# ON THE FIXED POINT INDEX FOR SUMS OF OPERATORS 

SALIM BENSLIMANE*, SMAÏL DJEBALI** AND KARIMA MEBARKI***<br>*Laboratory of Applied Mathematics, Faculty of Exact Sciences,<br>Bejaia University, 06000 Bejaia, Algeria<br>E-mail: salim.bens1993@gmail.com<br>** Department of Mathematics, Faculty of Sciences, Imam Mohammad Ibn Saud Islamic University (IMSIU),<br>PB 90950. Riyadh 11623, Saudi Arabia<br>and<br>Laboratoire "Théorie du Point Fixe et Applications"<br>ENS, PB 92 Kouba. Algiers, 16006. Algeria<br>E-mail: djebali@hotmail.com<br>${ }^{* * *}$ Laboratory of Applied Mathematics, Faculty of Exact Sciences, Bejaia University, 06000 Bejaia, Algeria<br>E-mail: mebarqi_karima@hotmail.fr


#### Abstract

We have obtained some new Krasoneslskii type fixed point theorems for the sum of two operators $T+F$, where $T$ is expansive with constant $h>1$ and $I-F$ is a $k$-set contraction with $k<h$. The existence results are based on a fixed point index for these classes of mappings. Various fixed point theorems are derived in cones and in translates of cones of some Banach spaces. Two examples of application are included to illustrate the theory. Key Words and Phrases: Sum of operators, cone, fixed point index, $k$-set contraction, expansive mapping, Krasnoselskii's Theorem. 2020 Mathematics Subject Classification: 37C25, 47H10, 58J20.


## 1. Introduction

As a very important part of nonlinear analysis, fixed point theory plays a key role with regards to the solvability of many complex systems from applied mathematics (chemicals reactors, neutron transport, population biology, infection diseases, economics, applied mechanics, ...). The theory itself developed quickly in many directions starting from Brouwer's fixed point theorem (1910), Banach's contraction principle (1922), and Schauder's fixed point theorem for compact mappings (1930). Krasnoselskii's fixed point theorem for sums of operators (1955) is considered as both an extension and a combination of these previous two results (see [7, 8, 13, 22]). It turns out to be a powerful tool to deal with several classes of nonlinear equations of the form $F x+G x=x$, in a suitable functional setting, where $F$ is a contraction and
$G$ is compact. Actually, many boundary value problems for differential equations can be recast in this abstract formulation.

Among the very rich and recent literature on the development of the fixed point theory for the sums of operators, we quote, e.g., [5, 18, 24, 23]. Another fixed point result established by Krasnoselskii in 1960 is the cone compression-expansion fixed point theorem; it is mostly used for proving existence, and localization, and multiplicity of positive solutions for various nonlinear problems in some conical shells of a Banach space (see [12, 14, 15]). During the last couple of years, its extension has attracted many researchers (see $[3,11,16,17,19]$ and references therein).

Note that the fixed point theory has also been greatly influenced by the parallel progress of the research works made on the topological degree for different classes of mappings (see, e.g., $[2,1,16,17]$ ). In this regards, the pioneer works of Petryshyn [20, 21] have initiated important steps in establishing the relationship between the fixed point theory and the index fixed point theory.

In [10], the last two authors of this paper have developed a new fixed point index for the sum of an expansive mapping and a $k$-set contraction defined in cones of some Banach spaces. Then some fixed point theorems, including Krasnoselskii type theorems, have been showed.

In this work, we continue to extend the theory to the sum $T+F$ of two mappings, where $T$ is an expansive mapping with constant $h>1$ and the perturbation $I-F$ is a $k$-set contraction with $0<k<h$. Our aim is to provide a new contribution to the fixed point index theory for this class of operators, and it is twofold: first, we define and compute a topological index and then we prove several fixed point results, by considering the fixed point index for sums of operators defined on translates of cones.

This paper contains five sections including this introduction. In section 2 , we have collected some basic concepts and auxiliary results needed throughout the paper. The main results are then presented in Section 3, where the fixed point index is defined and computed. Section 4 is devoted to the presentation of some cone compression and expansion fixed point theorems for sums of operators. In Section 5, two examples of application to nonlinear integral equations illustrate the abstract results obtained in Section 3.

## 2. Preliminaries

Let $(E,\|\|$.$) be a real Banach space.$
Definition 2.1. (a) A closed convex subset $\mathcal{P}$ of $E$ is called a cone if $\alpha \mathcal{P} \subset \mathcal{P}$ for all positive real number $\alpha$ and $\mathcal{P} \cap(-\mathcal{P})=\{0\}$.
(b) A cone $\mathcal{P}$ is called normal if there exists a positive constant $N$ such that, for all $x, y \in \mathcal{P}$, we have $x \leq y \Rightarrow\|x\| \leq N\|y\|$. The least positive constant $N$ is called the normal constant of $\mathcal{P}$.
(c) The partial order relation in $E$ induced by the cone $\mathcal{P}$ is given by $x \leq y$ if and only if $y-x \in \mathcal{P}$.

For some constant $r>0$, denote $\mathcal{P}_{r}=\mathcal{P} \cap \mathcal{B}_{r}$, where $\mathcal{B}_{r}=\{x \in E:\|x\|<r\}$ is the open ball centered at the origin with radius $r . \mathcal{P}^{*}=\mathcal{P} \backslash\{0\}$ will refer to the punctured cone.

Definition 2.2. Let $\Omega_{E}$ be the class of all bounded subsets of $E$. The Kuratowski measure of noncompactness (MNC for short) $\alpha: \Omega_{E} \rightarrow[0,+\infty)$ is defined as

$$
\alpha(V)=\inf \left\{\delta>0 \mid V=\bigcup_{i=1}^{n} V_{i} \text { and } \operatorname{diam}\left(\mathrm{V}_{\mathrm{i}}\right) \leq \delta, \forall \mathrm{i}=1, \ldots, \mathrm{n}\right\}
$$

where $\operatorname{diam}\left(V_{i}\right)=\sup \left\{\|\mathrm{x}-\mathrm{y}\|_{\mathrm{E}}, \mathrm{x}, \mathrm{y} \in \mathrm{V}_{\mathrm{i}}\right\}$ is the diameter of $V_{i}$.
The Kuratowski MNC $\alpha$ has the following properties (see [4]).
Proposition 2.1. (a) $\alpha(A)=0 \Leftrightarrow \bar{A}$ is compact.
(b) $A \subset B \Longrightarrow \alpha(A) \leq \alpha(B)$.
(c) $\alpha(A+B) \leq \alpha(A)+\alpha(B), \forall A, B \in \Omega_{E}$.
(d) $\alpha(A+x)=\alpha(A), \forall A \in \Omega_{E}, \forall x \in E$.
(e) $\alpha(\lambda A)=|\lambda| \alpha(A), \forall A \in \Omega_{E}, \forall \lambda \in \mathbb{R}$.
(f) If $\operatorname{dim}(E)=\infty, \alpha\left(\mathcal{B}_{r}\right)=2 r$.

In connection with Definition 2.2, we have
Definition 2.3. Let $A: D \subset E \rightarrow E$ be a continuous operator. $A$ is said to be:
(1) bounded if it maps bounded sets into bounded sets;
(2) compact if the set $A(D)$ is relatively compact;
(3) completely continuous if it maps bounded sets into relatively compact sets;
(4) $k$-set contraction, for some number $k \geq 0$, if it is bounded and $\alpha(A(V)) \leq k \alpha(V)$ for every bounded set $V \subset D$, and strict-set contraction whenever $k<1$.

For $k$-set-contractions, the following proposition holds:
Proposition 2.2. [20, Proposition 2] (a) If $A_{i}: G \rightarrow E$ is $k_{i}$-set contraction, $i=1,2$, and $A_{3}: A_{1}(G) \rightarrow E$ is $k_{3}$-set contraction, then $A_{1}+A_{2}: G \rightarrow E$ is $\left(k_{1}+k_{2}\right)$-set contraction, and $A_{3} A_{1}: G \rightarrow X$ is $k_{1} k_{3}$-set contraction.
(b) $A: G \rightarrow E$ is completely continuous if and only if $A$ is 0 -set contraction.
(c)If $A: G \rightarrow E$ is L-Lipschitzian (i.e., $\|A(x)-A(y)\| \leq L\|x-y\|$ for $x, y \in G$ ), then $A$ is $k$-set contraction with $k=L$.
(d) If $C: G \rightarrow E$ is completely continuous and $S: G \rightarrow E$ is L-Lipschitzian, then $C+S$ is $k$-set contraction with $k=L$.

Definition 2.4. A mapping $T: D \subset X \rightarrow X$, where $(X, d)$ is a metric space, is said to be expansive if there exists a constant $h>1$ such that

$$
d(T x, T y) \geq h d(x, y) \quad \text { for all } \quad x, y \in D
$$

The following fixed point result is proved in [24, Theorem 2.1] for expansive mappings.
Proposition 2.3. Let $(X, d)$ be a complete metric space and $D$ be a closed subset of $X$. Assume that the mapping $T: D \rightarrow X$ is expansive and $D \subset T(D)$. Then there exists a unique point $x^{*} \in D$ such that $T x^{*}=x^{*}$.

The proof is based on the following self-interesting result.

Lemma 2.4. Let $(E,\|\|$.$) be a linear normed space and D \subset X$. Assume that the mapping $T: D \rightarrow E$ is expansive with constant $h>1$. Then the inverse of $T: D \rightarrow$ $T(D)$ exists and

$$
\left\|T^{-1} x-T^{-1} y\right\| \leq \frac{1}{h}\|x-y\|, \quad \forall x, y \in T(D)
$$

We complete the preliminaries by a useful compactness criterion, where $C_{b}\left(\mathbb{R}^{+}, \mathbb{R}\right)$ stands for the Banach space of bounded continuous functions on the half-line.

Lemma 2.5. [6, Page 62] Let $M \subseteq C_{b}\left(\mathbb{R}^{+}, \mathbb{R}\right)$. Then $M$ is relatively compact in $C_{b}\left(\mathbb{R}^{+}, \mathbb{R}\right)$ if the following conditions hold:
(a) $M$ is uniformly bounded in $C_{b}\left(\mathbb{R}^{+}, \mathbb{R}\right)$.
(b) The functions belonging to $M$ are almost equicontinuous on $\mathbb{R}^{+}$, i.e. equicontinuous on every compact interval of $\mathbb{R}^{+}$.
(c) The functions from $M$ are equiconvergent, that is, given $\varepsilon>0$, there corresponds $T(\varepsilon)>0$ such that $|x(t)-l|<\varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.

## 3. Definition of a fixed point index

Given a real Banach space $(E,\|\cdot\|)$, let $Y \subset E$ be a closed convex subset. Let $\Omega$ be any subset of $Y$ and $U$ be a bounded open subset of $Y$. Consider an expansive mapping $T: \Omega \rightarrow E$ with constant $h>1$ and let $I-F: \bar{U} \rightarrow E$ be a $k$-set contraction with $0 \leq k<h$. By Lemma 2.4, the operator $T^{-1}$ is $\frac{1}{h}$-Lipschtzian on $T(\Omega)$. Suppose that

$$
(I-F)(\bar{U}) \subset T(\Omega)
$$

Then $T^{-1}(I-F): \bar{U} \rightarrow \Omega \subset Y$ is a strict set contraction. Actually, the mapping $T^{-1}(I-F)$ is continuous, bounded, and for all bounded subset $B \subset \bar{U}$, we have

$$
\alpha\left(\left(T^{-1}(I-F)\right)(B)\right) \leq \frac{1}{h} \alpha((I-F)(B)) \leq \frac{k}{h} \alpha(B)
$$

Note further that if $x \neq T x+F x$, for all $x \in \partial U \cap \Omega$, then $x \neq T^{-1}(I-F) x$, for all $x \in \partial U$.

As in [10], a fixed point index of the sum $T+F$ on $U \cap \Omega$ with respect to the closed convex set $Y$ can be defined by

$$
i_{*}(T+F, U \cap \Omega, Y)= \begin{cases}i\left(T^{-1}(I-F), U, Y\right), & \text { if } U \cap \Omega \neq \emptyset  \tag{3.1}\\ 0, & \text { if } U \cap \Omega=\emptyset\end{cases}
$$

Theorem 3.1. The fixed point index $i_{*}(T+F, U \cap \Omega, Y)$ defined in (3.1) has the following properties:
(i) (Normalization) If $U=Y \cap \mathcal{B}(\omega, r), \omega \in \Omega$, and $(I-F) x=z_{0}$ for all $x \in \bar{U}$, where $z_{0} \in Y \cap \in T(\Omega)$ and $\left\|z_{0}-T \omega\right\|<h r$, then

$$
i_{*}(T+F, U \cap \Omega, Y)=1
$$

(ii) (Additivity) For any pair of disjoint open subsets $U_{1}, U_{2} \subset U$ such that $T+F$ has no fixed point on $\left(\bar{U} \backslash\left(U_{1} \cup U_{2}\right)\right) \cap \Omega$, we have

$$
i_{*}(T+F, U \cap \Omega, Y)=i_{*}\left(T+F, U_{1} \cap \Omega, Y\right)+i_{*}\left(T+F, U_{2} \cap \Omega, Y\right)
$$

(iii) (Homotopy invariance) The generalized fixed point index $i_{*}(T+H(., t), U \cap \Omega, Y)$ does not depend on the parameter $t \in[0,1]$, where
(a) $(I-H):[0,1] \times \bar{U} \rightarrow E$ is continuous and $H(t, x)$ is uniformly continuous in $t$ with respect to $x \in \bar{U}$,
(b) $(I-H)([0,1] \times \bar{U}) \subset T(\Omega)$,
(c) $(I-H(t,)):. \bar{U} \rightarrow E$ is a $\ell$-set contraction with $0 \leq \ell<h$, for all $t \in[0,1]$,
(d) $T x+H(t, x) \neq x$ for all $t \in[0,1]$ and $x \in \partial U \cap \Omega$.
(iv) (Solvability) If $i_{*}(T+F, U \cap \Omega, Y) \neq 0$, then $T+F$ has a fixed point in $U \cap \Omega$.

Proof. We argue as in [10, Theorem 2.1]. Properties (ii), (iii), and (iv) are consequences of (3.1) and of the properties of the fixed point index for strict set contractions (see [11, Theorem 1.3.5]). It remains to check the normalization property. If $U=Y \cap \mathcal{B}(w, r)$, then

$$
i\left(T^{-1}(I-F), U, Y\right)=i\left(T^{-1} z_{0}, U, Y\right)=1
$$

For this purpose, we show that $y_{0}:=T^{-1} z_{0} \in \mathcal{B}(\omega, r) \cap \Omega$. $(I-F)(\bar{U})=\left\{z_{0}\right\} \subset T(\Omega)$ implies that $y_{0} \in \Omega$ and since $T$ is an expansive operator with $h>1$, then

$$
\left\|T y_{0}-T \omega\right\| \geq h\left\|y_{0}-\omega\right\|
$$

Then

$$
h\left\|y_{0}-\omega\right\| \leq\left\|T y_{0}-T \omega\right\|=\left\|z_{0}-T \omega\right\|<h r
$$

and thus $y_{0}=T^{-1} z_{0} \in U$. Using the normalization property of the index [11, Theorem 1.3.5], we find that

$$
i\left(T^{-1} z_{0}, U, Y\right)=1
$$

Finally $i_{*}(T+F, U \cap \Omega, Y)=1$, as claimed.
Remark 3.1. Let $\mathcal{P} \subset E$ be a cone, $0 \in \Omega$, and $U=\mathcal{P} \cap\{x \in E: \psi(x)<R\}$, where $\psi$ is a nonnegative continuous functional on $\mathcal{P}$ satisfying $\psi(x) \leq\|x\|$ for all $x \in \mathcal{P}$. If $(I-F) x=z_{0}$, for all $x \in \bar{U}$, where $z_{0} \in \mathcal{P}$ and $\left\|z_{0}-T 0\right\|<h R$, then we can prove as for the normalization property that

$$
i_{*}(T+F, U \cap \Omega, \mathcal{P})=1
$$

Remark 3.2. (1) Since $T$ and $I-T$ have the same properties in terms of invertibility and since $I-F$ is an $\ell$-set contraction with $\ell<h$, one could think that the fixed point index developed in this paper is a generalization of the one developed in [10]. Unfortunately the implication

$$
F(\bar{U}) \subset(I-T)(\Omega) \Rightarrow(I-F)(\bar{U}) \subset T(\Omega)
$$

does not in general hold. For example:
(a) Let $T:[0,1] \rightarrow \mathbb{R}$ be such that $T x=-\frac{5}{2} e^{x}$ and $F:[0,4] \rightarrow \mathbb{R}$ is $F x=e^{-x}+3$. Then, the conditions of the fixed point index developed in [10] are satisfied. Indeed, $T$ is a $\frac{5}{2}$-expansive mapping and $F$ is a 1 -set contraction. In addition

$$
F([0,4])=\left[e^{-4}+3,4\right] \subset(I-T)([0,1])=\left[\frac{5}{2}, 1+\frac{5 e}{2}\right]
$$

but

$$
(I-F)([0,4])=\left[-4,1-\frac{1}{e}\right] \not \subset T([0,1])=\left[-\frac{5 e}{2},-\frac{5}{2}\right]
$$

(b) Let $T:[0,1] \rightarrow \mathbb{R}$ be such that $T x=2 x$ and $F:[0,5] \rightarrow \mathbb{R}$ is $F x=-\frac{1}{10} x+g(x)$, where $g:[0,5] \rightarrow\left[-\frac{1}{2}, 0\right]$ is a $\frac{4}{5}$-set contraction such that the equation $g(x)+\frac{9}{10} x=0$ has a solution in $(0,1]$. Then the conditions of the fixed point index developed in [10] are satisfied. Indeed, $T$ is a 2 -expansive mapping and $F$ is a $\frac{9}{10}$-set contraction. In addition $F([0,5]) \subset[-1,0]=(I-T)([0,1])$ but $(I-F)([0,5]) \not \subset T([0,1])=[0,2]$.
(2) Conversely, define two mappings $T, F:[0,1] \rightarrow \mathbb{R}$ by $T x=\frac{3}{2} e^{x}$ and $F x=-2 e^{-x}$. Then $T$ is a $\frac{3}{2}$-expansive mapping, $(I-F) x=x+2 e^{-x}$ is a 1 -set contraction, and

$$
(I-F)([0,1])=\left[\frac{2+e}{e}, 2\right] \subset T([0,1])=\left[\frac{3}{2}, \frac{3}{2} e\right]
$$

It is clear that the conditions of the fixed point index developed in this paper are satisfied, while that of the index defined in [10] are not ( $F$ is a 2 -set contraction). Moreover, the equation $F x+T x=x$ cannot be rewritten in the abstract form

$$
\tilde{T} x+\tilde{F} x=x
$$

where $\tilde{T}$ is $\tilde{h}$-expansive and $\tilde{F} \not \equiv 0$ is $\tilde{k}$-set contraction with $\tilde{k}<\tilde{h}-1$.
(3) These two examples show that the fixed point index we present here and the one developed in [10] do not coincide and are not easily comparable. Even in the case where both approaches are applicable, we will present in this work new sufficient conditions allowing the computation of the index of the fixed point for the sum of two operators even on translates of cones.
3.1. Computation of a fixed point index. In this section, we show that the fixed point index can be computed in case of a translate of a cone, rather than in a cone, and in some cases even in an arbitrary closed convex subset. A fixed point index in translates of cones of Banach spaces is defined in [9] for completely continuous mappings and can be extended to the case of a strict set contractions. Let $\mathcal{P} \neq\{0\}$ be a cone in $E$ and $\mathcal{K}=\mathcal{P}+\theta(\theta \in E)$ a $\theta$-translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ be a subset and $U \subset \mathcal{K}$ be a bounded open subset such that $\Omega \cap U \neq \emptyset$. Since $\mathcal{K}$ is a closed convex of $E$, the fixed point index $i_{*}(T+F, U \cap \Omega, \mathcal{K})$ is well defined whenever $T: \Omega \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h,(I-F)(\bar{U}) \subset T(\Omega)$, and $x \neq T x+F x$ for all $x \in \partial U \cap \Omega$, where $\bar{U}$ and $\partial U$ denotes the closure and the boundary of $U$ in $\mathcal{K}$, respectively. For two real numbers $0<r<R$, define the sets:

$$
\begin{aligned}
\mathcal{K}_{r} & =\{x \in \mathcal{K}:\|x-\theta\|<r\} \\
\partial \mathcal{K}_{r} & =\{x \in \mathcal{K}:\|x-\theta\|=r\} \\
\mathcal{K}_{r, R} & =\{x \in \mathcal{K}: r<\|x-\theta\|<R\}
\end{aligned}
$$

Proposition 3.2. Let $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1$ and $I-F: \overline{\mathcal{K}_{r}} \rightarrow E$ be a $k$-set contraction with $0 \leq k<h$ such that $t(I-F)\left(\overline{\mathcal{K}_{r}}\right)+(1-t) \theta \subset T(\Omega)$, for all $t \in[0,1]$. Assume that $\theta \in \Omega,\|T \theta-\theta\|<h r$, and

$$
\begin{equation*}
T x \neq \lambda(x-F x)+(1-\lambda) \theta, \text { for all } x \in \partial \mathcal{K}_{r} \cap \Omega \text { and } 0 \leq \lambda \leq 1 \tag{3.2}
\end{equation*}
$$

Then $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{P}\right)=1$.
Proof. Define the line homotopy $H:[0,1] \times \overline{\mathcal{K}_{r}} \rightarrow E$ by

$$
H(t, x)=t F x+(1-t)(x-\theta)
$$

Then, the operator $(I-H)$ is continuous and uniformly continuous in $t$ for each $x$. Moreover the mapping $(I-H(t,)$.$) is a k$-set contraction for each $t$. Actually, for any bounded set $B$ in $\mathcal{K}_{r}$, we have

$$
\alpha((I-H(t, .))(B))=\alpha(t(I-F)(B)+(1-t) \theta) \leq k \alpha(B)
$$

In addition, $T+H(t,$.$) has no fixed point on \partial \mathcal{K}_{r} \cap \Omega$. If not, there exist some $x_{0} \in \partial \mathcal{K}_{r} \cap \Omega$ and $t_{0} \in[0,1]$ such that

$$
T x_{0}+t_{0} F x_{0}+\left(1-t_{0}\right)\left(x_{0}-\theta\right)=x_{0}
$$

Then $T x_{0}=t_{0}\left(x_{0}-F x_{0}\right)+\left(1-t_{0}\right) \theta$, leading to a contradiction with the hypothesis. By properties (i) and (iii) of the fixed point index in Theorem 3.1, we get

$$
i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=i_{*}\left(T+I-\theta, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=1
$$

From Proposition 3.2, we capture the following two results.
Corollary 3.3. Assume that $T: \Omega \subset \mathcal{K} \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \overline{\mathcal{K}_{r}} \rightarrow E$ is a $k$-set contraction with $0 \leq k<h$, and $\left(t(I-F)\left(\overline{\mathcal{K}_{r}}\right)+(1-t) \theta\right) \subset T(\Omega)$, for all $t \in[0,1]$. If $\theta \in \Omega,\|T \theta-\theta\|<h r$, and

$$
\|T x-\theta\| \geq\|x-F x-\theta\| \text { and } T x+F x \neq x, \text { for all } x \in \partial \mathcal{K}_{r} \cap \Omega
$$

Then $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=1$.
Proof. It is sufficient to prove that Assumption (3.2) holds. By contradiction, let $x_{0} \in \mathcal{K}_{r} \cap \Omega$ and let $0 \leq \lambda_{0} \leq 1$ satisfy $T x_{0}=\lambda_{0}\left(x_{0}-F x_{0}\right)+\left(1-\lambda_{0}\right) \theta$. If $\lambda_{0}=1$, then $x_{0}-F x_{0}=T x_{0}$ which is impossible. If $0 \leq \lambda_{0}<1$, then

$$
\left\|T x_{0}-\theta\right\|=\lambda_{0}\left\|x_{0}-T x_{0}-\theta\right\|<\left\|x_{0}-T x_{0}-\theta\right\|
$$

which is a contradiction.
Corollary 3.4. Let $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1$ and let $I-F: \overline{\mathcal{K}_{r}} \rightarrow E$ be a $k$-set contraction with $0 \leq k<h$ such that $\left((I-F)\left(\overline{\mathcal{K}_{r}}\right)+(1-t) \theta\right) \subset T(\Omega)$, for all $t \in[0,1]$. Assume further that $\theta \in \Omega$, $\|T \theta-\theta\|<h r$,

$$
x-F x \in \mathcal{K} \text { for all } x \in \partial \mathcal{K}_{r} \cap \Omega,
$$

and

$$
T x \not \leq x-F x \text { for all } x \in \partial \mathcal{K}_{r} \cap \Omega
$$

Then $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=1$.
Proof. Assumption (3.2) is readily checked for otherwise there would exist some $x_{0} \in$ $\mathcal{K}_{r} \cap \Omega$ and $0 \leq \lambda_{0} \leq 1$ such that $T x_{0}=\lambda_{0}\left(x_{0}-F x_{0}\right)+\left(1-\lambda_{0}\right) \theta$. Hence

$$
T x_{0}-\theta=\lambda_{0}\left(x_{0}-F x_{0}-\theta\right)
$$

Since $x_{0}-F x_{0}-\theta \in \mathcal{P}$, then $\lambda_{0}\left(x_{0}-F x_{0}-\theta\right) \leq x_{0}-F x_{0}-\theta$, which is a contradiction to our assumption.

Proposition 3.5. Let $\theta \in U \subset \mathcal{K}$ be a bounded open subset and $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Assume further that

$$
x-F x \neq T(\lambda x+(1-\lambda) \theta), \quad \text { for all } x \in \partial U, \lambda \geq 1 \text { and } \lambda x+(1-\lambda) \theta \in \Omega
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{K})=1$.
Proof. The mapping $T^{-1}(I-F): \bar{U} \rightarrow \mathcal{K}$ is a strict set contraction and it is clear that

$$
\begin{equation*}
T^{-1}(I-F) x-\theta \neq \lambda(x-\theta), \text { for all } x \in \partial U \text { and } \lambda \geq 1 \tag{3.3}
\end{equation*}
$$

Owing to [9, Proposition 2.2], $i\left(T^{-1}(I-F), U, \mathcal{K}\right)=1$. Then Equality (3.1) ends this proof.

Proposition 3.6. Let $U \subset \mathcal{K}$ be a bounded open subset, $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Assume that $\theta \in \Omega \cap U$,

$$
\begin{equation*}
\|x-F x-T \theta\| \leq h\|x-\theta\|, \quad \text { and } T x+F x \neq x, \quad \text { for all } x \in \partial U \cap \Omega \tag{3.4}
\end{equation*}
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{P})=1$.
Proof. According to Lemma 2.4, $T^{-1}(I-F): \bar{U} \rightarrow \mathcal{K}$ is a strict set contraction. From the inclusion $(I-F)(\bar{U}) \subset T(\Omega)$, for all $x \in \bar{U}$, we can find some $y \in \Omega$ such that $x-F x=T y$. For all $x \in \bar{U}$, we have $T^{-1}(x-F x) \in \Omega$ and

$$
T\left(\left(T^{-1}(x-F x)\right)=x-F x\right.
$$

which implies that

$$
\left\|T\left(T^{-1}(x-F x)\right)-T \theta\right\|=\|x-F x-T \theta\|
$$

Since $T$ is expansive with constant $h$, we have

$$
\left\|T\left(T^{-1}(x-F x)\right)-T \theta\right\| \geq h\left\|T^{-1}(x-F x)-\theta\right\|
$$

Hence

$$
\begin{equation*}
h\left\|T^{-1}(I-F) x-\theta\right\| \leq\|x-F x-T \theta\| . \tag{3.5}
\end{equation*}
$$

From (3.5) and Assumption (3.4), we get

$$
\left\|T^{-1}(I-F) x-\theta\right\| \leq \frac{1}{h}\|x-F x-T \theta\| \leq\|x-\theta\|, \forall x \in \partial U
$$

Therefore for all $x \in \partial U \cap \Omega$

$$
\left\|T^{-1}(I-F) x-\theta\right\| \leq\|x-\theta\| \text { and } T^{-1}(I-F) x \neq x
$$

Due to [9, Corollary 2.2], $i\left(T^{-1}(I-F), U, \mathcal{K}\right)=1$. Equality (3.1) completes the proof.

In case of a cone, i.e., $\theta=0$, Proposition 3.5 and Proposition 3.6 become

Corollary 3.7. Let $0 \in U \subset \mathcal{K}$ be a bounded open subset and $T: \Omega \subset \mathcal{P} \rightarrow E$ be an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ be a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Assume further that

$$
x-F x \neq T(\lambda x), \quad \text { for all } x \in \partial U \cap \Omega, \lambda \geq 1, \text { and } \lambda x \in \Omega
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{P})=1$.
Corollary 3.8. Assume that $T: \Omega \subset \mathcal{P} \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ is a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Let $0 \in \Omega \cap U$,

$$
\begin{equation*}
\|x-F x-T 0\| \leq h\|x\|, \quad \text { and } T x+F x \neq x, \text { for all } x \in \partial U \cap \Omega \tag{3.6}
\end{equation*}
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{P})=1$.
The following result can be directly proved by replacing the operator $A$ in [11, Corollary 1.3.1] by $T^{-1}(I-F)$.

Proposition 3.9. Assume that $T: \Omega \subset \mathcal{K} \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \overline{\mathcal{K}}_{r} \rightarrow E$ is a $k$-set contraction with $0 \leq k<h$, and $(I-F)\left(\overline{\mathcal{K}}_{r}\right) \subset T(\Omega)$.

In addition, if $T^{-1}(I-F)\left(\overline{\mathcal{K}}_{r}\right) \subset \mathcal{K}_{r}$, then $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=1$.
In particular, we have
Corollary 3.10. Assume that $T: \Omega \subset \mathcal{K} \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \overline{\mathcal{K}}_{r} \rightarrow E$ is a $k$-set contraction with $0 \leq k<h$, and $(I-F)\left(\overline{\mathcal{K}}_{r}\right) \subset T(\Omega)$. If $\theta \in \Omega$, and

$$
\begin{equation*}
\|x-F x-T \theta\|<h r, \quad \text { for all } x \in \overline{\mathcal{K}}_{r} . \tag{3.7}
\end{equation*}
$$

Then $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{K}\right)=1$.
Proof. From (3.5) and Assumption (3.7), for all $x \in \overline{\mathcal{K}}_{r}$, we conclude that

$$
\left\|T^{-1}(I-F) x-\theta\right\| \leq \frac{1}{h}\|x-F x-T \theta\|<r
$$

Hence $T^{-1}(I-F)\left(\overline{\mathcal{K}}_{r}\right) \subset \mathcal{K}_{r}$.
A special situation in Corollary 3.10 is
Corollary 3.11. Assume that $T: \Omega \subset \mathcal{K} \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \overline{\mathcal{K}}_{r} \rightarrow E$ is a $k$-set contraction with $0 \leq k<h, r$ is sufficiently large, and $(I-F)\left(\overline{\mathcal{K}}_{r}\right) \subset T(\Omega)$. If further $\theta \in \Omega$ and

$$
\begin{equation*}
\|x-F x\| \leq\|x-\theta\|, \text { for all } x \in \overline{\mathcal{K}}_{r} \tag{3.8}
\end{equation*}
$$

then $T+F$ has at least one fixed point in $\mathcal{K}_{r} \cap \Omega$.
Proof. Notice that

$$
\begin{aligned}
\|x-F x-T \theta\| & \leq\|x-F x\|+\|T \theta\| \\
& \leq\|x-\theta\|+\|T \theta\| \\
& \leq r+\|T \theta\| \\
& \leq h r
\end{aligned}
$$

for all $r>\frac{\|T \theta\|}{h-1}$. By Corollary 3.10, $i_{*}\left(T+F, \mathcal{K}_{r} \cap \Omega, \mathcal{P}\right)=1$. As a consequence, $T+F$ has a fixed point in $\mathcal{K}_{r} \cap \Omega$.

Before giving results for zero index $i_{*}$, we need an auxiliary lemma on index fixed point of strict set contractions.

Lemma 3.12. Let $\mathcal{K}$ be a translate of a cone $\mathcal{P} \neq \emptyset$ and $U$ be a bounded open subset of $\mathcal{K}$. Assume that $A: \bar{U} \rightarrow \mathcal{K}$ is a strict set contraction and there is $w_{0} \in \mathcal{P}^{*}$ such that

$$
\begin{equation*}
x-A x \neq \lambda w_{0}, \quad \text { for all } x \in \partial U, \lambda \geq 0 \tag{3.9}
\end{equation*}
$$

Then $i(A, U, \mathcal{K})=0$.
Proof. Define the homotopy $H:[0,1] \times \bar{U} \rightarrow \mathcal{K}$ by

$$
H(t, x)=A x+t \lambda_{0} w_{0}
$$

for some

$$
\begin{equation*}
\lambda_{0}>\sup _{x \in \bar{U}}\left(\left\|w_{0}\right\|^{-1}(\|x\|+\|A x\|)\right) . \tag{3.10}
\end{equation*}
$$

Such a choice is possible since $U$ is a bounded subset and so is $A(\bar{U})$. The operator $H$ is continuous and uniformly continuous in $t$ for each $x$, and the mapping $H(t,$.$) is$ a strict set contraction for each $t \in[0,1]$. In addition, $H(t,$.$) has no fixed point on$ $\partial U$. On the contrary, there would exist some $x_{0} \in \partial U$ and $t_{0} \in[0,1]$ such that

$$
x_{0}=A x_{0}+t_{0} \lambda_{0} w_{0}
$$

contradicting the hypothesis. By [11, Theorem 1.3.5], we get

$$
\begin{equation*}
i(A, U, \mathcal{K})=i(H(0, .), U, \mathcal{K})=i(H(1, .), U, \mathcal{K})=0 \tag{3.11}
\end{equation*}
$$

Indeed, suppose that $i(H(1,), U,. \mathcal{K}) \neq 0$. Then there exists $x_{0} \in U$ such that

$$
A x_{0}+\lambda_{0} w_{0}=x_{0}
$$

which implies that $\lambda_{0} \leq\left\|w_{0}\right\|^{-1}\left(\left\|x_{0}\right\|+\left\|A x_{0}\right\|\right)$, contradicting (3.10).
Proposition 3.13. Let $U \subset \mathcal{K}$ be a bounded open subset, $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Let $u_{0} \in \mathcal{P}^{*}$ be such that

$$
\begin{equation*}
x-F x \neq T\left(x-\lambda u_{0}\right), \text { for all } x \in \partial U \cap\left(\Omega+\lambda u_{0}\right) \text { and } \lambda \geq 0 \tag{3.12}
\end{equation*}
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{K})=0$.
Proof. The mapping $T^{-1}(I-F): \bar{U} \rightarrow \mathcal{K}$ is a strict set contraction and in view of (3.12), we have

$$
x-T^{-1}(I-F) x \neq \lambda u_{0} \text { for all } x \in \partial U \text { and } \lambda \geq 0
$$

By (3.1) and Lemma 3.12, we deduce that

$$
i_{*}(T+F, U \cap \Omega, \mathcal{P})=i\left(T^{-1}(I-F), U, \mathcal{P}\right)=0
$$

The following two propositions are direct consequences of Proposition 3.13; the proofs are omitted.

Proposition 3.14. $U \subset \mathcal{K}$ be a bounded open subset and $T: \Omega \subset \mathcal{K} \rightarrow E$ be an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Suppose further that there exists $u_{0} \in \mathcal{P}^{*}$ such that $T\left(x-\lambda u_{0}\right) \in \mathcal{P}$, for all $x \in \partial U \cap\left(\Omega+\lambda u_{0}\right)$ and

$$
F x \not \leq x, \text { for all } x \in \partial U \text { and } \lambda \geq 0 .
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{K})=0$.
Proposition 3.15. Let $U \subset \mathcal{K}$ be a bounded open subset. Assume that $T: \Omega \rightarrow E$ is an expansive mapping with constant $h>1, I-F: \bar{U} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$, and $(I-F)(\bar{U}) \subset T(\Omega)$. Let $u_{0} \in \mathcal{P}^{*}$ satisfy $T\left(x-\lambda u_{0}\right) \in \mathcal{P}$, for all $x \in \partial U \cap\left(\Omega+\lambda u_{0}\right)$ and $\lambda \geq 0$. Suppose that $P$ is a normal cone with constant $N$ and the following conditions hold:

$$
F x \in \mathcal{K}, \text { and }\|F x-\theta\|>N\|x-\theta\|, \text { for all } x \in \partial U
$$

Then $i_{*}(T+F, U \cap \Omega, \mathcal{K})=0$.
Remark 3.3. (1) Letting $\theta=0$, we obtain computations of the index in case of the cone.
(2) Proposition 3.2 and Corollary 3.3 remain valid in the more general setting of $Y \cap \mathcal{B}(\theta, R)$, where $Y \subset E$ is an arbitrary closed convex subset and

$$
\mathcal{B}(\theta, R)=\{x \in E:\|x-\theta\|<R\}
$$

(3) Proposition 3.5 holds in the framework of any closed convex subset $Y$ of $E$ containing $\theta$.

## 4. FixEd point theorems of cone compression and expansion type

Some results from the previous section are next combined to establish three fixed point theorems of cone compression and expansion type. The proofs are based on the properties of the topological index $i_{*}$. We omit the details.

Theorem 4.1. (Homotopy version). Let $E$ be a Banach space, $\mathcal{P} \subset E$ a cone, and $\mathcal{K}=\mathcal{P}+\theta$ a translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ with $\theta \in \Omega$. Let $U_{1}$ and $U_{2}$ be two open subsets of $\mathcal{K}$ such that $\theta \in \overline{U_{1}} \subset U_{2}$. Let $T: \Omega \rightarrow E$ be an expansive mapping with constant $h>1$ and $I-F: \overline{U_{2}} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$ such that $(I-F)\left(\overline{U_{2}}\right) \subset T(\Omega)$. Assume that $\left(\overline{U_{2}} \backslash U_{1}\right) \cap \Omega \neq \emptyset$ and there exists $u_{0} \in \mathcal{P}^{*}$ such that either one of the following conditions holds:
(i) $x-F x \neq T\left(x-\lambda u_{0}\right)$, for all $x \in \partial U_{1} \cap\left(\Omega+\lambda u_{0}\right)$, and

$$
x-F x \neq T(\lambda x+(1-\lambda) \theta)
$$

for all $x \in \partial U_{2}, \lambda \geq 1$ and $\lambda x+(1-\lambda) \theta \in \Omega$.
(ii) $x-F x \neq T\left(x-\lambda u_{0}\right)$ for all $x \in \partial U_{2} \cap\left(\Omega+\lambda u_{0}\right)$, and

$$
x-F x \neq T(\lambda x+(1-\lambda) \theta)
$$

for all $x \in \partial U_{1}, \lambda \geq 1$ and $\lambda x+(1-\lambda) \theta \in \Omega$.
Then $T+F$ has a fixed point $x \in\left(\overline{U_{2}} \backslash U_{1}\right) \cap \Omega$.

Proof. Without loss of generality, suppose that $T x+F x \neq x$ on $\partial U_{1} \cap \Omega$ and on $\partial U_{2} \cap \Omega$, for otherwise we are finished. If condition (i) holds, by Propositions 3.5 and 3.13, we have

$$
i_{*}\left(T+F, U_{1} \cap \Omega, \mathcal{K}\right)=1 \text { and } i_{*}\left(T+F, U_{2} \cap \Omega, \mathcal{K}\right)=0
$$

The additivity property of the index yields

$$
i_{*}\left(T+F,\left(\bar{U}_{2} \backslash U_{2} \cap \Omega, \mathcal{K}\right)=-1\right.
$$

By the existence property of the index, the sum $T+F$ has at least one fixed point in the closed set $\left(\overline{U_{2}} \backslash U_{1}\right) \cap \Omega$. The proof is similar in case (ii).

Theorem 4.2. (Norm version). Let $E$ be a Banach space, $\mathcal{P} \subset E$ a normal cone with constant $N$, and $\mathcal{K}=\mathcal{P}+\theta$ a translate of $\mathcal{P}$. Let $\theta \in \Omega \subset \mathcal{K}$ and $U_{1}, U_{2}$ be two bounded open subsets of $\mathcal{K}$ such that $\theta \in \overline{U_{1}} \subset U_{2}$. Let $T: \Omega \rightarrow E$ be an expansive mapping with constant $h>1$ and $I-F: \overline{U_{2}} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$ such that $(I-F)\left(\overline{U_{2}}\right) \subset T(\Omega)$. Assume that $\left(\overline{U_{2}} \backslash U_{1}\right) \cap \Omega \neq \emptyset$ and there are $u_{0} \in$ $\mathcal{P}^{*}$ with $T\left(x-\lambda u_{0}\right) \in \mathcal{P}$, for all $\lambda \geq 0$ and $x \in \partial U_{1} \cap \partial U_{2} \cap\left(\Omega+\lambda u_{0}\right)$. Let one of the following conditions holds:
(i) $\|x-F x-T \theta\|<h\|x-\theta\|$, for all $x \in \partial U_{1} \cap \Omega$ and $F x \in \mathcal{K}$,

$$
\|F x-\theta\|>N\|x-\theta\|
$$

for all $x \in \partial U_{2}$,
(ii) $\|x-F x-T \theta\|<h\|x-\theta\|$ for all $x \in \partial U_{2} \cap \Omega$ and $F x \in \mathcal{K}$,

$$
\|F x-\theta\|>N\|x-\theta\|
$$

for all $x \in \partial U_{1}$.
Then $T+F$ has a fixed point $x \in\left(\overline{U_{2}} \backslash U_{1}\right) \cap \Omega$.
Proof. The proof uses Propositions 3.6 and 3.15.
Theorem 4.3. (Order version). Let $E$ be a Banach space, $\mathcal{P} \subset E$ a cone, and $\mathcal{K}=\mathcal{P}+\theta$ a translate of $\mathcal{P}$. Let $\Omega \subset \mathcal{K}$ with $\theta \in \Omega, \gamma, \beta>0, \gamma \neq \beta, r=\min \{\gamma, \beta\}$, and $R=\max \{$ gamma, $\beta\}$. Let $T: \Omega \rightarrow E$ be an expansive mapping with constant $h>1$ such that $\|T \theta-\theta\|<h \gamma$, and $I-F: \overline{\mathcal{K}}_{R} \rightarrow E$ be a $k$-set contraction with $0 \leq k<h$. Assume that $\mathcal{K}_{r, R} \cap \Omega \neq \emptyset$,

$$
(I-F)\left(\partial \mathcal{K}_{\gamma} \cap \Omega\right) \subset \mathcal{K}
$$

and there is

$$
u_{0} \in \mathcal{P}^{*} \text { with } T\left(x-\lambda u_{0}\right) \in \mathcal{P}, \text { for all } \lambda \geq 0, x \in \partial \mathcal{K}_{\beta} \cap\left(\Omega+\lambda u_{0}\right)
$$

If further
then $T+F$ has a fixed point $x \in \overline{\mathcal{K}}_{r, R} \cap \Omega$.
Proof. The proof uses Corollary 3.4 and Proposition 3.14.
Clearly, the following result on a cone is a particular case of Theorem 4.1.

Corollary 4.4. Let $E$ be a Banach space, $\mathcal{P} \subset E$ a cone, and $\Omega \subset \mathcal{P}$ with $0 \in \Omega$. Let $U_{1}$ and $U_{2}$ be two open subsets of $\mathcal{P}$ such that $0 \in \bar{U}_{1} \subset U_{2}$. Let $T: \Omega \rightarrow E$ be an expansive mapping with constant $h>1$ and $I-F: \bar{U}_{2} \rightarrow E$ a $k$-set contraction with $0 \leq k<h$. Assume that $\left(\bar{U}_{2} \backslash U_{1}\right) \cap \Omega \neq \emptyset$ and

$$
(I-F)\left(\bar{U}_{2}\right) \subset T(\Omega)
$$

Assume that there exists $u_{0} \in \mathcal{P}^{*}$ such that either one of the following conditions holds:
(i) $x-F x \neq T(\lambda x)$, for all $x \in \partial U_{1} \cap \lambda \geq 1$ and $\lambda x \in \Omega$, and

$$
(I-F) x \neq T\left(x-\lambda u_{0}\right),
$$

for all $x \in \partial U_{2} \cap\left(\Omega+\lambda u_{0}\right), \lambda \geq 0$,
(ii) $x-F x \neq T(\lambda x)$, for all $x \in \partial U_{2} \cap \Omega$ and $\lambda \geq 1$, and

$$
(I-F) x \neq T\left(x-\lambda u_{0}\right),
$$

for all $x \in \partial U_{1} \cap\left(\Omega+\lambda u_{0}\right), \lambda \geq 0$.
Then $T+F$ has a fixed point $x \in\left(\bar{U}_{2} \backslash U_{1}\right) \cap \Omega$.

## 5. Applications

5.1. Example 1. Consider the nonlinear equation

$$
\begin{equation*}
p(t) x^{3}(t)-x(t)=g(t, x(t)), \quad 0<t<1 \tag{5.1}
\end{equation*}
$$

where
$\left(\mathcal{H}_{1}\right) \quad p:[0,1] \rightarrow \mathbb{R}^{+}$is continuous, $g:[0,1] \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$is continuous, and for each bounded function $x$ on $[0,1]$, the superposition operator $g(\cdot, x(\cdot))$ is equicontinuous on $[0,1]$.

Let

$$
p_{1}:=\min _{0 \leq t \leq 1} p(t) \text { and } p_{2}=: \max _{0 \leq t \leq 1} p(t)
$$

Assume that
$\left(\mathcal{H}_{2}\right) \quad 1 \leq p_{1} \leq p_{2}<1+2 p_{1}$.
$\left(\mathcal{H}_{3}\right)$ There exists $R>0$ such that

$$
\begin{equation*}
p(t)-1 \leq g(t, x) \leq p_{1} R^{3}-R, \quad \forall(t, x) \in[0,1] \times[0, R+1] \tag{5.2}
\end{equation*}
$$

and

$$
\begin{equation*}
3 p_{1} R-p_{1} R^{3} \geq p_{2}-1 \tag{5.3}
\end{equation*}
$$

Remark 5.1 (Discussion of Hypothesis $\left(\mathcal{H}_{3}\right)$ ). (a) A sufficient condition for $\left(\mathcal{H}_{3}\right)$ to hold is that $g$ is uniformly bounded and

$$
\begin{equation*}
0<p_{2}-1 \leq\|g\|_{0}<\frac{3 p_{1}-1}{2} \sqrt{\frac{3 p_{1}+1}{2 p_{1}}} \tag{5.4}
\end{equation*}
$$

where $\|g\|_{0}=\sup _{0 \leq t \leq 1, x \geq 0} g(t, x)$.
To see this, let the functions $\phi(R)=3 p_{1} R-p_{1} R^{3}$ and $\psi(R)=p_{1} R^{3}-R$. Then the function $\phi$ is positive on $(0, \sqrt{3})$ and assumes $2 p_{1}$ as a maximum at the point $R=1$. The function $\psi$ is positive increasing function over $\left(\frac{1}{\sqrt{p_{1}}},+\infty\right)$. The functions $\phi$ and
$\psi$ intersect at the point $R_{0}=\sqrt{\frac{3 p_{1}+1}{2 p_{1}}}$ with $\phi\left(R_{0}\right)=\psi\left(R_{0}\right)=\frac{3 p_{1}-1}{2} \sqrt{\frac{3 p_{1}+1}{2 p_{1}}}$. As a consequence, (5.2) and (5.3) hold for all $R \in\left(R_{1}, R_{2}\right)$, where $R_{1}=\psi^{-1}\left(\|g\|_{0}\right)$ and $R_{2}=\phi^{-1}\left(p_{2}-1\right)$ (actually $1<R_{1}<R_{2}<\sqrt{3}$ ).
(b) As for the first inequality in (5.2), it suffices that it holds for $(t, x) \in[0,1] \times[0,+\infty)$.

Our main existence result is
Theorem 5.1. Under Assumptions $\left(\mathcal{H}_{1}\right)-\left(\mathcal{H}_{3}\right)$, Equation (5.1) has at least one solution $x \in C([0,1])$ such that $x(t) \geq 1$, for $0 \leq t \leq 1$.
Proof. Consider the Banach space $E=C([0,1], \mathbb{R})$ with the sup-norm

$$
\|x\|_{0}=\max _{t \in[0,1]}|x(t)| .
$$

Let the cone

$$
\mathcal{K}=\{x \in E: x(t) \geq 1\}
$$

and the conical shell

$$
\mathcal{K}_{R}=\mathcal{K} \cap \mathcal{B}(1, R)=\left\{x \in \mathcal{K}:\|x-1\|_{0}<R\right\}
$$

where $R$ is defined in $\left(\mathcal{H}_{3}\right)$. In view of Proposition 3.6, we introduce the operators $T, F: \overline{\mathcal{K}}_{R} \rightarrow E$ by

$$
(T x)(t)=x(t)-p(t) x^{3}(t)
$$

and

$$
(F x)(t)=x(t)+g(t, x(t))
$$

respectively, for $t \in[0,1]$. Then equation (5.8) is equivalent to the abstract equation $x=T x+F x$.
Step 1. (a) $T$ and $F$ clearly map $\overline{\mathcal{K}}_{R}$ into $E$. Moreover

$$
\|T x-T y\|_{0} \geq\left(3 p_{1}-1\right)\|x-y\|_{0}, \forall x, y \in \overline{\mathcal{K}}_{R}
$$

that is $T: \overline{\mathcal{K}}_{R} \rightarrow E$ is expansive with constant $h=3 p_{1}-1>1$.
(b) If $x \in \overline{\mathcal{K}}_{R}$, then $\|x-1\|_{0} \leq R$ and

$$
\begin{equation*}
\|x-F x\|_{0} \leq \sup _{0 \leq t \leq 1 ; 1 \leq u \leq 1+R} g(t, u)<+\infty \tag{5.5}
\end{equation*}
$$

which implies that $(I-F)\left(\overline{\mathcal{K}}_{R}\right)$ is uniformly bounded. $\left(\mathcal{H}_{1}\right)$ further implies that $(I-F)\left(\overline{\mathcal{K}}_{R}\right)$ is equicontinuous in $E$. By Arzéla-Ascoli Lemma, $(I-F)$ maps bounded sets of $\overline{\mathcal{K}}_{R}$ into relatively compact sets. Since $g$ is continuous, then so is $(I-F)$. Hence $I-F: \overline{\mathcal{K}}_{R} \rightarrow E$ is completely continuous, i.e., is a 0 -set contraction. (c) By (5.3), for all $x \in \partial \mathcal{K}_{R}$ and $t \in[0,1]$, we have

$$
\begin{aligned}
|x-F x(t)-T \theta(t)| & =|-g(t, x(t))+p(t)-1| \\
& \leq p_{1} R^{3}-R+p_{2}-1 \\
& \leq\left(3 p_{1}-1\right) R=h R \\
\|x-F x+T \theta\|_{0} \leq & h\|x-\theta\|_{0}, \quad \forall x \in \partial \mathcal{K}_{R} .
\end{aligned}
$$

i.e.,

Step 2. We claim that

$$
\begin{equation*}
(I-F)\left(\overline{\mathcal{K}}_{R}\right) \subset T\left(\overline{\mathcal{K}}_{R}\right) \tag{5.6}
\end{equation*}
$$

Let $y \in(I-F)\left(\overline{\mathcal{K}}_{R}\right)$ and $x \in \overline{\mathcal{K}}_{R}$ be such that $y=(I-F) x$.
(a) First we claim that

$$
\begin{equation*}
\overline{\mathcal{K}}_{R} \subset y+(I-T)\left(\overline{\mathcal{K}}_{R}\right) \tag{5.7}
\end{equation*}
$$

Let $u \in \overline{\mathcal{K}}_{R}$ and

$$
v(t)=\sqrt[3]{\frac{u(t)+g(t, x(t))}{p(t)}}, \quad t \in[0,1]
$$

Using Assumptions $\left(\mathcal{H}_{2}\right)-\left(\mathcal{H}_{3}\right)$, for all $t \in[0,1]$, we obtain the estimates

$$
1 \leq \sqrt[3]{\frac{1+g(t, x(t))}{p(t)}} \leq v(t) \leq \sqrt[3]{\frac{1+R+g(t, x(t))}{p(t)}} \leq \sqrt[3]{\frac{p_{1} R^{3}+1}{p(t)}} \leq R+1
$$

Thus, $v \in \overline{\mathcal{K}}_{R}$ and

$$
u(t)=-g(t, x(t))+p(t) v^{3}(t), \quad t \in[0,1]
$$

Since $y=x-F x=-g\left(\cdot, x(\cdot)\right.$, then $u=y+(I-T)(v)$ with $v \in \overline{\mathcal{K}}_{R}$, that is $u \in y+(I-T)\left(\overline{\mathcal{K}}_{R}\right)$, proving (5.7).
(b) To show (5.6), notice that the mapping $y+(I-T): \overline{\mathcal{K}}_{R} \rightarrow E$ is $3 p_{1}$-expansive. Owing to Lemma 2.3 with $D=\overline{\mathcal{K}}_{R}$ and using (5.7), we conclude that $y+(I-T)$ has a unique fixed point, i.e., there exists $w \in \overline{\mathcal{K}}_{R}$ such that

$$
y+(I-T)(w)=w \Longleftrightarrow y=T(w)
$$

that is $y \in T\left(\overline{\mathcal{K}}_{R}\right)$, proving (5.6). Finally, assume that $T x+F x \neq x$ on $\partial \mathcal{K}_{R}$, otherwise we are done. Letting $U=\mathcal{K}_{R}$ and $\Omega=\overline{\mathcal{K}}_{R}$ in Proposition 3.6, we obtain

$$
i_{*}\left(T+F, \mathcal{K}_{R}, \mathcal{K}\right)=1
$$

By the existence property of the index, the mapping $T+F$ has at least one positive fixed point $x$ in $\overline{\mathcal{K}}_{R}$, solution of Equation (5.1).
5.2. Example 2. Consider the nonlinear integral equation

$$
\begin{equation*}
x(t)=\int_{0}^{+\infty} G(t, s) f(s, x(s)) d s, \quad t \geq 0 \tag{5.8}
\end{equation*}
$$

where $f, G \in C\left(\mathbb{R}^{+} \times \mathbb{R}^{+}, \mathbb{R}^{+}\right)$and $\lim _{t \rightarrow+\infty} G(t, s)=\ell$, for all positive $s$. Suppose that the following conditions hold:
$\left(\mathcal{H}_{1}\right) \quad \exists p>0, p \neq 1,0 \leq f(t, x) \leq a(t)+b(t) x^{p}, \forall(t, x) \in[0,+\infty) \times[0,+\infty)$, where the coefficients $a, b \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$.
$\left(\mathcal{H}_{2}\right)$ Assume that

$$
\left\{\begin{array}{l}
M_{1}:=\sup _{t \in[0,+\infty)} \int_{0}^{+\infty} G(t, s) a(s) d s<\infty \\
M_{2}:=\sup _{t \in[0,+\infty)} \int_{0}^{+\infty} G(t, s) b(s) d s<\infty
\end{array}\right.
$$

and there exist $\varepsilon \in(0,1)$ and $R>\frac{1+\varepsilon}{2}$ such that

$$
M_{1}+M_{2} R^{p}<\frac{1+\varepsilon}{2}
$$

Remark 5.2. As example, the values $M_{1}=\frac{1}{20}, M_{2}=\frac{1}{10}, p \in \mathbb{R}, \varepsilon=\frac{1}{2}$, and $R=1$ validate the inequality in Assumption $\left(\mathcal{H}_{2}\right)$.

Theorem 5.2. Under Assumptions $\left(\mathcal{H}_{1}\right)$ and $\left(\mathcal{H}_{2}\right)$, the integral equation (5.8) has at least one positive solution $x \in C[0,+\infty)$ such that $0<x(t) \leq R, \forall t \geq 0$.

Proof. Consider the Banach space

$$
E=\left\{x \in C([0,+\infty), \mathbb{R}): \lim _{t \rightarrow+\infty} x(t) \text { exists }\right\}
$$

with norm

$$
\|x\|=\sup _{t \in[0,+\infty)}|x(t)|
$$

and the positive cone

$$
\mathcal{P}=\{x \in E: x(t) \geq 0, t \geq 0\} .
$$

Let $R_{1}=\frac{\varepsilon R+M_{1}+M_{2} R^{p}}{1+\varepsilon}$ and let $\mathcal{B}_{R}=\mathcal{B}(0, R)$ denote the open ball centered at the origin with radius $R$. Consider the open sets:

$$
\begin{aligned}
U & =\mathcal{B}_{R} \cap\left\{x \in E: x(t) \geq \frac{1+\varepsilon}{2}, \forall t \in J\right\} \\
\Omega & =\mathcal{B}_{R_{1}} \cap \mathcal{P}
\end{aligned}
$$

for some compact sub-interval $J \subset[0,+\infty)$. Since $R<\frac{1+\varepsilon}{2}$, then $U \neq \emptyset$. On $E$, define the operators

$$
\begin{aligned}
& T x(t)=(1+\varepsilon) x(t) \\
& F x(t)=(1-\varepsilon) x(t)-\int_{0}^{+\infty} G(t, s) f(s, x(s)) d s
\end{aligned}
$$

Then the integral equation (5.8) is equivalent to the operator equation $x=T x+F x$. Next, we check that all assumptions of Corollary 3.7 are satisfied. First we have $T: \Omega \rightarrow E$ and

$$
\|T x-T y\|=(1+\varepsilon)\|x-y\|,
$$

for all $x, y \in \Omega$, i.e., $T: \Omega \rightarrow E$ is an expansive operator with a constant $h=1+\varepsilon$. Step 1. We have $I-F: \bar{U} \rightarrow E$ is continuous, bounded mapping and for $x \in \bar{U}$,

$$
\begin{aligned}
\int_{0}^{+\infty} G(t, s)|f(s, x(s))| d s & \leq \int_{0}^{+\infty} G(t, s) a(s)+b(s) s(s) d s \\
& \leq M_{1}+M_{2} R^{p}<\infty
\end{aligned}
$$

Hence, by the properties of the kernel $G$, Lebesgue's dominated convergence theorem yields

$$
\begin{aligned}
& \left|\int_{0}^{+\infty} G\left(t_{1}, s\right) f(s, x(s)) d s-\int_{0}^{+\infty} G\left(t_{2}, s\right) f(s, x(s)) d s\right| \\
\leq & \int_{0}^{\infty}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| f(s, x(s)) d s
\end{aligned}
$$

which tends to 0 , uniformly in $x \in \mathcal{B}_{R}$, as $\left|t_{1}-t_{2}\right| \rightarrow 0$. Moreover

$$
\begin{aligned}
& \lim _{t \rightarrow+\infty}\left|\int_{0}^{+\infty} G(t, s) f(s, x(s)) d s-\lim _{y \rightarrow+\infty} \int_{0}^{+\infty} G(y, s) f(s, x(s)) d s\right| \\
= & \lim _{t \rightarrow+\infty}\left|\int_{0}^{+\infty} G(t, s) f(s, x(s)) d s-l\right|=0
\end{aligned}
$$

As a consequence, Corduneanu's compactness criterion lemma 2.5 assures that for all $t \in[0,+\infty)$ and every bounded subset $B \subset \bar{U}$, the set

$$
\left\{t \mapsto \int_{0}^{+\infty} G(t, s) f(s, x(s)) d s, x \in B\right\}
$$

is relatively compact. Furthermore, the operator $I-F$ is written as sum of a $\varepsilon$ contraction and a completely continuous mapping. Thus, $I-F: \bar{U} \rightarrow E$ is a $\varepsilon$-set contraction.
Step 2. Let $y \in \mathcal{B}_{R}$ be arbitrarily chosen. For $t \geq 0$, take

$$
z(t)=\frac{\varepsilon y+\int_{0}^{+\infty} G(t, s) f(s, y(s)) d s}{1+\varepsilon}
$$

Then

$$
0 \leq z(t) \leq \frac{\varepsilon R+M_{1}+M_{2} R^{p}}{1+\varepsilon}=R_{1}
$$

i.e., $z \in \Omega$ and

$$
\varepsilon y+\int_{0}^{+\infty} G(t, s) f(s, y(s)) d s=(1+\varepsilon) z(t)=T z(t), \quad t \geq 0
$$

Therefore $(I-F)(\bar{U}) \subset T(\Omega)$.
Step 3. Assume that there exist some $x_{0} \in \partial U$ and $\lambda_{0} \geq 1$ such that $\lambda_{0} x_{0} \in \Omega$ and

$$
x_{0}(t)-F x_{0}(t)=T\left(\lambda_{0} x_{0}(t)\right), \quad t \geq 0
$$

Then

$$
\left.\varepsilon x_{0}(t)+\int_{0}^{+\infty} G(t, s) f\left(s, x_{0}(s)\right) d s=\lambda_{0}(1+\varepsilon) x_{0}(t), \quad t \geq 0\right)
$$

Hence

$$
\int_{0}^{+\infty} G(t, s) f\left(s, x_{0}(s)\right) d s=\left(\lambda_{0}+\left(\lambda_{0}-1\right) \varepsilon\right) x_{0}(t)
$$

Let $t_{1} \in J$ be such that

$$
x_{0}\left(t_{1}\right) \geq \frac{1+\varepsilon}{2}
$$

Since $\lambda_{0} \geq 1$, we have the estimates

$$
\begin{aligned}
\frac{1+\varepsilon}{2} \leq x_{0}\left(t_{1}\right) & \leq\left(\lambda_{0}+\left(\lambda_{0}-1\right) \varepsilon\right) x_{0}\left(t_{1}\right) \\
& =\int_{0}^{+\infty} G\left(t_{1}, s\right) f\left(s, x_{0}(s)\right) d s \\
& \leq M_{1}+M_{2} R^{p}<\frac{1+\varepsilon}{2}
\end{aligned}
$$

which is a contradiction. By Corollary 3.7, the integral equation (5.8) has a non trivial positive solution $x$ in $\mathcal{C}([0,+\infty))$ such that $0 \leq x(t) \leq R$, for all $x \in[0,+\infty)$. This completes the proof of Theorem 5.1.

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