastic production-inventory model to determine optimal production rates, i.e., the ones which minimize a discounted quadratic loss function. Their solution has three terms: the initial inventory, a steady state of the solution and a correction term which kicks in when time approaches maturity. This work was extended from a deterministic to a stochastic framework by [3] and [19] which added randomness to the dynamics of the inventory process. In a follow up work [22] added constraints on the production inventory. The work of [12] looks at the infinite horizon stochastic production planning problem in which a continuous-time Markov chain models the demand. A more recent work on this topic is [5]. They look at optimal production-inventory problem when there is a stochastic demand which depends on the business cycle.

The aforementioned papers consider in general the production planning problem with one economic good only. The extension to several economic goods makes the problem more mathematically involved as one can see from the recent works of [8] and [9]. Moreover, [9] characterized the solution through dynamic programming/HJB equation; using regularity and estimate results from the area of partial differential equations a classical solution of the HJB was established, and the verification result was proved. Since these works deal with the infinite horizon, a transversality condition was imposed on the value function, and it was shown that the value function verifies it. The paper [13] is within the paradigm of multiple goods' production. Because of the complexity of HJB equations, the goal is not to solve the HJB equations, but to offer an approximate solution.

Still in the realm of production-inventory management is [4]. The authors look at finding the optimal inventory given a mean-reverting inventor.

In this paper we specialized the model of [8] and [9] to make it more tractable and to obtain quantitative results. Our main contribution is that we solved in closed form the HJB equation and the optimal production rate. The solution displays a mean field structure; the optimal production rate of some good is a function of the number of that specific produced good and an average of all the goods produced (this average is expressed by a norm of the vector of goods produced). By exploiting the structure of our closed form solution we can see that the optimal production rates adjusted for demand are the same across all goods and they do not depend on some model parameters. Moreover, the optimal production rates are zeros when there are no goods produced, and they are of order $O(\frac{1}{N})$ (N here stands for the number of goods). We show that production rates are increasing in the aggregate number of goods produced, and they are also uniformly bounded. Numerical experiments reveal that the production rate is a decreasing function of the number of goods' type N and, the variance of the number of goods produced.

Finally, the HJB equation characterizing the optimal production rates appears in other practical applications as we mention in the last section of the paper.

A new class of optimal decisions has arisen in financial and economic problems. They are called time consistent decisions and are a substitute of the optimal decisions when the later are time inconsistent; for more on this see [11], [10] and [16]. The time consistent decisions are the fixed points of certain functionals. In our setting, since the time discounting is exponential, the optimal production rates coincide with the time consistent ones. This allows for a fixed point characterization of the optimal production rates.

Now we are ready to present the organization of this paper. Section 2 describes the model. Section 3 provides the methodology. Section 4 presents other practical applications of the mathematics developed.

2. The model

Consider a factory producing N types of economic goods which stores them in an inventory designated place.

Next, we describe the model mathematically. There exists a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq\infty}, P)$, on which lives a N-dimensional Brownian motion denoted by $w = (w_1, ..., w_N)$. The filtration $\{\mathcal{F}_t\}_{0\leq t\leq\infty}$, is the natural filtration of the Brownian motion.

Let

$$p\left(t\right) = \left(p_1(t), ..., p_N(t)\right),$$

represent the production rate at time t (control variable). Next, let us introduce the control variables. Let the threshold

$$p^0 = \left(p_1^0, ..., p_N^0\right),$$

be a vector standing for the factory optimal production level. This level can be optimal from a technological standpoint, but its implementation may not be optimal because of inventory costs.

Next, let

$$l = \left(l_1, ..., l_N\right),$$

be the factory-optimal inventory level which can be attained but not maintained since there is noise in the system. In order to simplify the notations we assume that

$$p^0 = l = (0, ..., 0)\,.$$

This simplification is obtained by considering deviations from the factory-optimal inventory level and the factory-optimal production level. The deviations may be negative.

Next, let us describe the inventories. There exists a constant demand rate for every economic good, demand rate represented by the vector

$$\xi = \left(\xi_1, ..., \xi_N\right).$$

Again, to simplify the notations we take $\xi = (0, ..., 0)$ meaning that we consider deviations from the constant demand rate. In other words we consider demand adjusted production rates.

Let y_i^0 denote the initial inventory level of good i, and $y_i(t)$ the inventory level of good i, at time t. These inventory levels are modelled by the following system of stochastic differential equations

$$dy_i\left(t\right) = (p_i - \xi_i)dt + \sigma dw_i,\ y_i\left(0\right) = y_i^0,\ i = 1, ..., N, \tag{2.1}$$

where σ is a constant (non-zero) diffusion coefficient. Let us recall that the stochasticity here is due to inventory spoilages which are random in nature. Another cause of stochasticity may be a random demand rate which is often the case in several practical applications.

Let τ be the stopping time representing the moment when the inventory level reaches some threshold R, i.e.,

$$\tau = \inf_{t>0}\{|y(t)| \geq R\}.$$

Here, $|\cdot|$ stands for the Euclidian norm, and this way of limiting the inventory level is imposed for tractability. The factory may consider stopping the production when the inventory level R is attained and/or exceeded.

2.1. **The Objective.** The performance over time of a production

$$p\left(t\right)=\left(p_1(t),...,p_N(t)\right),$$

is measured by means of its cost. At this point we introduce the cost functional which yields the cost:

$$J\left(p_1,...,p_N\right):=\ E\int_0^{\tau}(|p(t)|^2+|y(t)|^2)dt, \tag{2.2}$$

which measures the quadratic loss. Again let us recall that we measure deviations from an optimal state, whence the loss. At this point we are ready to frame our objective, which is to minimize the cost functional. i.e.,

$$\inf_{p\in\mathbb{R}^N}\{J\left(p_1,...,p_N\right)\}, \tag{2.3}$$

subject to the Itô equation (2.1).

3. The methodology

Having presented the problem we want to solve, now we provide our means to tackle it. Our approach is based on the value function and dynamic programming which leads to the HJB equation. Let z denote the value function, i.e.,

$$z(y_1^0,y_2^0,\dots y_N^0)=\inf_{p\in\mathbb{R}^N}\{J\left(p_1,...,p_N\right)\},$$

subject to the Itô equation (2.1). We apply probabilistic techniques to characterize the value function; that is we search for a function $U\left(x\right)$ such that the stochastic process $M^p(t)$ defined below

$$M^p\left(t\right)=U\left(y\left(t\right)\right)-\int_0^t\left[|p(s)|^2+|y(s)|^2\right]ds,$$

is supermartingale for all

$$p\left(t\right)=\left(p_1(t),...,p_N(t)\right),$$

and martingale for the optimal control

$$p^*\left(t\right)=\left(p_1^*(t),...,p_N^*(t)\right).$$

Once such a function is found it turns out that $-U=z$.

We search for U a $C^2\left[0,R\right]$ function and the supermartingale/martingale requirement yields by means of Itô's Lemma the Hamilton-Jacobi-Bellman (HJB) equation which characterizes the value function

$$-\frac{\sigma^2}{2}\Delta z-|x|^2=\inf_{p\in\mathbb{R}^N}\{p\nabla z+|p|^2\}, \tag{3.1}$$

where $x\in\mathbb{R}^N$ assumes values $\left(y_1\left(0\right),...,y_N\left(0\right)\right)$. This HJB can be turned into a partial differential equation (PDE) since a simple calculation yields

$$\inf_{p\in\mathbb{R}^N}\{p\nabla z+|p|^2\}=-\frac{1}{4}\left|\nabla z\right|^2. \tag{3.2}$$

Thus, the HJB equation becomes the PDE

$$-\frac{\sigma^2}{2}\Delta z - |x|^2 = -\frac{1}{4}|\nabla z|^2 \text{ for } x \in \mathbb{R}^N,\ |x| \le R,$$

or, equivalently

$$2\sigma^2\Delta z + 4|x|^2 = |\nabla z|^2 \text{ for } x \in \mathbb{R}^N,\ |x| \le R. \tag{3.3}$$

The change of variable $z = -v$, yields the PDE

$$\Delta v = \frac{4|x|^2 - |\nabla v|^2}{2\sigma^2} \text{ for } x \in \mathbb{R}^N,\ |x| \le R. \tag{3.4}$$

The gradient term in the above PDE can be removed by the change of variable $u(x) = e^{\frac{v(x)}{2\sigma^2}}$, to get a simpler PDE

$$\begin{cases} \Delta u(x) = \frac{1}{\sigma^4}|x|^2 u(x) \text{ for } x \in \mathbb{R}^N,\ |x| \le R, \\ u(x) > 0 \text{ for } x \in \mathbb{R}^N,\ |x| \le R. \end{cases} \tag{3.5}$$

The partial derivatives of the value function will give us in turn the candidate optimal control. The first order optimality conditions on the lefthand side of (3.2) are sufficient for optimality since we deal with a quadratic (convex) function and they produce the candidate optimal control as follows:

$$p_i^* = \overline{p}_i(y_1(t), \ldots, y_N(t)),\ i = 1, ..., N,$$

and

$$\overline{p}_i(x_1, ..., x_N) = \frac{1}{2}\frac{\partial v}{\partial x_i}(x_1, ..., x_N),\ \text{for } i = 1, ..., n. \tag{3.6}$$

Remark 3.1. As we already mentioned in the introduction, the optimal production rates equal the time consistent ones, so they are time consistent. Following [10] and [11] the optimal production rates, being time consistent, are the fixed points of the following problem

$$p_i^* = \overline{p}_i(y_1(t), \ldots, y_N(t)), \qquad \overline{p}_i(x_1, ..., x_N) = \frac{1}{2}\frac{\partial v}{\partial x_i}(x_1, ..., x_N),\ \text{for } i = 1, ..., n, \tag{3.7}$$

and

$$v(x_1, ..., x_N) = -J(p_1^*, ..., p_N^*). \tag{3.8}$$

The fixed point nature of the optimal production rates is due to the fact that in the above equation, the value function v is also present in the left hand side functional through $p_i^*, i = 1, ..., n$, of (3.7).

3.1. **The Equation of Value Function .** Let $B_R(0)$ be the ball in $\mathbb{R}^N$ centered at the origin and radius $R > 0$. The equation of the value function according to (3.5) is

$$\Delta u(x) = \frac{1}{\sigma^4}|x|^2 u(x) \text{ in } B_R(0). \tag{3.9}$$

The initial condition is taken to be

$$u(0) = \alpha, \tag{3.10}$$

where α is a positive constant. The following result concerns the equation of value function.

Theorem 3.1. *Given the positive constant α, there exists a unique positive radially symmetric solution $u_\alpha \in C^2[0,R]$, to the problem (3.9) subject to the initial condition (3.10). Moreover, the solution is convex and increasing, and the following holds true*

$$u'_\alpha(0) = 0, \tag{3.11}$$

$$u_\alpha(r) = \alpha\left(1+\sum_{j=1}^{\infty}\frac{1}{j!\,(N+2)\,(N+6)\,...(N+4j-2)}\left(\frac{r^2}{2\sigma^2}\right)^{2j}\right), \tag{3.12}$$

$$u'_\alpha(r) = \alpha\sum_{j=1}^{\infty}\frac{4jr}{2\sigma^2 j!\,(N+2)\,(N+6)\,...(N+4j-2)}\left(\frac{r^2}{2\sigma^2}\right)^{2j-1}, \tag{3.13}$$

for all $r := |x| \in [0,R]$. In addition,

$$u_\alpha(r) \leq \alpha e^{\frac{r^4}{4\sigma^4(N+2)}},\ r\in[0,R], \tag{3.14}$$

$$(u_\alpha)'(r) \leq \frac{\alpha r^3}{\sigma^4(N+2)}e^{\frac{r^4}{4\sigma^4(N+2)}},\ r\in[0,R], \tag{3.15}$$

hold.

Proof. We consider the radial form of the problem (3.9) subject to the initial condition (3.10), i.e.,

$$\begin{cases} u''_\alpha(r)+\frac{N-1}{r}u'_\alpha(r)=\frac{1}{\sigma^4}r^2u_\alpha(r) \text{ in } (0,R], \\ u_\alpha(0)=\alpha. \end{cases} \tag{3.16}$$

This is equivalent to an integral equation of Volterra type of the following form

$$u_\alpha(r)=\frac{1}{\sigma^4}\int_0^r K(t,u_\alpha(t))\,dt,\ r\in[0,R]$$

where

$$K(t,u_\alpha(t)) = t^{1-N}\int_0^t s^{N+1}u_\alpha(s)\,ds.$$

It is easy to see that K is continuous and Lipschitz in the second variable (by Bielecki norm technique). Thus, by the existence and uniqueness theorem for such integral equations (via Banach's Contraction Principle) the integral equation (and equivalently, the initial value problem (3.9)-(3.10) has a unique solution.

Next, we need to show that the solution $u_\alpha(r)$ of (3.16) can be obtained succesively in the following way

$$\begin{cases} u^0_\alpha(r)=u_\alpha(0)=\alpha, \\ u^k_\alpha(r)=\alpha+\int_0^r t^{1-N}\int_0^t s^{N+1}\frac{1}{\sigma^4}u^{k-1}_\alpha(s)\,dsdt, \quad 0<r\leq R,\ k\in\mathbb{N}^*. \end{cases} \tag{3.17}$$

It is easy to see that $\{u^k_\alpha(r)\}_{k\geq 0}$ is a nondecreasing sequence of functions satisfying

$$u^{k+1}_\alpha(r)-u^k_\alpha(r) \leq \frac{\alpha}{(k+1)!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^{k+1} \tag{3.18}$$

$$\leq \frac{\alpha}{(k+1)!}\left(\frac{R^4}{4\sigma^4(N+2)}\right)^{k+1} \overset{k\to\infty}{\to} 0, \tag{3.19}$$

for all $r\in[0,R]$. Then $\{u^k_\alpha(r)\}_{k\geq 0}$ is a Cauchy sequence of functions on $[0,R]$.

It is a straightforward argument to prove that $u_\alpha^k \in C^2[0,R]$, $k \in \mathbb{N}$. Since a Cauchy sequence of functions is convergent, it has a limit function $u_\alpha(r)$ and the convergence is uniform. Moreover, since an uniformly Cauchy sequence of continuous functions has a continuous limit, then $u_\alpha(r)$ is a continuous function on $[0,R]$.

By passing to the limit in (3.17) we obtain that $u_\alpha(r)$ verifies the integral form of the problem (3.9) subject to the initial condition (3.10)

$$u_\alpha(r) = \alpha + \int_0^r t^{1-N} \int_0^t s^{N+1} \frac{1}{\sigma^4} u_\alpha(s)\, ds dt, \text{ in } [0,R]. \tag{3.20}$$

Hence, the limit function $u_\alpha(r)$ is the solution of (3.9) subject to the initial condition (3.10).

Next, we examine the sequence $\{(u_\alpha^k(r))'\}_{k\geq 0}$. We note first that

$$0 \leq \left(u_\alpha^k\right)'(r) = r^{1-N} \int_0^r t^{N+1} u_\alpha^{k-1}(t)\, dt. \tag{3.21}$$

Thus, the function $r \to u_\alpha^k(r)$ is nondecreasing for all $k \in \mathbb{N}$. Using (3.18) and (3.19) we get

$$\begin{aligned} \left|\left(u_\alpha^{k+1}\right)'(r) - \left(u_\alpha^k\right)'(r)\right| &\leq \frac{r^3\alpha}{\sigma^4(N+2)k!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^k \\ &\leq \frac{R^3\alpha}{\sigma^4(N+2)k!}\left(\frac{R^4}{4\sigma^4(N+2)}\right)^k \overset{k\to\infty}{\to} 0. \end{aligned} \tag{3.22}$$

Consequently, $(u_\alpha^k(r))' \overset{k\to\infty}{\to} (u_\alpha(r))'$ uniformly in $[0,R]$, which implies that $(u_\alpha(r))'$ is a continuous function on $[0,R]$.

A direct computation shows that $u_\alpha \in C^2[0,R]$. Next, let us prove (3.14). To do this we use (3.18) succesively

$$\begin{aligned} u_\alpha^{k+1}(r) &\leq \frac{\alpha}{(k+1)!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^{k+1} + u_\alpha^k(r) \\ &\leq \frac{\alpha}{(k+1)!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^{k+1} + \frac{\alpha}{k!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^k + u_\alpha^{k-1}(r) \\ &\cdots \\ &\leq \sum_{j=0}^{k+1} \frac{\alpha}{j!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^j. \end{aligned} \tag{3.23}$$

On the other hand, we note that

$$u_\alpha(r) = \lim_{k\to\infty} u_\alpha^{k+1}(r) \leq \sum_{j=0}^{\infty} \frac{\alpha}{j!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^j = \alpha e^{\frac{r^4}{4\sigma^4(N+2)}}, \tag{3.24}$$

for all $r \in [0,R]$.

Next, let us prove (3.15). We observe that $\{(u_\alpha^k)'(r)\}_{k\geq 0}$ is a nondecreasing sequence of continuous functions. Following the proof in (3.23), and using (3.22) successively it can be shown the inequality

$$\begin{aligned} \left(u_\alpha^{k+1}\right)'(r) &\leq \frac{\alpha r^3}{\sigma^4(N+2)k!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^k + \left(u_\alpha^k\right)'(r) \\ &\cdots \\ &\leq \frac{\alpha r^3}{\sigma^4(N+2)}\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^j. \end{aligned}$$

Repeating the arguments of (3.24) we notice that

$$\begin{aligned} \left(u_\alpha\right)'(r) &= \lim_{k\to\infty}\left(u_\alpha^{k+1}\right)'(r) \\ &\leq \frac{\alpha r^3}{\sigma^4(N+2)}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{r^4}{4\sigma^4(N+2)}\right)^j \\ &= \frac{\alpha r^3}{\sigma^4(N+2)}e^{\frac{r^4}{4\sigma^4(N+2)}}, \end{aligned}$$

for all $r \in [0, R]$.

Next, let us prove (3.12). To do this, we observe that

$$\begin{aligned} u_\alpha^1(r) &= \alpha + \int_0^r t^{1-N}\int_0^t s^{N+1}\frac{1}{\sigma^4}u_\alpha^0(r)\,dsdt \\ &= \alpha + \int_0^r t^{1-N}\int_0^t s^{N+1}\frac{1}{\sigma^4}\alpha dsdt \\ &= \alpha\left(1 + \frac{1}{\sigma^4}\int_0^r \frac{t^3}{N+2}dt\right) \\ &= \alpha\left(1 + \frac{1}{4\sigma^4}\frac{r^4}{N+2}\right). \end{aligned}$$

Substituting $u_\alpha^1(r)$ into

$$u_\alpha^2(r) = \alpha + \int_0^r t^{1-N}\int_0^t s^{N+1}\frac{1}{\sigma^4}u_\alpha^1(r)\,dsdt,$$

we obtain

$$u_\alpha^2(r) = \alpha\left(1 + \frac{1}{4\sigma^4}\frac{r^4}{(N+2)} + \frac{r^8}{\sigma^8\cdot 4\cdot 8\cdot (N+2)(N+6)}\right).$$

Continuing this process we get

$$\begin{aligned} u_\alpha^k(r) &= \alpha + \frac{1}{\sigma^4}\int_0^r t^{1-N}\int_0^t s^{N+1}u_\alpha^{k-1}(s)\,dsdt \\ &= \alpha\left(1 + \sum_{j=1}^{k}\frac{1}{j!(N+2)(N+6)...(N+4j-2)}\left(\frac{r^2}{2\sigma^2}\right)^{2j}\right). \end{aligned}$$

Since the sequence of functions $\{u_\alpha^k\}_{k\geq 0}$ is uniform convergent to the limit function $u_\alpha(r)$ then (3.12) is proved.

The power series representation of function $u_\alpha(r)$ can be differentiated to obtain a power series representation of its derivative $u'_\alpha(r)$. Thus, we obtain that $u_\alpha(r)$ is differentiable on $[0,R]$ and (3.13) holds true. In addition, the term-by-term derivative of a power series has the same interval of convergence as the original power series.

Next, (3.13) leads to $u'_\alpha(0)=0$, whence (3.11) is proved. A direct computation shows that

$$u_\alpha \in C^2([0,R]).$$

The convexity of the solution is proved in [8]. The monotonicity of the solution is now obvious. This completes the proof of the Theorem.

3.2. **Verification.** In this subsection we show that the control of (3.26) is indeed optimal. In a first step let us show that $M^p(t)$

$$M^p(t) = U(y(t)) - \int_0^t \left(|p(s)|^2 + |y(s)|^2\right) ds,$$

is supermartingale for all

$$p(t) = (p_1(t), ..., p_N(t)),$$

and martingale for the optimal control

$$p^*(t) = (p_1^*(t), ..., p_N^*(t)).$$

Indeed, Itô Lemma yields for the optimal control candidate

$$dM^p(t) = (\frac{\sigma^2}{2}\Delta U(y(s)) - |y(s)|^2 + p(s)\nabla U(s) - |p(s)|^2)ds + \sigma p(s)\nabla z(y(s))dw(s).$$

Then, the claim yields in light of HJB equation (3.1). In a second step let us establish the optimality of $(p_1^*, ..., p_N^*)$.

The martingale/supermartingale principle yields

$$EU(y^*(\tau*)) - E\int_0^{\tau*} (|p^*(u)|^2 + |y^*(u)|^2)du = U(y^*(0)) = U(y(0)),$$

and

$$EU(y(\tau)) - E\int_0^{\tau} (|p(u)|^2 + |y(u)|^2)du \le U(y(0)),$$

where $\tau* = \inf_{t>0}\{|y^*(t))| \ge R\}$ and $\tau = \inf_{t>0}\{|y(t)| \ge R\}$.

Moreover, $EU(y^*(\tau*)) = EU(y(\tau)) = 2\sigma^2 \ln u(R)$, and this finishes the proof.

3.3. **Optimal Control.** Let us notice that equations (3.6) become

$$\overline{p}_i(y_1, \ldots, y_N) = \sigma^2 \frac{u'_\alpha(r)}{r u_\alpha(r)} y_i,\ r \neq 0,\ i = 1, 2 \cdots N, \tag{3.25}$$

and $r = |y|$.

The optimal control is given by $p_i^* = \overline{p}_i(y_1(t), \ldots, y_N(t))$, $i = 1, ..., N$, and

$$dy_i(t) = p_i^* dt + \sigma dw_i,\ y_i(0) = y_i^0,\ i = 1, ..., N. \tag{3.26}$$

This SDE system has a unique solution since the map $y \to \bar{p}_i(y)$, $i = 1, ..., N$, is Lipschitz on $[0, R]$. Let us notice that the production rate adjusted for demand

$$\frac{\overline{p}_i}{y_i} = \sigma^2 \frac{u'_\alpha(r)}{r u_\alpha(r)}, \; r \neq 0, \tag{3.27}$$

is the same across all goods. Let us notice the resemblance with *mean field models*, [1] with the mean quantity being $r = |y|$.

Remark 3.2. The choice of $\alpha > 0$ is irrelevant because the value function equation admits the following symmetry; if u is the solution with $\alpha = 1$, then αu is the solution for arbitrary $\alpha > 0$. However, both u and αu yield the same optimal control (see (3.27)). Let us notice that if we impose the boundary condition $\bar{u}(R) = \widetilde{\alpha} > 0$ instead of (3.10) then we get a solution $\bar{u}$ which is a scalar multiple of u, i.e., $\bar{u} = Ku$, for some constant $K > 0$. Thus, $\bar{u}$ yields the same optimal control (see (3.27)) and the optimal control does not depend on the choices of α, $\widetilde{\alpha}$ and R.

Therefore, the following result holds.

Remark 3.3. The problem (3.9) subject to the boundary condition

$$u(x) = \widetilde{\alpha} \in (0, \infty) \; \text{ for } x \in \partial B_R, \tag{3.28}$$

has a unique radially symmetric solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$. Clearly $\overline{u}(x) = \widetilde{\alpha}$ is a super solution for the problem (3.9)-(3.28) and $\underline{u}(x) = \widetilde{\alpha} e^{-\frac{1}{2\sigma^2}\left(R^2 - |x|^2\right)}$ is a sub solution for (3.9)-(3.28). Then, the sub and super solution method (see for example [6, 7]) implies that (3.9)-(3.28) has at least one solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ with

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \; \text{ for } x \in \overline{B}_R.$$

By the maximum principle the solution is unique. If $u(x) \neq u(|x|)$ (i.e., u is not a radially symmetric solution) we obtain a contradiction, since a different solution can be obtained by rotating $u(x)$.

In light of these remarks we set $\alpha = 1$, so that

$$u(r) := u_1(r) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!\,(N+2)(N+6)\,...(N+4j-2)} \left(\frac{r^2}{2\sigma^2}\right)^{2j}, \tag{3.29}$$

for all $r \geq 0$, whence we can get the production rate $\sigma^2 \frac{u'(r)}{ru(r)}, r \neq 0$, in closed form. Moreover, from (3.12) we get that $\lim_{r \to 0} \frac{u'(r)}{ru(r)} = 0$, thus the optimal production rates are zeros when there are no goods produced.

Using (3.29) and operations with power series (see [20] Chapter 1), we get the optimal production rate in closed form.

Theorem 3.2. *The optimal production rate (adjusted for demand) is given by*

$$\frac{\bar{p}_i}{y_i} = \sigma^2 \frac{u'(r)}{ru(r)} = \frac{4\sigma^2}{r^2} \sum_{j=0}^{\infty} c_j \left[\frac{r^4}{4\sigma^4}\right]^j, \; r \neq 0,$$

[1] Mean field modelling of high-dimensional problems relies upon reducing the dimensionality by considering mean type variables.

where

$$\begin{aligned} a_0 &= 1,\ c_0 = 0,\ c_j = \frac{1}{a_0}\left[b_j - \sum_{i=1}^{j} c_{j-i} a_i\right],\ j = 1, 2, 3, ... \\ a_j &= \frac{1}{j!\,(N+2)\,(N+6)\,...(N+4j-2)},\ j = 1, 2, ... \\ b_j &= \frac{j}{j!\,(N+2)\,(N+6)\,...(N+4j-2)},\ j = 1, 2, ... \end{aligned}$$

The production rate (adjusted for demand) is increasing and bounded. This fact will me made precise in the following Lemma. The intuition for a monotonous production rate comes from the fact that an increasing production rate will yield a lower variance for $y_i(t)$ in SDE (2.1).

Lemma 3.1. *The function $r \to \frac{u'(r)}{ru(r)}$ is increasing and*

$$\frac{u'(r)}{ru(r)} \leq \frac{1}{\sigma^2}. \tag{3.30}$$

Proof. The first part of the claim yields if the derivative of this function is positive which boils down to

$$u''(r) \geq \frac{(u'(r))^2}{u(r)} + \frac{u'(r)}{r}.$$

Next we use the fact that u solves the following ODE

$$u''(r) + \frac{N-1}{r} u'(r) = \frac{1}{\sigma^4} r^2 u(r),$$

whence, the claim becomes

$$\frac{1}{\sigma^4} r^2 u(r) \geq N \frac{u'(r)}{r} + \frac{(u'(r))^2}{u(r)}.$$

This is equivalent to

$$\frac{u'(r)}{u(r)} \leq \frac{\sqrt{\frac{N^2}{r^2} + \frac{4r^2}{\sigma^4}} - \frac{N}{r}}{2},$$

or

$$\frac{u'(r)}{ru(r)} \leq \frac{\sqrt{\frac{N^2}{r^2} + \frac{4r^2}{\sigma^4}} - \frac{N}{r}}{2r}. \tag{3.31}$$

This argument shows that

$$r \to \frac{u'(r)}{ru(r)},$$

is increasing if and only if (3.31) holds true. However, the function

$$r \to \frac{\sqrt{\frac{N^2}{r^2} + \frac{4r^2}{\sigma^4}} - \frac{N}{r}}{2r},$$

is increasing, both functions are 0 when $r = 0$ (since $u'(0) = 0$) and

$$r \to \frac{u'(r)}{ru(r)},$$

is increasing on some small interval $[0, \epsilon]$ in light of u being convex (for this see Theorem 3.1). This shows that

$$r \to \frac{u'(r)}{ru(r)},$$

is increasing and (3.31) holds true. Moreover, since

$$r \to \frac{\sqrt{\frac{N^2}{r^2} + \frac{4r^2}{\sigma^4}} - \frac{N}{r}}{2r},$$

is increasing and has as asymptote at infinity $\frac{1}{\sigma^2}$ we also get the second part of the claim. □

3.4. **Asymptotic Analysis.**

Let us recall the estimate for large N from [8]

$$\frac{u'(r)}{r} \leq \frac{K}{N-1}, \; r \neq 0.$$

Thus, for big N an approximate solution is

$$\frac{u'(r)}{ru(r)} \leq \frac{K}{u_0(N-1)} \approx 0, \; r \neq 0,$$

which says that the optimal control $p^* \approx 0$, since

$$\overline{p}_i = \sigma^2 \frac{u'(r)}{ru(r)} x_i, \; r \neq 0, \; i = 1, ..., N.$$

This means that if the number of goods is big then $p^* = 0$ is an approximate solution.

Next, we prove an asymptotical result.

Lemma 3.2. *The following result hold true*

$$\lim_{r \to \infty} \left[\frac{u'(r)}{ru(r)} \right] = \frac{1}{\sigma^2}. \tag{3.32}$$

Proof. Because the function u is convex and increasing (for this see Theorem 3.1) it follows that

$$\lim_{r \to \infty} u(r) = \lim_{r \to \infty} u'(r) = \infty.$$

In light of Lemma 3.1 the limit exists and is finite. Let us denote it by l. L'Hospital rule yields

$$l = \lim_{r \to \infty} \left[\frac{u'(r)}{ru(r)} \right] = \lim_{r \to \infty} \left[\frac{u''(r)}{u(r) + ru'(r)} \right]. \tag{3.33}$$

Next we use the fact that u solves the following ODE

$$u''(r) + \frac{N-1}{r} u'(r) = \frac{1}{\sigma^4} r^2 u(r),$$

whence

$$u''(r) = \frac{1}{\sigma^4} r^2 u(r) - \frac{N-1}{r} u'(r).$$

Inserting this into (3.33) we get

$$l = \frac{1}{l\sigma^4}.$$

Therefore

$$l = \lim_{r\to\infty}\left[\frac{u'(r)}{ru(r)}\right] = \frac{1}{\sigma^2}. \qquad \square$$

3.5. **Simulation of the optimal inventory.** Let us recall the SDE system

$$dy_i\left(t\right) = p_i^* dt + \sigma dw_i,\ y_i\left(0\right) = y_i^0,\ i = 1, ..., N, \tag{3.34}$$

governing the optimal inventory.

This SDE system can be simulated numerically. It can be done using a Euler scheme as follows: start with

$$y_i^0, i = 1, ..., N, \text{ and } r = \Sigma_{i=1}^N \left[y_i^0\right]^2, r \neq 0.$$

On $[0, \Delta t]$ we approximate

$$y_i\left(\Delta t\right) \simeq \sigma^2 \frac{u'\left(r\right)}{ru\left(r\right)} y_i^0 + \sigma\sqrt{\Delta t} Z_i^0,\ r \neq 0,$$

where Z_i^0 is standard normal.

Next repeat this on $[\Delta t, 2\Delta t]$ as follows:

$$r\left(\Delta t\right) = \Sigma_{i=1}^N \left[y_i^{\Delta t}\right]^2,$$

and

$$y_i\left(2\Delta t\right) \simeq \sigma^2 \frac{u'\left(r\left(\Delta t\right)\right)}{r\left(\Delta t\right) u\left(r\left(\Delta t\right)\right)} y_i^{\Delta t} + \sigma\sqrt{\Delta t} Z_i^1,$$

where Z_i^1 is standard normal.

The process is then repeated on $[2\Delta t, 3\Delta t]$ and so on.

In the following we present two plots resulting from this simulation procedure. We considered $N = 2$ (two economic goods) and $\sigma = 2$ in the first plot $\sigma = 5$ in the second plot.

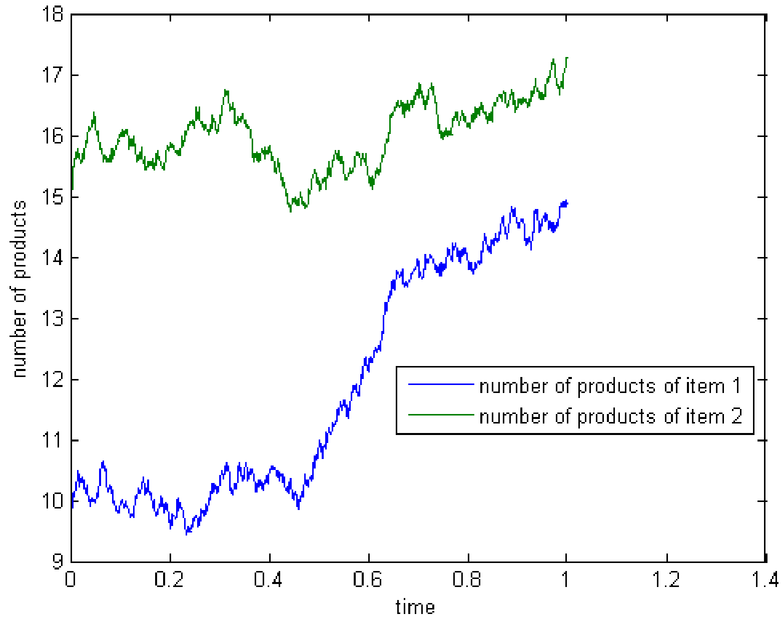

Figure 1. $\sigma = 2$.

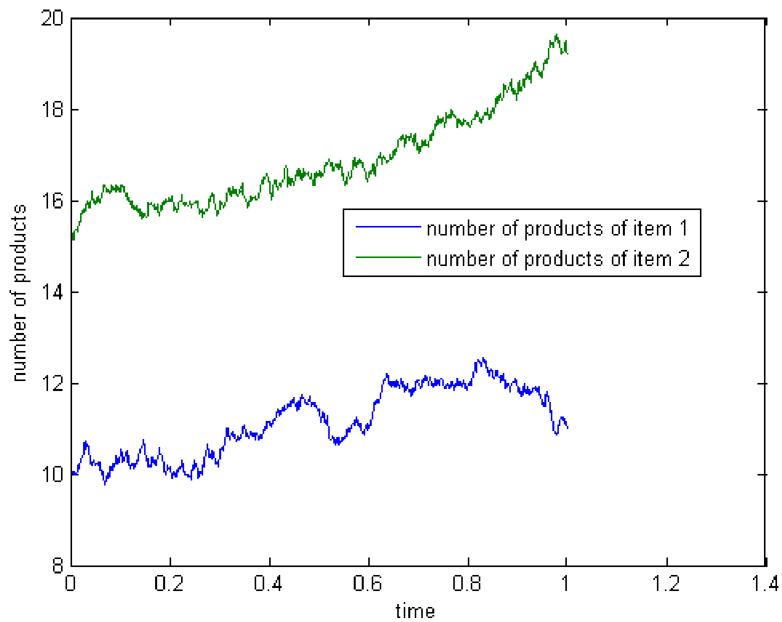

Figure 2. $\sigma = 5$.

3.6. **Numerical Experiments.** In the first set of experiments we set $\sigma = 0.5$, and vary N the number of goods' type.

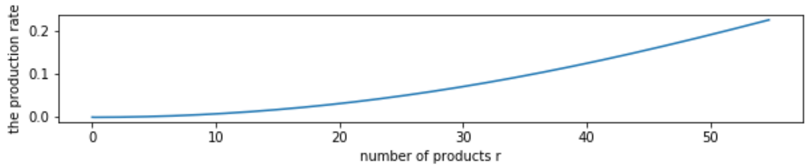

Figure 3. $N = 50000$.

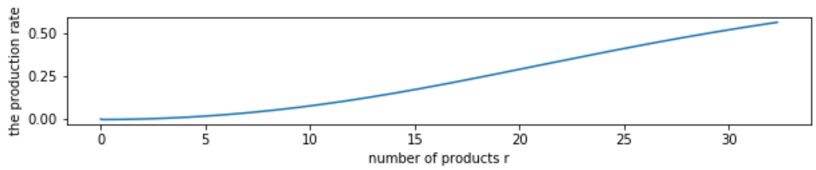

Figure 4. $N = 5000$.

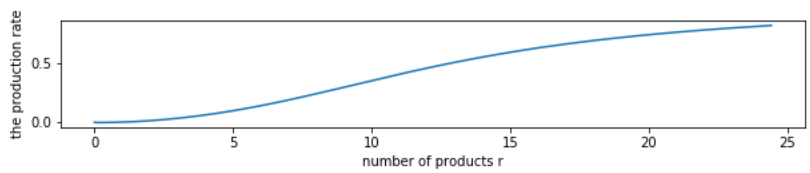

Figure 5. $N = 1000$.

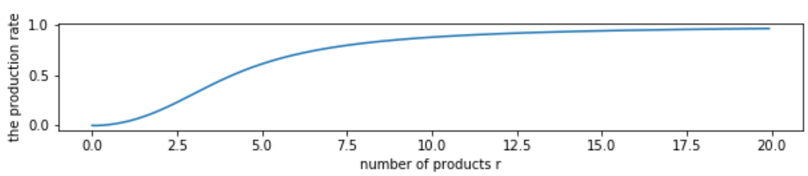

Figure 6. $N = 100$.

We observe from these set of plots the following patterns:

1) the production rate is an increasing function of the total number of goods produced, fact explained by Lemma 3.1;

2) when the total number of goods produced exceed a certain threshold the production rate converges to 1, fact explained by Lemma 3.2;

3) the production rate is a decreasing function of the total number of goods produced.

In the next set of plots we set $N = 100$, and vary σ.

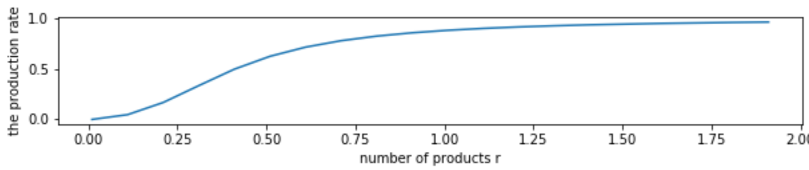

Figure 7. $\sigma = 0.05$.

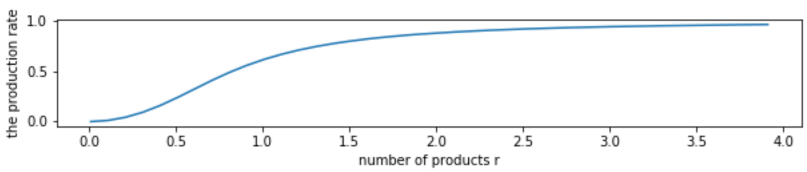

Figure 8. $\sigma = 0.1$.

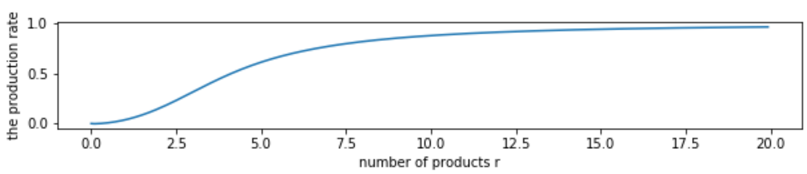

Figure 9. $\sigma = 0.5$.

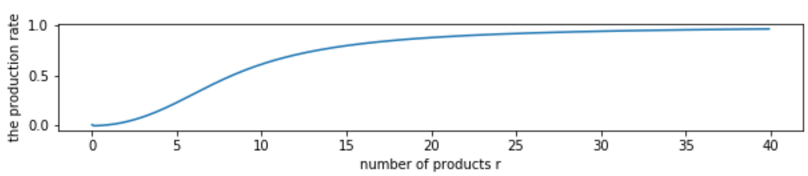

Figure 10. $\sigma = 1$.

We observe from these set of plots the following patterns:

1) the production rate is an increasing function of the total number of goods produced, fact explained by Lemma 3.1;

2) when the total number of goods produced exceed a certain threshold the production rate converges, fact explained by Lemma 3.2;

3) the production rate is a decreasing function of σ.

4. Other applications

The value function equation characterizing the optimal control, i.e., (3.9), appears naturally in other practical applications. There is by now a vast literature concerning on the existence of positive solutions and their behaviour for the partial differential equation

$$\Delta u(x) = f(x, u(x)) \text{ for } x \in \Omega, \tag{4.1}$$

where Ω is a bounded or unbounded domain of $\mathbb{R}^N$ $(N \geq 1)$ or the all space $\mathbb{R}^N$ and f is a function suitable chosen.

The interest in studying the above equation comes, for instance, from various applications of physics, such as quantum mechanics, quantum optics, nuclear physics and reaction-diffusion processes (cf. [1, 15, 17, 18]). For instance, a basic preoccupation for the study of problem (4.1) is the time-independent Schrödinger equation (single non-relativistic particle)

$$\Delta u = \frac{2m}{\hbar^2}(V(x) - E)u, \; \hbar = h/2\pi, \tag{4.2}$$

where h is Planck's constant, $\hbar$ is the reduced Planck constant (or the Dirac constant), E and $V(x)$ are the total (non relativistic) and potential energies of a particle of mass m, respectively.

Besides the importance in applications, the equation (4.1) also raises many difficult mathematical problems that need to be solved. In general, the existence of the solutions and numerical approximation of the elliptic problem (4.1) is widely open. See the paper of Santos, Zhou and Santos [18], which includes a nice survey and recent progresses for Eq. (4.1).

Let us mention this result which is interesting in itself.

Theorem 4.1. (see [23, Theorem 2.1, p. 199]) *The problem (3.9) subject to the boundary condition*

$$u(x) \to \infty \text{ as } |x| \to R, \tag{4.3}$$

has no positive solutions.

Even if the next result has no importance in economic theories, it helps us to understand the beauty of this problem and to discover other questions that will need to be solved by the researchers.

Theorem 4.2. (see [18]) *The problem (3.9) with $B_R(0)$ replaced with $\mathbb{R}^N$, admits a sequence of symmetric radial solutions $u_k(|x|) \in C^2(\mathbb{R}^N)$ with*

$$u_k(0) = \infty \text{ as } k \to \infty.$$

Besides this, $u_k' \geq 0$ in $[0, \infty)$.

In the next, we provide two exact solutions for the problem (3.9) with $B_R(0)$ replaced with $\mathbb{R}^4 \setminus \{0_{\mathbb{R}^4}\}$. They are:

$$u_1(x) = \alpha e^{\frac{1}{2\sigma^2}|x|^2} |x|^{-2}, \ \alpha \in \mathbb{R} \text{ and } x \in \mathbb{R}^4 \setminus \{0_{\mathbb{R}^4}\}, \tag{4.4}$$

$$u_2(x) = \alpha e^{-\frac{1}{2\sigma^2}|x|^2} |x|^{-2}, \ \alpha \in \mathbb{R} \text{ and } x \in \mathbb{R}^4 \setminus \{0_{\mathbb{R}^4}\}. \tag{4.5}$$

The solutions (4.4) and (4.5) were determined by analyzing the series in (3.12) and can be used by physicists in the study of the time-independent Schrödinger equation (4.2). Moreover, reasoning in the same manner we think that similar solutions can be constructed for the total (non relativistic) and potential energies of a particle of mass m in (4.2).

Next, we posit the following open problems inspired by the two solutions and [18].

Problem 4.1. Assume that $g \in C^1([0,\infty),[0,\infty))$ is a non-decreasing function satisfying

$$\int_\gamma^\infty \frac{1}{\sqrt{\int_0^t g(s)\,ds}} dt = \infty, \text{ for } t \geq \gamma > 0,$$

and p is a non-negative continuous symmetric radially function such that

$$\int_0^\infty t^{1-N} \int_0^t s^{N-1} p(s)\,ds dt = \infty.$$

Then, there exists at least one positive radially symmetric solution

$$u \in C^2\left(\mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}\right)$$

for the problem

$$\Delta u(x) = p(r)\, g(u(x)) \ \text{ in } \mathbb{R}^N, \ r = |x|, \tag{4.6}$$

subject to the boundary condition

$$u(x) \to \infty \text{ as } |x| \to \infty, \tag{4.7}$$

such that

$$u(x) \to \infty \text{ as } |x| \to 0.$$

Moreover, $\partial u/\partial r \geq 0$ on $[t_0,\infty)$ and $\partial u/\partial r < 0$ on $[0,t_0)$, for some $t_0 \geq 0$.

Problem 4.2. Under the same assumptions on p and g as in Problem 4.1, there exists at least one positive radially symmetric solution $u \in C^2\left(\mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}\right)$ of (4.6) subject to the boundary condition

$$u(x) \to 0 \text{ as } |x| \to \infty, \tag{4.8}$$

such that

$$u(x) \to \infty \text{ as } |x| \to 0.$$

Moreover, $\partial u/\partial r \leq 0$ on $[0,\infty)$.

Example of solutions for problems 4.1, and 4.2 are the ones given in (4.4), and (4.5). To the best of our knowledge the only result for the problems 4.1, and 4.2 is Theorem 4.2.

Acknowledgements. The authors are grateful to the referees and editors for useful remarks that greatly improved the presentation of the paper. This work was supported by a mobility grant of the Romanian Ministery of Research and Innovation, CNCS-UEFISCDI, project number PN-III-P1-1.1-MCD-2019-0151, within PNCDI III, and NSERC grant 5-36700. The second author dedicates this work to the memory of his father, Mathematics Teacher Constantin Covei, who graduated from Babes-Bolyai University.

References

[1] O. Alvarez, *A quasilinear elliptic equation in* R^N, Proc. Roy. Soc. Edinburgh Sect. A, **126**(1996), 911-921.

[2] L. Arnold, *Stochastic Differential Equations*, Wiley, New York, 1974.

[3] A. Bensoussan, S.P. Sethi, R. Vickson, N. Derzko, *Stochastic production planning with production constraints*, SIAM J. Control Optim., **22**(1984), 920-935.

[4] A. Cadenillas, P. Lakner, M. Pinedo, *Optimal control of a mean-reverting inventory*, Operations Research, **58**(2010), no. 6, 1046-1062.

[5] A. Cadenillas, P. Lakner, M. Pinedo, *Optimal production management when demand depends on the business cycle*, Operations Research, **61**(2013), no. 4, 1046-1062.

[6] D. -P. Covei, T.A. Pirvu, *A stochastic control problem with regime switching*, Carpathian J. Math., **37** (2021), no 3, 427 - 440

[7] D.-P. Covei, *An elliptic partial differential equation modeling the production planning problem*, J. Appl. Anal. Comput., 11(2021), no. 2, 903-910.

[8] D.-P. Covei, *Symmetric solutions for an elliptic partial differential equation that arises in stochastic production planning with production constraints*, Appl. Math. Comput., **350**(2019), 190-197.

[9] D.-P. Covei, T.A. Pirvu, *An elliptic partial differential equation and its application*, Appl. Math. Lett., **101**(2020), 1-7.

[10] I. Ekeland, O. Mbodji, T.A. Pirvu, *Time consistent portfolio management*, SIAM Journal of Mathematical Finance, **3**(2012), no.1, 1-32.

[11] I. Ekeland, T.A. Pirvu, *Investment and consumption without commitment*, Mathematics and Financial Economics, **71**(2014), 142-150.

[12] W.H. Fleming, S.P. Sethi, H.M. Soner, *An optimal stochastic production planning problem with randomly fluctuating demand*, SIAM J. Control Optim., **25**(1987), 1494-1502.

[13] A. Gharbi, J.P. Kenne, *Optimal production control problem in stochastic multiple-product multiple-machine manufacturing systems*, IIE Transactions, **35**(2003), 941-952.

[14] J.M. Lasry, P.L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints*, Math. Ann., **283**(1989), 583-630.

[15] T. Leonori, *Large solutions for a class of nonlinear elliptic equations with gradient terms*, Adv. Nonlinear Stud., **7**(2007), 237-269.

[16] T.A. Pirvu, H. Zhang, *Investment-consumption with regime-switching discount rates*, Math. Social Sci., **71**(2014), 142-150.

[17] A. Porretta, *Some uniqueness results for elliptic equations without condition at infinity*, Commun. Contemp. Math., **5**(2003), 705-717.

[18] C.A. Santos, J. Zhou, J.A. Santos, *Necessary and sufficient conditions for existence of blow-up solutions for elliptic problems in Orlicz-Sobolev spaces*, Complex Var. Elliptic Equ., **62**(2017), 887-899.

[19] S.P. Sethi, G.L. Thompson, *Applied Optimal Control: Applications to Management Science*, Nijhoff, Boston, 1981.

[20] D.G. Simpson, *Power Series*, NASA Goddard Space Flight Center Greenbelt, Maryland 20771.

[21] G.L. Thompson, S.P. Sethi, *Turnpike horizons for production planning*, Management Sci., **26**(1980), 229-241.

[22] G.L. Thompson, S.P. Sethi, J. Teng, *Strong planning and forecast horizons for a model with simultaneous price and production decisions*, European Journal of Operational Research, **16**(1984), 378-388.

[23] H. Yang, *On the existence and asymptotic behavior of large solutions for a semilinear elliptic problem in* $\mathbb{R}^N$, Commun. Pure Appl. Anal., **4**(2005), 187-198.

Received: March 29, 2020; Accepted: May 16, 2020.