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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS TO THE CAPUTO-TYPE NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY VALUE CONDITIONS

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Abstract. In this article, the existence criteria of at least one or at least three positive solutions to the Caputo-type nonlinear fractional differential equation with integral boundary value conditions has been established. The method applied in this study is formulated by the well-known Guo-Krasnoselskii's fixed point theorem and Leggett-Williams fixed point theorem. First, the Green's function for corresponding linear fractional differential equation of the main nonlinear fractional differential equation under same boundary value conditions has been constructed. Next, several essential properties of that Green's function have been proved. Finally, in cone spaces some new existence and multiplicity results for the Caputo-type nonlinear fractional differential equation with integral boundary value conditions are obtained. To support the analytic proof appropriate illustrative examples has also been discussed.
Key Words and Phrases: Caputo-type nonlinear fractional differential equation, integral boundary value condition, positive solution, Guo-Krasnoselskii's fixed point theorem, Leggett-Williams fixed point theorem.
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1. Introduction

The purpose of this study is to proclaim some new existence criteria of at least one or at least three positive solutions to the following boundary value problem (for short BVP) for Caputo-type nonlinear fractional differential equation (for short NLFDE) with integral boundary value conditions (for short IBVCs):

$$\begin{cases} {}^C D_{0^+}^{\alpha} u(t) + a(t) f(t,\, u(t)) = 0,\ t \in [0,\, 1], \\ u(0) = u'(0) = 0,\ u(1) = \int_0^1 u(t) dA(t), \end{cases} \tag{1.1}$$

where, ${}^C D_{0^+}^{\alpha}$ is the Caputo fractional differential operator of order $\alpha \in (2,\, 3]$, $A(t)$ is right continuous on $[0,\, 1)$, left continuous at $t = 1$ and non-decreasing on $[0,\, 1]$ with

$A(0) = 0$, $\int_0^1 u(t)dA(t)$, denotes the Riemann-Stieltjes integral of u with respect to A and $f(t, u(t))$, $a(t)$ satisfy the following hypothesis:

(H_1) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous function.

(H_1) $a \in L^\infty[0, 1]$ is non-negative and not identically zero on any compact subset of $(0, 1)$ and $\lambda = \int_0^1 t^2 dA(t) < 1$.

The BVP given by (1.1) is a resonance problem, because $\int_0^1 u(t)dA(t) = 1$, if and only if its associated linear homogeneous BVP

$$\begin{cases} {}^C D_{0^+}^\alpha u(t) = 0, \ t \in [0, 1], \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(t)dA(t), \end{cases}$$

have non-trivial solution.

Boundary value problems (for short BVPs) for NLFDE have been addressed by several researchers during last few decades. The necessity of fractional order differential equations (for short FDEs) lies in the fact that fractional order models are more accurate than integer order models, that is, there are more degree of freedom in the fractional order models. Furthermore, fractional order derivatives provide an excellent mechanism for the description of memory and hereditary properties of various materials and processes. In applied sense, FDEs arise in various engineering and scientific disciplines for mathematical modeling in the fields of physics, chemistry, biology, electrochemistry of corrosion, fluid flow, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electromagnetic theory, electrodynamics of complex medium or polymer rheology, control theory of dynamical system, electrical network, statistics, economics, signal and image processing, viscoelasticity, aerodynamics and porous media, see for instance [13, 19, 23, 24, 29, 32] and there cited references. Consequently, the topics in FDEs are taking an important part in various applied research day by day. Some recent development of FDEs can be found in [5, 8, 9, 10, 15, 18, 26, 27, 28, 30, 31, 34, 37, 39, 40, 41].

Now a days, many researchers dedicate themselves to determine the solvability of nonlinear fractional order differential equations (for short NLFDEs) with different boundary conditions, particularly to the study of existence of positive solutions to BVPs for NLFDEs, see for instance [3, 4, 6, 7, 11, 12, 16, 18, 20, 22, 26, 27, 35, 37, 36, 39].

On the other hand, IBVCs are applied in various fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. Further, detailed description of the IBV conditions can be found in [1, 21, 33, 38] and their references. BVPs with IBVCs generate a very interesting and important class of problems. For more detailed description on BVPs with IBVCs and comments on their importance, we refer the reader to [1, 2, 20, 26, 28] and their cited references.

In current literature, several researchers have been studied the existence of positive solutions for NLFDEs with IBVCs, see for instance [2, 6, 7, 20, 26, 28, 34, 39, 41] and their cited references. Most recently, Sun *et al.* [34] studied the similar BVP as like (1.1), but they used Riemann-Liouville-type NLFDE. All above, to the best of our knowledge, there is no any work considering the existence of positive solutions to the

Caputo-type NLFDE with IBVCs given by (1.1) using Guo- Krasnoselskii's fixed point theorem (see [17]) and Leggett-Williams fixed point theorem (see [25]). Therefore, the main objective of this study is to establish the existence criteria of multiple positive solutions to the Caputo-type NLFDE with IBVCs given by (1.1), using Guo-Krasnoselskii's fixed point theorem and Leggett-Williams fixed point theorem. At the end, two illustrative examples are worked out to validate the main results. The rest of this work is furnished as follows:

In Section 2, we provide some basic definitions, lemmas and state Guo-Krasnoselskii's fixed point theorem and Leggett-Williams fixed point theorem. Section 3 is used to state and prove our main results, which provide the techniques to check the existence of at least one or at least three positive solutions of Caputo-type NLFDE with IBVCs given by (1.1). Finally, in Section 4, we give two examples to explain our main results.

2. Preliminary notes

In this section, we introduce some necessary definitions and preliminary facts which will be used throughout this paper.

Among the different definitions of fractional derivative and fractional integration the most frequent used definitions are Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative [13, 23, 29].

Definition 2.1. (see [23, 29]). For a continuous function $f : (0, \infty) \to \mathbf{R}$ (set of real numbers), the Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ t > 0,$$

where $\Gamma(\alpha)$ is the Gamma function of α and provided that the integral exists.

Definition 2.2. (see [23, 29]) For a continuous function $f : (0, \infty) \to \mathbf{R}$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α and provided that the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. (see [23, 29]) For a continuous function $f : (0, \infty) \to \mathbf{R}$, the Caputo fractional derivative of order $\alpha > 0$ is given by

$${}^{C}D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,\ n-1 < \alpha \le n,\ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α and provided that the right-hand side is point-wise defined on $(0, \infty)$.

In general, the Caputo fractional derivative and the Riemann-Liouville fractional derivative of a function are not same. In particular, the solution space of Caputo-type fractional differential equation ${}^{C}D_{0^+}^{\alpha} u(t) = 0$ is spanned by $\{1, t, t^2, \cdots, t^{n-1}\}$, while the solution space of Riemann-Liouville-type fractional differential equation $D_{0^+}^{\alpha} u(t) = 0$ is spanned by $\{t^{\alpha-1}, t^{\alpha-2}, \cdots, t^{\alpha-n}\}$. Regarding this concept, the

Green's functions of BVP associated to these two kinds of fractional differential equations are different and the Green's function is one of the major tools for studying the BVP of fractional differential equations.

Lemma 2.1. (see [23, 29]) *Let $n-1<\alpha\leq n$, $n\in \boldsymbol{N}$ (set of natural numbers), $u\in C^n[0,\,1]$. Then*

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

for some $c_i\in\boldsymbol{R}$, $i=0,\,1,\,2,\,\cdots,\,n-1,\ n=[\alpha]+1$.

Lemma 2.2. (see [23, 29]) *Let $\alpha>0$ and $y\in C[a,\,b]$. Then holds*

$$\left({}^{C}D_{0+}^{\alpha}I_{0+}^{\alpha}y\right)(t)=y(t)$$

holds on $[a,\,b]$.

Lemma 2.3. (see [23, 29]) *Let $\alpha>\gamma>0$. If one assumes that*

$$u(t)\in C(0,\,1)\cap L^{\infty}(0,\,1),$$

then

$${}^{C}D_{0+}^{\gamma}I_{0+}^{\alpha}u(t)=I_{0+}^{\alpha-\gamma}u(t).$$

Lemma 2.4. (see [23, 29]) *Let $n-1<\alpha\leq n,\ n\in\boldsymbol{N}$. Then the fractional differential equation*

$${}^{C}D_{0+}^{\alpha}u(t)=0$$

has a solution

$$u(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},$$

for some $c_i\in\boldsymbol{R}$, $i=0,\,1,\,2,\,\cdots,\,n-1,\ n=[\alpha]+1$.

Lemma 2.5. *Let $2<\alpha\leq 3$ and $h(t)\in C[0,\,1]$, for all $t\in(0,\,1)$. Then the following BVP*

$$\begin{cases}{}^{C}D_{0+}^{\alpha}u(t)+h(t)=0,\ t\in[0,\,1],\\ u(0)=u'(0)=0,\ u(1)=\displaystyle\int_0^1 u(t)dA(t),\end{cases}\tag{2.1}$$

has a unique solution

$$u(t)=\int_0^1 G_1(t,\,s)h(s)ds,$$

where,

$$G_1(t,\,s)=G(t,\,s)+\frac{t^2}{1-\lambda}\int_0^1 G(r,\,s)dA(r),\tag{2.2}$$

and

$$G(t,\,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^2(1-s)^{\alpha-1}-(t-s)^{\alpha-1}; & 0\leq s\leq t\leq 1,\\ t^2(1-s)^{\alpha-1}; & 0\leq t\leq s\leq 1.\end{cases}\tag{2.3}$$

Proof. In view of Lemma 2.1, the BVP given by (2.1) is equivalent to the integral equation

$$u(t)=-I_{0+}^{\alpha}h(t)+c_0+c_1t+c_2t^2,\tag{2.4}$$

for some $c_i\in\mathbf{R}$, $i=0,\,1,\,2$. So, we have

$$u'(t)=-I_{0+}^{\alpha-1}h(t)+c_1+2c_2t,\tag{2.5}$$

From the boundary conditions $u(0) = u'(0) = 0,\ u(1) = \int_0^1 u(t)dA(t)$ and Definition 2.1, we have

$$c_0 = 0,\ c_1 = 0,\ and\ c_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \int_0^1 u(s)dA(s).$$

Thus, we get

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + t^2 \left[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \int_0^1 u(s)dA(s) \right] \\
&= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t t^2(1-s)^{\alpha-1} h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^1 t^2(1-s)^{\alpha-1} h(s)ds + t^2 \int_0^1 u(s)dA(s) \\
&= \int_0^1 G(t,\ s)h(s)ds + t^2 \int_0^1 u(s)dA(s), \qquad (2.6)
\end{aligned}$$

where

$$G(t,\ s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^2(1-s)^{\alpha-1} - (t-s)^{\alpha-1}; & 0 \le s \le t \le 1, \\ t^2(1-s)^{\alpha-1}; & 0 \le t \le s \le 1. \end{cases}$$

Then, we yield that

$$\begin{aligned}
\int_0^1 u(s)dA(s) &= \int_0^1 \left[\int_0^1 G(s,\ r)h(r)dr + t^2 \int_0^1 u(r)dA(r) \right] dA(s) \\
&= \int_0^1 \int_0^1 G(s,\ r)h(r)drdA(s) + \int_0^1 u(s)dA(s) \cdot \int_0^1 t^2 dA(t),
\end{aligned}$$

That is,

$$\int_0^1 u(s)dA(s) = \frac{\int_0^1 \int_0^1 G(s,\ r)h(r)drdA(s)}{1 - \int_0^1 t^2 dA(t)}. \qquad (2.7)$$

Therefore, the unique solution of the BVP given by (2.1) is

$$\begin{aligned}
u(t) &= \int_0^1 G(t,\ s)h(s)ds + t^2 \frac{\int_0^1 \int_0^1 G(s,\ r)h(r)drdA(s)}{1 - \int_0^1 t^2 dA(t)} \\
&= \int_0^1 \left[G(t,\ s) + \frac{t^2}{1-\lambda} \int_0^1 G(r,\ s)h(r)dA(r) \right] h(s)ds \qquad (2.8) \\
&= \int_0^1 G_1(t,\ s)h(s)ds,
\end{aligned}$$

where $\lambda = \int_0^1 t^2 dA(t)$.

This completes the proof.

Lemma 2.6. *The Green function $G(t,\ s)$ defined as (2.3) satisfies the following conditions:*

(i) $G(t,\ s) \in C([0,\ 1] \times [0,\ 1])$ and $G(t,\ s) \ge 0$, for all $t,\ s \in [0,\ 1]$,

(ii) $q(1-t)q(s) \le \Gamma(\alpha)G(t,\ s) \le (\alpha-1)q(s)$, for all $t,\ s \in [0,\ 1]$ and $q(t) = t(1-t)^{\alpha-1}$,

(iii) $\dfrac{G(t,\, s)}{t}$ *is continuous for all* $t \in (0,\, 1]$ *and* $s \in [0,\, 1]$.

Proof. (i) It is obvious that $G(t,\, s)$ is continuous on $[0,\, 1] \times [0,\, 1]$, hence

$$G(t,\, s) \in C([0,\, 1] \times [0,\, 1]).$$

For $0 \le s \le t \le 1$ and $\alpha \in (2,\, 3]$, we have

$$G(t,\, s) = \frac{1}{\Gamma(\alpha)} \left[t^2(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \ge 0.$$

And for $0 \le t \le s \le 1$, we have

$$G(t,\, s) = \frac{t^2(1-s)^{\alpha-1}}{\Gamma(\alpha)} \ge 0.$$

Hence, $G(t,\, s) \ge 0$, for all $t,\, s \in [0,\, 1]$.

(ii) For $0 \le s \le t \le 1$, we have

$$\Gamma(\alpha)G(t,\, s) = t^2(1-s)^{\alpha-1} - (t-s)^{\alpha-1}.$$

Since $1 < \alpha - 1 \le 2$, so we have

$$\begin{aligned}
\Gamma(\alpha)G(t,\, s) &\le t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (\alpha-1)\int_{t-s}^{t(1-s)} x^{\alpha-2}dx \\
&\le (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}[t(1-s)-(t-s)] \\
&\le (\alpha-1)(1-s)^{\alpha-2}s(1-t) \\
&\le (\alpha-1)(1-s)^{\alpha-1}s = (\alpha-1)q(s).
\end{aligned}$$

On the other hand, since $2 < \alpha \le 3$, we have

$$\begin{aligned}
\Gamma(\alpha)G(t,\, s) &\ge t^{\alpha}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \ge t \cdot \left[t^{\alpha}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\
&\ge t \cdot \left[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] \\
&\ge t \cdot [t(1-s)]^{\alpha-2}\left[t(1-s)-(t-s)\right] \\
&= t \cdot t^{\alpha-2}(1-s)^{\alpha-2}s(1-t) = t^{\alpha-1}(1-s)^{\alpha-2}s(1-t) \\
&\ge t^{\alpha-1}(1-s)^{\alpha-1}s(1-t) = q(1-t)q(s).
\end{aligned}$$

Hence the inequality $q(1-t)q(s) \le \Gamma(\alpha)G(t,\, s) \le (\alpha-1)q(s)$ holds for $0 \le s \le t \le 1$.

Now, for $0 \le t \le s \le 1$, we have

$$\begin{aligned}
\Gamma(\alpha)G(t,\, s) &= t^2(1-s)^{\alpha-1} \\
&\le t(1-s)^{\alpha-1} \le s(1-s)^{\alpha-1} \\
&\le (\alpha-1)s(1-s)^{\alpha-1} = (\alpha-1)q(s).
\end{aligned}$$

On the other hand, since $2 < \alpha \le 3$, we have

$$\begin{aligned}
\Gamma(\alpha)G(t,\, s) &= t^2(1-s)^{\alpha-1} \\
&\ge t^{\alpha}(1-s)^{\alpha-1} = t \cdot t^{\alpha-1}(1-s)^{\alpha-1} \\
&\ge (1-t)s \cdot t^{\alpha-1}(1-s)^{\alpha-1} = q(1-t)q(s).
\end{aligned}$$

Hence the inequality $q(1-t)q(s) \le \Gamma(\alpha)G(t,\, s) \le (\alpha-1)q(s)$ holds for $0 \le t \le s \le 1$.

(iii) Since $G(t,\, s)$ is continuous for all $t \in (0,\, 1]$ and $s \in [0,\, 1]$, then it is obvious that $\dfrac{G(t,\, s)}{t}$ is continuous for all $t \in (0,\, 1]$ and $s \in [0,\, 1]$.

This completes the proof.

Lemma 2.7. *If (H_1) is satisfied, then $G_1(t,\, s)$ defined as (2.2) satisfies the following conditions:*

(i) $G_1(t,\, s)$ and $\dfrac{G_1(t,\, s)}{t}$ are two continuous functions for all $t,\, s \in [0,\, 1]$,

(ii) $q(1-t)q(s) \leq \Gamma(\alpha)G_1(t,\, s) \leq Rq(s)$, for all $t,\, s \in [0,\, 1]$ and $q(t) = t(1-t)^{\alpha-1}$. where $R = (\alpha - 1)\left(1 + \dfrac{\int_0^1 dA(s)}{(1-\lambda)}\right)$.

Proof. The proof is easily obtainable by Lemma 2.6. So, here we omit it.

Definition 2.4. (see [17]) Let B be a real Banach space and P be a nonempty closed convex subset of B. Then we say that P is a cone on B if it satisfies the following properties:

(i) $\xi c \in P$ for $c \in P$, $\xi \geq 0$,

(ii) $c,\, -c \in P$ implies $c = \theta$,

where θ denotes the null element of B.

Let $B = C[0,\, 1]$ be a Banach space with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and define a cone $P \subseteq B$ as

$$P = \{u \in B : u(t) \geq 0,\, t \in [0,\, 1]\}.$$

Remark 2.1. By Lemma 2.5, we can convert the BVP given by (1.1) as the following integral equation

$$u(t) = \int_0^1 G_1(t,\, s)a(s)f(s,\, u(s))ds,\ \forall\, t \in [0,\, 1], \tag{2.9}$$

where $G_1(t,\, s)$ is given by (2.2).

Obviously, $u \in [0,\, 1] \times [0,\, 1]$ is a solution of the BVP given by (1.1), if and only if it is a solution of the integral equation (2.9). Furthermore, if we consider a cone P on $B = C[0,\, 1]$ and define an integral operator $T : P \to P$ by

$$Tu(t) = \int_0^1 G_1(t,\, s)a(s)f(s,\, u(s))ds,\ \forall\, u \in P, \tag{2.10}$$

then it is easy to see that the BVP given by (1.1) has a solution u if and only if u is a fixed point of the operator T defined by (2.10).

Lemma 2.8. *If (H_1) and (H_2) are satisfied, then the operator $T : P \to P$ given by (2.10) is completely continuous.*

Proof. From the continuity and non-negativity of $G(t,\, s)$, $G_1(t,\, s)$, $f(t,\, u(t))$ and $a(t)$ on their domain of definition, we have that if $u \in P$ then $Tu \in B$ and $Tu(t) \geq 0$ for all $t \in [0,\, 1]$.

Moreover,

$$\frac{Tu(t)}{t} = \int_0^1 \frac{G_1(t,\, s)}{t} a(s)f(s,\, u(s))ds,$$

and according to the Lemma 2.7, this is continuous for all $u \in B$.

Now, let $u \in P$, then for all $t \in [0, 1]$ we have

$$Tu(t) = \int_0^1 G_1(t, s)a(s)f(s, u(s))ds \geq \frac{q(1-t)}{\Gamma(\alpha)} \int_0^1 q(s)a(s)f(s, u(s))ds. \tag{2.11}$$

On the other hand, we have

$$\|Tu(t)\| = \max_{0\leq t\leq 1} |Tu(t)| \leq \frac{R}{\Gamma(\alpha)} \left| \int_0^1 q(s)a(s)f(s, u(s))ds \right|. \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$Tu(t) \geq \frac{q(1-t)}{R} \|Tu(t)\|.$$

which implies that the operator $T : P \to P$ is continuous.

Let $\Psi \subset P$ be bounded, that is there exists a positive constant $m > 0$ such that $\|u\| \leq m$ for all $u \in \Psi$.

Now, if we set $L = \max_{0\leq t\leq 1,\, 0\leq u\leq m} |a(t)f(t, u(t))|$, then for all $u \in \Psi$ we have

$$|Tu(t)| \leq \int_0^1 G_1(t, s)|a(s)f(s, u(s))|ds \leq L \int_0^1 G_1(t, s)ds, \ \forall\, t \in [0, 1],$$

that is, $T(\Psi)$ is bounded in B.

For each $u \in \Psi$, we have

$$\begin{aligned}
|(Tu)'(t)| = |&-\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} a(s)f(s, u(s))ds \\
&+ 2t\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s)f(s, u(s))ds + 2t \int_0^1 u(s)dA(s)| \\
\leq& \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |a(s)f(s, u(s))|ds \\
&+ \frac{2}{\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-1} |a(s)f(s, u(s))|ds + 2\int_0^1 t|u(s)|dA(s) \\
\leq& \frac{L}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} ds + \frac{2L}{\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-1} ds + 2\|u\| \int_0^1 t dA(s) \\
\leq& \frac{L}{\Gamma(\alpha-1)} + \frac{2L}{\Gamma(\alpha)} + 2m = L_1(say).
\end{aligned}$$

Consequently, for all $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$|Tu(t_1) - Tu(t_2)| \leq \int_{t_1}^{t_2} |(Tu)'(s)|ds \leq L_1(t_1 - t_2).$$

Hence, $T(\Psi)$ is equicontinuous.

Now, by an application of Arzela-Ascoli theorem (see [14]) we conclude that $\overline{T(\Psi)}$ is compact, that is the operator $T : P \to P$ is completely continuous.

This completes the proof.

Now, we list the following fixed point theorems which will be used as the tools to establish our main results.

Theorem 2.1. (Guo-Krasnoselskii's fixed point theorem) (see [17]) *Let B be a Banach space and $P \subseteq B$ be a cone in B. Suppose that Ψ_1 and Ψ_2 are two bounded open subsets of B with $0 \in \Psi_1$ and $\overline{\Psi_1} \subset \Psi_2$. Assume that $T : P \cap (\overline{\Psi_2} \setminus \Psi_1) \to P$ is a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|, \forall\, u \in P \cap \partial\Psi_1$ and $\|Tu\| \ge \|u\|, \forall\, u \in P \cap \partial\Psi_2$

or

(ii) $\|Tu\| \ge \|u\|, \forall\, u \in P \cap \partial\Psi_1$ and $\|Tu\| \le \|u\|, \forall\, u \in P \cap \partial\Psi_2$

Then T has at least one fixed point and $u^* \in P \cap (\overline{\Psi_2} \setminus \Psi_1)$ and $u^* > 0$.

Before, stating Leggett-Williams fixed point theorem, we define continuous concave functional.

Definition 2.5. (see [25]) A mapping β is said to be a non-negative continuous concave functional on a cone P if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(\delta v + (1-\delta)w) \ge \delta\beta(v) + (1-\delta)\beta(w),\ v,\, w \in P,\, \delta \in [0,\, 1].$$

Theorem 2.2. (Leggett-Williams fixed point theorem) (see [25]) *Let P be a cone in a real Banach space B and*

$$P_{d_4} = \{u \in P : \|u\| < d_4\}.$$

Assume that $T : \overline{P_{d_4}} \to \overline{P_{d_4}}$ is a completely continuous operator, β is a non-negative continuous concave functional on P such that $\beta(u) \le \|u\|$ for all $u \in \overline{P_{d_4}}$ and

$$P(\beta,\, d_2,\, d_3) = \{u \in P : \beta(u) \ge d_2,\, \|u\| \le d_3\}.$$

Suppose that there exist constants $0 < d_1 < d_2 < d_3 \le d_4$ such that the following conditions are satisfied:

(i) $\{u \in P(\beta,\, d_2,\, d_3) : \beta(u) > d_2\} \ne \emptyset$ and $\beta(Tu) > d_2$ for $u \in P(\beta,\, d_2,\, d_3)$;

(ii) $\|Tu\| < d_1$ for $\|u\| < d_1$;

(iii) $\beta(Tu) > d_2$ for any $u \in P(\beta,\, d_2,\, d_4)$ with $\|Tu\| > d_3$.

Then T has at least three fixed points $u_1,\, u_2,\, u_3$ in $\overline{P_{d_4}}$ satisfying

$$\|u_1\| < d_1,\, d_2 < \beta(u_2),\, \|u_3\| > d_1 \, with \, \beta(u_3) < d_2.$$

Remark 2.2. If $d_3 = d_4$, then the condition (i) of Theorem 2.2 implies the condition (iii) of that theorem.

3. Main results

This section is devoted to establish the existence criteria of at least one or at least three positive solutions of the BVP given by (1.1).

Let the non-negative continuous concave functional β on the cone P be defined by

$$\beta(u) = \min_{t \in [\eta,\, 1-\eta]} |u(t)|. \tag{3.1}$$

Throughout this paper, we suppose that

$$g = \min_{\eta \le t \le 1-\eta} q(1-t),\ where\, \eta \in \left(0,\, \frac{1}{2}\right), \tag{3.2}$$

$$M_1 = \frac{\Gamma(\alpha)}{R|\int_0^1 q(s)a(s)ds|}, \tag{3.3}$$

$$M_2 = \frac{\Gamma(\alpha)}{g\left|\int_\eta^{1-\eta} q(s)a(s)ds\right|} \tag{3.4}$$

We are now in position to present and prove our main results.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *are hold. Suppose that there exist two positive constants* c_1 *and* c_2 *with* $c_2 > c_1 > 0$ *such that the following assumptions hold:*

(A_1) $f(t, u) \leq M_1 c_2,\ (t, u) \in [0, 1] \times [0, c_2]$;

(A_2) $f(t, u) \geq M_2 c_1,\ (t, u) \in [0, 1] \times [0, c_1]$.

Then the BVP given by (1.1) has least one positive solution $u \in P$ *such that*

$$c_1 \leq \|u\| \leq c_2.$$

Proof. According to the Remark 2.1, we know that the solution of the BVP given by (1.1) is equivalent to the fixed point of the integral operator T defined by (2.10) and by Lemma 2.8, we know that the operator T defined by (2.10) is completely continuous.

Now, if we let $\Psi_2 = \{u \in P : \|u\| \leq c_2\}$, then for any $u \in P \cap \partial\Psi_2$, we have $0 \leq u(t) \leq c_2$, for all $t \in [0, 1]$. It follows from assumption (A_1), Lemma 2.7 and (3.3) that, for all $t \in [0, 1]$, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,\,1]}\left|\int_0^1 G_1(t, s)a(s)f(s, u(s))ds\right| \leq M_1c_2 \max_{t\in[0,\,1]}\left|\int_0^1 G_1(t, s)a(s)ds\right| \\ &\leq \frac{M_1c_2R}{\Gamma(\alpha)}\left|\int_0^1 q(s)a(s)ds\right| = M_1c_2\cdot\frac{1}{\dfrac{\Gamma(\alpha)}{R|\int_0^1 q(s)a(s)ds|}} = c_2 = \|u\|.\end{aligned} \tag{3.5}$$

This implies that $\|Tu\| \leq \|u\|,\ \forall\, u \in P \cap \partial\Psi_2$. Hence, the condition (i) of Theorem 2.1 is satisfied.

On the other hand, if we Let $\Psi_1 = \{u \in P : \|u\| \leq c_1\}$, then for any $u \in P \cap \partial\Psi_1$, we have $0 \leq u(t) \leq c_2$ for all $t \in [0, 1]$. It follows from assumption (A_2), Lemma 2.7, (3.2) and (3.4) that, for all $t \in [0, 1]$, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,\,1]}\left|\int_0^1 G_1(t, s)a(s)f(s, u(s))ds\right| \geq \max_{t\in[\eta,\,1-\eta]}\left|\int_0^1 G_1(t, s)a(s)f(s, u(s))ds\right| \\ &\geq \frac{M_2c_1}{\Gamma(\alpha)}\max_{t\in[\eta,\,1-\eta]} q(1-t)\left|\int_0^1 q(s)a(s)ds\right| \\ &\geq \frac{M_2c_1}{\Gamma(\alpha)}\min_{t\in[\eta,\,1-\eta]} q(1-t)\left|\int_0^1 q(s)a(s)ds\right| \\ &\geq \frac{M_2c_1g}{\Gamma(\alpha)}\left|\int_\eta^{1-\eta} q(s)a(s)ds\right| \\ &= M_2c_1\cdot\frac{1}{\dfrac{\Gamma(\alpha)}{g|\int_\eta^{1-\eta} q(s)a(s)ds|}} = c_1 = \|u\|.\end{aligned} \tag{3.6}$$

This implies that $\|Tu\| \geq \|u\|,\ \forall\, u \in P \cap \partial\Psi_1$. Hence, the condition (ii) of Theorem 2.1 is satisfied.

Thus, in view of Theorem 2.1, the integral operator T has at least one fixed point in $P \cap (\overline{\Psi_2} \setminus \Psi_1)$, which is positive and this means that the BVP given by (1.1) has at least one positive solution $u \in P \cap (\overline{\Psi_2} \setminus \Psi_1)$, where $c_1 \leq \|u\| \leq c_2$.

This completes the proof.

Theorem 3.2. *Assume that* (H_1) *and* (H_2) *are hold. Suppose that there exist four positive constants* d_1, d_2, d_3 *and* d_4 *with* $0 < d_1 < d_2 < \left(\frac{g}{R}\right) d_3 < d_3 < d_4$ *such that the following assumptions hold:*

(A_3) $\quad f(t,\, u) < M_1 d_1,\ (t,\, u) \in [0,\, 1] \times [0,\, d_1]$;

(A_4) $\quad f(t,\, u) > M_2 d_2,\ (t,\, u) \in [\eta,\, 1-\eta] \times [d_2,\, d_3]$;

(A_5) $\quad f(t,\, u) \leq M_1 d_4,\ (t,\, u) \in [0,\, 1] \times [0,\, d_4]$.

Then the BVP given by (1.1) has least three positive solutions u_1, u_2, u_3 *such that*

$$\begin{aligned}
&\max_{t\in[0,\,1]} |u_1(t)| < d_1, \quad d_2 < \min_{t\in[\eta,\,1-\eta]} |u_2(t)| < \max_{t\in[0,\,1]} |u_2(t)| < d_4,\\
&d_1 < \max_{t\in[0,\,1]} |u_3(t)| \leq d_4, \quad \min_{t\in[\eta,\,1-\eta]} |u_3(t)| < d_2.
\end{aligned} \tag{3.7}$$

Proof. To prove this theorem, we have to show that all the conditions of Theorem 2.2 are satisfied by the given assumptions.

First, if we let

$$u \in \overline{P_{d_4}} = \{u \in P : \|u\| \leq d_4\}$$

then $\|u\| \leq d_4$. From assumption (A_5) Lemma 2.7 and (3.3), we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,\,1]} \left| \int_0^1 G_1(t,\, s) a(s) f(s,\, u(s)) ds \right|\\
&\leq M_1 d_4 \max_{t\in[0,\,1]} \left| \int_0^1 G_1(t,\, s) a(s) ds \right|\\
&\leq \frac{M_1 d_4 R}{\Gamma(\alpha)} \left| \int_0^1 q(s) a(s) ds \right|\\
&= M_1 d_4 \cdot \frac{1}{\dfrac{\Gamma(\alpha)}{R|\int_0^1 q(s)a(s)ds|}} = d_4,
\end{aligned} \tag{3.8}$$

that is $\|Tu\| \leq d_4$ for $u \in \overline{P_{d_4}}$. Hence, $T\left(\overline{P_{d_4}}\right) \subseteq \overline{P_{d_4}}$. Thus, from Lemma 2.8, we have $T : \overline{P_{d_4}} \to \overline{P_{d_4}}$ is completely continuous.

Next, by using the similar argument, it follows from assumption (A_3) that $\|Tu\| \leq d_1$ for $u \in \overline{P_{d_4}}$, and this proves that the condition (ii) of Theorem 2.2 is satisfied.

Now, by setting $u(t) = \frac{(d_2+d_3)}{2}$ for $t \in [0,\, 1]$ we have $u(t) \in P(\beta,\, d_2,\, d_3)$ and

$$\{u \in P(\beta,\, d_2,\, d_3) : \beta(u) > d_2\} \neq \emptyset.$$

Hence if $u \in P(\beta,\, d_2,\, d_3)$, then $d_2 \leq u(t) \leq d_3$ for $t \in [\eta,\, 1-\eta]$.

By assumption (A_4), we have $f(t,\, u(t)) > M_2 d_2$ for $t \in [\eta,\, 1-\eta]$. So, from Lemma 2.7, (3.1) and (3.4), we have

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\eta,\,1-\eta]} |Tu(t)| = \min_{t\in[\eta,\,1-\eta]} \left| \int_0^1 G_1(t,\, s)a(s)f(s,\, u(s))ds \right| \\
&\geq \min_{t\in[\eta,\,1-\eta]} \frac{q(1-t)}{\Gamma(\alpha)} \left| \int_0^1 q(s)a(s)f(s,\, u(s))ds \right| \\
&> \frac{M_2 d_2}{\Gamma(\alpha)} \min_{t\in[\eta,\,1-\eta]} q(1-t) \left| \int_0^1 q(s)a(s)ds \right| \\
&\geq \frac{M_2 d_2 g}{\Gamma(\alpha)} \left| \int_\eta^{1-\eta} q(s)a(s)ds \right| = d_2,
\end{aligned} \tag{3.9}$$

that is $\beta(Tu) > d_2$ for $u \in P(\beta,\, d_2,\, d_3)$. This proves that the condition (i) of Theorem 2.2 is satisfied.

On the other hand, since

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,\,1]} \left| \int_0^1 G_1(t,\, s)a(s)f(s,\, u(s))ds \right| \\
&\leq \frac{R}{\Gamma(\alpha)} \left| \int_0^1 q(s)a(s)f(s,\, u(s))ds \right|,
\end{aligned} \tag{3.10}$$

hence by Lemma 2.7, we have

$$\begin{aligned}
|Tu(t)| &= \left| \int_0^1 G_1(t,\, s)a(s)f(s,\, u(s))ds \right| \\
&\geq \frac{q(1-t)}{\Gamma(\alpha)} \left| \int_0^1 q(s)a(s)f(s,\, u(s))ds \right| \\
&= \frac{q(1-t)}{R} \|Tu\|.
\end{aligned} \tag{3.11}$$

So, if we consider $u \in P(\beta,\, d_2,\, d_4)$ with $\|Tu\| > d_3$, then using (3.2) and (3.11), we obtain that

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[\eta,\,1-\eta]} |Tu(t)| \\
&\geq \frac{\min_{t\in[\eta,\,1-\eta]} q(1-t)}{R} \|Tu\| \\
&> \left(\frac{g}{R}\right) d_3 > d_2
\end{aligned} \tag{3.12}$$

and this proves that the condition (iii) of Theorem 2.2 is satisfied.

Therefore, in view of Theorem 2.2, the integral operator T has at least three fixed points $u_1,\, u_2,\, u_3$ which are positive and this means that the BVP given by (1.1) has at least three positive solutions satisfying (3.7).

This completes the proof.

4. Examples

In this section, we provide two illustrative examples to support the analytic proof of the Theorem 3.1 and Theorem 3.2.

Example 4.1. Consider the following BVP for Caputo-type NLFDE with IBVCs:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{2}}u(t) + a(t)f(t,\, u(t)) = 0, \ t \in [0,\, 1], \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(t)dA(t), \end{cases} \tag{4.1}$$

where $f(t,\, u(t)) = \frac{1}{4}\tan t + u(t) + 4$, $a(t) = 1$, $u(t) = t$ and $A(t) = t$.

Now, by taking $\eta = \frac{1}{3}$, we have

$$g = 0.128,\ R = 3.75,\ M_1 = \frac{7}{4}\sqrt{\pi} \approx 3.10,\ M_2 \approx 183.84.$$

Then for any $(t,\, u) \in [0,\, 1] \times [0,\, 2]$, we have

$$f(t,\, u(t)) = \frac{1}{4}\tan t + t + 4 \leq 5.00 \leq M_1 c_2 \approx 6.2,$$

and for any $(t,\, u) \in [0,\, 1] \times [0,\, 0.02]$, we have

$$f(t,\, u(t)) = \frac{1}{4}\tan t + t + 4 \geq 4 \geq M_2 c_1 \approx 3.68.$$

That is all the assumptions of Theorem 3.1 are satisfied.

Therefore, the BVP given by (4.1) has at least one positive solution $u \in P$ such that $0.02 \leq \|u\| \leq 2$.

Example 4.2. Consider the following BVP for Caputo-type NLFDE with IBVCs:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{5}{2}}u(t) + a(t)f(t,\, u(t)) = 0, \ t \in [0,\, 1], \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 u(t)dA(t), \end{cases} \tag{4.2}$$

where $f(t,\, u(t)) = \begin{cases} \frac{t}{4} + u^2, & 0 < u \leq 1, \\ 183 + \frac{t}{4} + u, & u > 1, \end{cases}$ $a(t) = 1$, $u(t) = t$ and $A(t) = t$.

By taking $\eta = \frac{1}{3}$, we have

$$g = 0.128,\ R = 3.75,\ M_1 = \frac{7}{4}\sqrt{\pi} \approx 3.10,\ M_2 \approx 183.84.$$

Now, if we let $d_1 = 0.5$, $d_2 = 1$, $d_3 = 35$, $d_4 = 100$, then for any $(t,\, u) \in [0,\, 1] \times [0,\, \frac{1}{2}]$, we have

$$f(t,\, u(t)) = \frac{t}{4} + u^2 \leq 0.50 < M_1 d_1 \approx 1.55,$$

for any $(t,\, u) \in [\frac{1}{3},\, \frac{2}{3}] \times [1,\, 35]$, we have

$$f(t,\, u(t)) = 183 + \frac{t}{4} + u \geq 184.08 > M_2 d_2 \approx 184.48,$$

and for any $(t,\, u) \in [0,\, 1] \times [0,\, 100]$, we have

$$f(t,\, u(t)) = 183 + \frac{t}{4} + u \leq 283.25 < M_1 d_4 \approx 310.$$

That is all the assumptions of Theorem 3.2 are satisfied.

Therefore, the BVP given by (4.2) has at least three positive solutions u_1, u_2, u_3 satisfying (3.7).

5. Conclusion

In this article, we have proven two new existence criteria for positive solution of the Caputo-type NLFDE with IBVCs given by (1.1) by applying Guo-Krasnoselskii's fixed point theorem and Leggett-Williams fixed point theorem. Using the Theorem 3.1, one can easily be checked the existence of at least one positive solution of BVP given by (1.1). And the Theorem 3.2 will be used to check the existence of at least three positive solutions of BVP given by (1.1). The established results of this article provided an easy and straightforward technique to cheek the existence of positive solutions to the Caputo-type NLFDE with IBVCs given by (1.1). Furthermore, the results of this research extend the corresponding results of Cabada and Wang [6] and Caballero *et al.* [7], Ma *et al.* [28], Ma [27] and Sun *et al.* [34].

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