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ON SYSTEM OF SPLIT GENERALISED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS FOR MULTIVALUED MAPPINGS WITH NO PRIOR KNOWLEDGE OF OPERATOR NORM

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Abstract. In this paper, we introduce the System of Split Generalized Mixed Equilibrium Problem (SSGMEP), which is more general than the existing well known split equilibrium problem and its generalizations, split variational inequality problem and several other related problems. We propose a new iterative algorithm of inertial form which is independent on the operator norm for solving SSGMEP in real Hilbert spaces. Motivated by the adaptive step size technique and inertial method, we incorporate self adaptive step size and inertial technique to overcome the difficulty of having to compute the operator norm and to accelerate the convergence of the proposed method. Under standard and mild assumptions on the control sequences, we establish the strong convergence of the algorithm, obtain a common solution of the SSGMEP and fixed point of finite family of multivalued demicontractive mappings. We obtain some consequent results which complement several existing results in this direction in the literature. We also apply our results to finding solution of split convex minimisation problems. Numerical example is presented to illustrate the performance of our method as well as comparing it with its non-inertial version.
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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $||\cdot||$, and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a nonlinear bifunction, $P : C \to H$ a nonlinear mapping, and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function. The *generalized Mixed Equilibrium Problem* (*GMEP*) (see [29, 49]) is to find a point $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle P\hat{x}, y - \hat{x}\rangle + \phi(y) - \phi(\hat{x}) \geq 0, \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(f, P, \phi)$. If $P = 0$, then the *GMEP* (1.1) reduces to the following *Mixed Equilibrium Problem* (*MEP*) (see [52]), find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \phi(y) - \phi(\hat{x}) \geq 0, \text{ for all } y \in C. \tag{1.2}$$

If $\phi = 0$, then the *GMEP* (1.1) reduces to the following *generalized Equilibrium Problem* (*GEP*) (see [18]), find $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle P\hat{x}, y - \hat{x}\rangle \geq 0, \text{ for all } y \in C. \tag{1.3}$$

In particular, if $P = \phi = 0$, then the *GMEP* (1.1) reduces to the classical *Equilibrium Problem* (*EP*) introduced by Blum and Oettli [11], which is defined as finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \text{ for all } y \in C. \tag{1.4}$$

The EP and its generalisations are known to have wide area of applications in a large variety of problems arising in the fields of linear and nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed-point problems and have been widely applied in physics, structural analysis, management sciences and economics, etc. (see, for example [11, 14, 19, 24, 47, 42]). Several algorithms have been developed for solving the EP and its related optimization problems, see [1, 4, 30, 14, 15, 20, 22, 26, 32, 37, 46, 50], and the references therein.
Let H_1 and H_2 be Hilbert spaces, and let C and D be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $f : C \times C \to \mathbb{R}$, $g : D \times D \to \mathbb{R}$ be nonlinear bifunctions, $P : C \to H_1$, $Q : D \to H_2$, be nonlinear mappings, and $\phi : C \to \mathbb{R} \cup \{+\infty\}$, $\varphi : D \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator. The *Split generalized Mixed Equilibrium Problem* (*SGMEP*) (see, for example [25]) is to find a point $\hat{x} \in C$ such that

$$f(\hat{x}, x) + \langle P\hat{x}, x - \hat{x}\rangle + \phi(x) - \phi(\hat{x}) \geq 0, \text{ for all } x \in C, \tag{1.5}$$

and $\hat{y} = A\hat{x} \in D$ solves

$$g(\hat{y}, y) + \langle Q\hat{y}, y - \hat{y}\rangle + \varphi(y) - \varphi(\hat{y}) \geq 0, \text{ for all } y \in D. \tag{1.6}$$

We denote the solution set of (1.5)-(1.6) by $\Gamma = \{\hat{x} \in GMEP(f, P, \phi) : A\hat{x} \in GMEP(g, Q, \varphi)\}$. If $P = Q = 0$, then (1.5)-(1.6) reduces to the following *Split Mixed Equilibrium Problem* (*SMEP*) introduced by Onjai-uea and Phuengrattana [38] in 2017: Find $\hat{x} \in C$ such that

$$f(\hat{x}, x) + \phi(x) - \phi(\hat{x}) \geq 0, \text{ for all } x \in C, \tag{1.7}$$

and $\hat{y} = A\hat{x} \in D$ solves

$$g(\hat{y}, y) + \varphi(y) - \varphi(\hat{y}) \geq 0, \text{ for all } y \in D. \tag{1.8}$$

Also, if $\phi = \varphi = 0$ in (1.5)-(1.6), we have the following *Split generalized Equilibrium Problem* (*SGEP*) (see, for example [40, 13]): Find $\hat{x} \in C$ such that

$$f(\hat{x}, x) + \langle P\hat{x}, x - \hat{x} \rangle \geq 0, \text{ for all } x \in C, \tag{1.9}$$

and $\hat{y} = A\hat{x} \in D$ solves

$$g(\hat{y}, y) + \langle Q\hat{y}, y - \hat{y} \rangle \geq 0, \text{ for all } y \in D. \tag{1.10}$$

Furthermore, if $P = Q = 0$ and $\phi = \varphi = 0$, then the *SGMEP* (1.5)-(1.6) reduces to the *Split Equilibrium Problem* (*SEP*) (see, for example [2, 16, 17]), defined as follows: Find a point $\hat{x} \in C$ such that

$$f(\hat{x}, x) \geq 0, \text{ for all } x \in C, \tag{1.11}$$

and $\hat{y} = A\hat{x} \in D$ solves

$$g(\hat{y}, y) \geq 0, \text{ for all } y \in D. \tag{1.12}$$

Let $S : C \to C$ be a nonlinear mapping. A point $x^* \in C$ is called a fixed point of S if $Sx^* = x^*$. We denote by $F(S)$, the set of all fixed points of S, i.e.

$$F(S) = \{x^* \in C : Sx^* = x^*\}. \tag{1.13}$$

If S is a multivalued mapping, i.e. $S : C \to 2^C$, then $x^* \in C$ is called a fixed point of S if

$$x^* \in Sx^*. \tag{1.14}$$

The fixed point theory for multivalued mappings can be utilized in various areas such as game theory, control theory, mathematical economics, etc.

Recently, Onjai-uea and Phuengrattana [38] introduced the following iterative scheme for solving *SMEP* and fixed point of $\lambda-$hybrid multivalued mappings in real Hilbert spaces:

Algorithm 1.1.

$$\begin{cases} x_1 \in C, \\ u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)w_n, \quad w_n \in Su_n, \\ x_{n+1} = \beta_n w_n + (1 - \beta_n)z_n, \quad z_n \in Sy_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{1.15}$$

where $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset (0,1), \{r_n\} \subset (0,\infty)$ and $\gamma \in (0, \frac{1}{L})$ such that L is the spectral radius of A^*A and A^* is the adjoint of the bounded linear operator $A, C \subset H_1, D \subset H_2, S : C \to K(C)$ a $\lambda-$hybrid multivalued mapping, $F_1 : C \times C \to \mathbb{R}$ and $F_2 : D \times D \to \mathbb{R}$ are two bifunctions. The authors established under certain conditions that the sequence $\{x_n\}$ generated by the Algorithm 1 converges weakly to a common solution of the *SMEP* and fixed point of the $\lambda-$hybrid multivalued mapping.

Bauschke and Combettes [9] pointed out that in solving optimization problems, strong convergence of iterative schemes are more desirable and useful than their weak convergence counterparts. Hence, the need to develop algorithms that generate strong convergence sequence.

Very recently, Khan *et al.* [31] proposed the following shrinking projection algorithm for approximating a common solution for a finite family of SEPs and fixed point for a finite family of total asymptotically nonexpansive mappings in the setting of Hilbert spaces:

Algorithm 1.2.

$$\begin{cases} x_1 \in C_1 = C, \\ u_{n,i} = T^{F_i}_{r_{n,i}}(I - \gamma A_i^*(I - T^{G_i}_{s_{n,i}})A_i)x_n, \\ y_{n,i} = \alpha_{n,i}x_n + (1-\alpha_{n,i})S_i^n u_{n,i}, \\ C_{n+1} = \{z \in C_n : ||y_{n,i} - z||^2 \leq ||x_n - z||^2 + \theta_{n,i}\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \ n \geq 1, \end{cases}$$

where

$$\theta_{n,i} = (1-\alpha_{n,i})\{\lambda_n \xi_n(M_n) + \lambda_n M_n^* D_n + \mu_n\},$$

$$D_n = \sup\{||x_n - p|| : p \in \bigcap_{i=1}^{N} F(S_i)\},$$

M_n and M_n^* are positive real numbers, $S_i \pmod N) : C \to C$ is a finite family of total asymptotically nonexpansive mappings, $F_i \pmod N) : C \times C \to \mathbb{R}$ and $G_i \pmod N) : Q \times Q \to \mathbb{R}$ are two finite families of bifunctions, $A_i \pmod N) : H_1 \to H_2$ is a finite family of bounded linear operators, $\{r_{n,i}\}, \{s_{n,i}\}$ are two positive real sequences, $\{\alpha_{n,i}\} \subset (0,1), \gamma \in \left(0\frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \ldots, L_N\}$ and L_i is the spectral radius of the operator $A_i^*A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, \ldots, N\}$. Under mild conditions on the control parameters, they obtained strong convergence result for the proposed iterative scheme.

We need to point out at this point that the step size γ of the above algorithms plays an essential role in the convergence properties of iterative methods. The results obtained by Onjai-uea and Phuengrattana [38], Khan et al.[31] and other related results in literature involve step size that requires prior knowledge of the operator norm $||A||$. Such algorithms are usually not easy to implement because they require computation of the operator norm $||A||$, which is very difficult if not impossible to calculate or even estimate. Moreover, the step size defined by such algorithms are often very small and deteriorates the convergence rate of the algorithm. In practice, a larger step size can often be used to yield better numerical results.

Based on the heavy ball methods of a two-order time dynamical system, Polyak [41] first proposed an inertial extrapolation as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, several researchers have constructed some fast iterative schemes by employing inertial technique (e.g. see [6, 5, 7, 8, 10, 12, 27, 33, 35, 36]).

For approximating the null point of a maximal monotone operator, Alvarez and Attouch [8] introduced the following inertial proximal algorithm:

Algorithm 1.3.

$$x_{n+1} = J^A_{\mu_n}(x_n + \alpha_n(x_n - x_{n-1})), \quad n \geq 1.$$

They obtained a weak convergence of the algorithm under the following conditions:

(B1) There exists $\mu > 0$ such that for all $n \in \mathbb{N}, \ \mu_n \geq \mu$.

(B2) There exists $\alpha \in [0,1)$ such that for all $n \in \mathbb{N},\ 0 \leq \alpha_n \leq \alpha$.
(B3) $\sum_{n=1}^{\infty} \alpha_n |x_n - x_{n-1}|^2 < \infty$.
Recently, authors have pointed some of the drawbacks of the summability condition (B3) of the Algorithm 1, that is, to satisfy the summability condition, it is necessary to first calculate α_n at each step (see [36]).
Motivated by the above results and the ongoing research interest in this direction, we introduce the notion of *System of Split generalized Mixed Equilibrium Problem* and propose a new iterative scheme to find a common solution of the (*SSGMEP*) and fixed point problem *(FPP)* for multivalued mappings. We define SSGMEP as follows:
Definition 1.4. Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $i = 1 \leq i \leq m$. Let $A_i : H_1 \to H_2$ be bounded linear operators, $f_i : C_i \times C_i \to \mathbb{R}$ and $g_i : D_i \times D_i \to \mathbb{R}$, nonlinear bifunctions, $P_i : C_i \to H_1,\ Q_i : D_i \to H_2$, nonlinear mappings, and let $\phi_i : C_i \to \mathbb{R} \cup \{+\infty\}$, $\varphi_i : D_i \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions such that $\cap_{i=1}^m C_i \neq \emptyset$ and $\cap_{i=1}^m D_i \neq \emptyset$. The SSGMEP is to find $\hat{x} \in C = \cap_{i=1}^m C_i$ such that

$$f_i(\hat{x}, x) + \langle P_i \hat{x}, x - \hat{x} \rangle + \phi_i(x) - \phi_i(\hat{x}) \geq 0, \text{ for all } x \in C_i, \tag{1.16}$$

and $\hat{y} = A_i \hat{x} \in D = \cap_{i=1}^m D_i$ solves

$$g_i(\hat{y}, y) + \langle Q_i \hat{y}, y - \hat{y} \rangle + \varphi_i(y) - \varphi_i(\hat{y}) \geq 0, \text{ for all } y \in D_i. \tag{1.17}$$

We denote the solution set of (1.16)-(1.17) by

$$\Omega = \{\hat{x} \in \bigcap_{i=1}^m GMEP(f_i, P_i, \phi_i) : A_i \hat{x} \in \bigcap_{i=1}^m GMEP(g_i, Q_i, \varphi_i)\}.$$

If $P_i = Q_i = 0$, then (1.16)-(1.17) reduces to the following *System of Split Mixed Equilibrium Problem* (*SSMEP*) introduced by Karahan [28] in 2019:
Find $\hat{x} \in C = \cap_{i=1}^m C_i$ such that

$$f_i(\hat{x}, x) + \phi_i(x) - \phi_i(\hat{x}) \geq 0, \text{ for all } x \in C_i, \tag{1.18}$$

and $\hat{y} = A_i \hat{x} \in D = \cap_{i=1}^m D_i$ solves

$$g_i(\hat{y}, y) + \varphi_i(y) - \varphi_i(\hat{y}) \geq 0, \text{ for all } y \in D_i, \tag{1.19}$$

with the solution set given as

$$\Omega_{\phi,\varphi} = \{\hat{x} \in \bigcap_{i=1}^m GMEP(f_i, \phi_i) : A_i \hat{x} \in \bigcap_{i=1}^m GMEP(g_i, \varphi_i)\}.$$

Also, if $\phi_i = \varphi_i = 0$ in (1.16)-(1.17), we have the following *System of Split generalized Equilibrium Problem* (*SSGEP*): Find $\hat{x} \in C = \cap_{i=1}^m C_i$ such that

$$f_i(\hat{x}, x) + \langle P_i \hat{x}, x - \hat{x} \rangle \geq 0, \text{ for all } x \in C_i, \tag{1.20}$$

and $\hat{y} = A_i \hat{x} \in D = \cap_{i=1}^m D_i$ solves

$$g_i(\hat{y}, y) + \langle Q_i \hat{y}, y - \hat{y} \rangle \geq 0, \text{ for all } y \in D_i, \tag{1.21}$$

with solution set

$$\Omega_{P,Q} = \{\hat{x} \in \bigcap_{i=1}^m GMEP(f_i, P_i) : A_i \hat{x} \in \bigcap_{i=1}^m GMEP(g_i, Q_i)\}.$$

Furthermore, if $P_i = Q_i = 0$ and $\phi_i = \varphi_i = 0$, then the *SSGMEP* (1.16)-(1.17) reduces to the *System of Split Equilibrium Problem* (*SSEP*) introduced by Ugwunnadi and Ali [51], defined as follows: Find a point $\hat{x} \in C = \cap_{i=1}^m C_i$ such that

$$f_i(\hat{x}, x) \geq 0, \text{ for all } x \in C_i, \tag{1.22}$$

and $\hat{y} = A_i\hat{x} \in D = \cap_{i=1}^m D_i$ solves

$$g_i(\hat{y}, y) \geq 0, \text{ for all } y \in D_i, \tag{1.23}$$

with solution set

$$\Omega_{0,0} = \{\hat{x} \in \bigcap_{i=1}^m GMEP(f_i) : A_i\hat{x} \in \bigcap_{i=1}^m GMEP(g_i)\}.$$

Remark 1.5. Observe that if $m = 1$, the new problem introduced reduces to the SGMEP (1.5)-(1.6). Hence, our new problem is a generalization of SGMEP.

In this article, we introduce an inertial iterative scheme which does not require prior knowledge of the operator norm and obtain strong convergence result for approximating a common solution of SSGMEP (1.16)-(1.17) which also solves a fixed-point problem for a finite family of multivalued demicontractive mappings. We obtain some consequent results which complement and generalise several existing results in this direction in the literature.

More precisely, we consider the following problem: find $x^* \in \cap_{i=1}^m F(S_i)$, such that

$$f_i(x^*, x) + \langle P_i x^*, x - x^*\rangle + \phi_i(x) - \phi_i(x^*) \geq 0, \text{ for all } x \in C_i \tag{1.24}$$

and $\hat{y} = A_i\hat{x} \in D = \cap_{i=1}^m D_i$ solves

$$g_i(y^*, y) + \langle Q_i y^*, y - y^*\rangle + \psi_i(y) - \psi_i(y^*) \geq 0, \text{ for all } y \in D_i \tag{1.25}$$

where $S_i : C_i \to CB(C_i)$ is a finite family of multivalued demicontractive mappings.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$. The nearest point projection of H onto C is denoted by P_C, that is, $||x - P_Cx|| \leq ||x - y||$ for all $x \in H$ and $y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is firmly nonexpansive, i.e.,

$$||P_Cx - P_Cy||^2 \leq \langle P_Cx - P_Cy, x - y\rangle, \tag{2.1}$$

for all $x, y \in H$. Moreover $\langle x - P_Cx, y - P_Cx\rangle \leq 0$ holds for all $x \in H$ and $y \in C$, see [21, 39].

We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For a given sequence $\{x_n\} \subset$H, $w_\omega(x_n)$ denotes the set of weak limits of $\{x_n\}$, that is,

$$w_\omega(x_n) := \{x \in H : x_{n_j} \rightharpoonup x\} \text{ for some subsequence } \{n_j\} \text{ of } \{n\}.$$

Definition 2.1. Let H be a Hilbert space. A function $h : H \to H$ is called a *contraction* if there exists $\mu \in [0, 1)$ such that

$$||h(x) - h(y)|| \leq \mu||x - y||, \quad \text{for all } x, y \in H.$$

Let $\{h_n\}$ be a sequence of functions defined on a bounded subset K of H. Then $\{h_n\}$ is said to *converge uniformly* to the function h on K if, given $\epsilon > 0$, there exists n_0 such that

$$||h_n(x) - h(x)|| < \epsilon, \quad \text{for all } n \geq n_0,\ x \in K.$$

Let $\{h_n\}$ ($h_n : K \to H$) be a uniformly convergent sequence of contraction mappings. Then there exists a sequence of real numbers $\{\mu_n\} \subset (0, 1)$ such that

$$||h_n(x) - h_n(y)|| \leq \mu_n ||x - y|| \text{ for all } x, y \in K.$$

A subset K of H is called *proximal* if for each $x \in H$, there exists $y \in K$ such that

$$||x - y|| = d(x, K).$$

We denote by $CB(C), CC(C), K(C)$ and $P(C)$ the families of all nonempty closed bounded subsets of C, nonempty closed convex subset of C, nonempty compact subsets of C, and nonempty proximal bounded subsets of C, respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for all $A, B \in CB(C)$. Let $S : C \to 2^C$ be a multivalued mapping. We say that S satisfies the endpoint condition if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : C \to 2^C (i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfies the common endpoint condition if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{\infty} F(S_i)$. We recall some basic and useful definitions on multivalued mappings.

Definition 2.2. A multivalued mapping $S : C \to CB(C)$ is said to be

(i) *nonexpansive* if

$$H(Sx, Sy) \leq ||x - y||, \text{ for all } x, y \in C,$$

(ii) *quasi-nonexpansive* if $F(S) \neq \emptyset$ and

$$H(Sx, Sp) \leq ||x - p||, \text{ for all } x \in C, p \in F(S),$$

(iii) *nonspreading* if

$$2H(Sx, Sy)^2 \leq d(y, Sx)^2 + d(x, Sy)^2, \text{ for all } x, y \in C,$$

(iv) $\lambda-$*hybrid* if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)H(Sx, Sy)^2 \leq (1 - \lambda)||x - y||^2 + \lambda d(y, Sx)^2 + \lambda d(x, Sy)^2, \text{ for all } x, y \in C,$$

(v) $k-$*demicontractive* for $0 \leq k < 1$ if $F(S) \neq \emptyset$ and

$$H(Sx, Sp)^2 \leq ||x - p||^2 + kd(x, Sx)^2, \text{ for all } x \in C, p \in F(S).$$

We note that $0-$hybrid is nonexpansive, $1-$hybrid is nonspreading, and if S is $\lambda-$hybrid with $F(S) \neq \emptyset$, then S is quasi-nonexpansive. Similarly, if S is nonspreading with $F(S) \neq \emptyset$, then S is quasi-nonexpansive. We point out that the class of $k-$demicontractive mappings is more general and includes all the other types of mappings defined above. The *best approximation operator* P_S for a multivalued mapping $S : C \to P(C)$ is defined by

$$P_S(x) := \{y \in Sx : ||x - y|| = d(x, Sx)\}.$$

It is known that $F(S) = F(P_S)$ and P_S satisfies the endpoint condition. Song and Cho [44] gave an example of a best approximation operator P_S which is nonexpansive, but where S is not necessarily nonexpansive.

Definition 2.3. Let $S : C \to CB(C)$ be a multivalued mapping. The multivalued mapping $I - S$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset C$ which converges weakly to q and the sequence $\{||x_n - u_n||\}$ converges strongly to 0, where $u_n \in Sx_n$, then $q \in F(S)$.

The following results will be needed in the sequel:

Lemma 2.4. *For all $x, y \in H$, we have the following statements* [23, 50]*:*

(i) $||x + y||^2 \le ||x||^2 + 2\langle y, x + y\rangle$;
(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(iii) $||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2$.

Lemma 2.5. [48] *For each $x_1, \ldots, x_m \in H$ and $\alpha_1, \ldots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^m \alpha_i = 1$, the following holds:*

$$||\alpha_1 x_1 + \ldots + \alpha_m x_m||^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{1 \le i < j \le m} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.6. [43] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n, \text{ for all } n \ge 1,$$

if $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. [34] *Let $\{a_n\}, \{c_n\} \subset \mathbb{R}_+, \{\sigma_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ be sequences such that*

$$a_{n+1} \le (1 - \sigma_n)a_n + b_n + c_n \text{ for all } n \ge 0.$$

Assume $\sum_{n=0}^\infty |c_n| < \infty$. Then the following results hold:

(1) If $b_n \le \beta\sigma_n$ for some $\beta \ge 0$, then $\{a_n\}$ is a bounded sequence.

(2) If we have

$$\sum_{n=0}^{\infty} \sigma_n = \infty \text{ and } \limsup_{n\to\infty} \frac{b_n}{\sigma_n} \le 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Assumption 2.8. In solving EP (1.4), the bifunction f is assumed to satisfy the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e. $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C, \lim_{t\to 0} f(tz + (1 - t)x, y) \le f(x, y)$;
(A4) for each $x \in C, y \to f(x, y)$ is convex and lower semicontinuous.

It is known (see [53]), that if $f(x,y)$ satisfies (A1)-(A4) then the function

$$F(x,y) := f(x,y) + \langle Px, y-x\rangle + \phi(y) - \phi(x)$$

also satisfies (A1)-(A4) and $GMEP(f,P,\phi)$ is closed and convex.

Lemma 2.5. [53] *Let C be a nonempty closed closed convex subset of a Hilbert space H. Let $P : C \to H$ be a continuous and monotone mapping, $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, and $f : C \times C \to \mathbb{R}$ be a bifunction that satisfies $(A1)-(A4)$. For $r > 0$ and $x \in H$, there exists $u \in C$ such that*

$$f(u,y) + \langle Pu, y-u\rangle + \phi(y) - \phi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \text{ for all } y \in C. \tag{2.2}$$

Define a resolvent function $T_r^f : H \to C$ as follows:

$$T_r^f(x) = \Big\{u \in C : f(u,y)+\langle Pu, y-u\rangle+\phi(y)-\phi(u)+\frac{1}{r}\langle y-u, u-x\rangle \geq 0, \text{ for all } y \in C\Big\}. \tag{2.3}$$

Then the following conclusions hold:

1. *for each $x \in H$, $T_r^f(x) \neq \emptyset$,*
2. *T_r^f is single-valued,*
3. *T_r^f is firmly nonexpansive, i.e. for any $x, y \in H$,*
$$||T_r^f(x) - T_r^f(y)||^2 \leq \langle T_r^f(x) - T_r^f(y), x-y\rangle,$$
4. *$F(T_r^f) = GMEP(f,P,\phi)$,*
5. *$GMEP(f,P,\phi)$ is closed and convex.*

3. Main results

In this section, we present our algorithm and prove some strong convergence theorems of the proposed algorithm for solving the SSGMEP and fixed point problems.

Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_i \pmod m : H_1 \to H_2$ be a finite family of bounded linear operators with adjoint A_i^*, $f_i \pmod m : C_i \times C_i \to \mathbb{R}$ and $g_i \pmod m : D_i \times D_i \to \mathbb{R}$ are two finite families of bifunctions satisfying conditions *(A1) - (A4)* and g_i is upper semicontinuous in the first argument for each $i = 1, 2, \ldots, m$.

Let $\phi_i \pmod m : C_i \to \mathbb{R} \cup +\infty$ and $\varphi_i \pmod m : D_i \to \mathbb{R} \cup +\infty$ be proper lower semicontinuous and convex functions, and $P_i \pmod m : C_i \to H_1$ and $Q_i \pmod m :$ $D_i \to H_2$ are continuous and monotone mappings. Let $S_i \pmod m : C_i \to CB(C_i)$ be a finite family of multivalued demicontractive mappings with constant k_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, and $k = \max\{k_i\}$, and let $\{h_n\}$ $(h_n : H_1 \to H_1)$ be a sequence of μ_n-contractive mappings with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ for any $x \in K$, where K is any bounded subset of H_1. Suppose that the solution set $\Omega \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$. We establish the convergence of the algorithm under the following assumptions on the control parameters:

- (B1) $\{\beta_{n,i}\}, \{\delta_{n,i}\} \subset (0,1), \sum_{i=0}^m \beta_{n,i} = \sum_{i=0}^m \delta_{n,i} = 1$;
- (B2) $\liminf_n \beta_{n,0}\beta_{n,i} > 0$, and $\liminf_n(\delta_{n,0} - k)\delta_{n,i} > 0$, for each $1 \leq i \leq m$;
- (B3) $\{\alpha_n\}, \{\xi_n\}$ and $\{\gamma_n\}$ are real sequences in $\subset (0,1)$ such that $\alpha_n+\xi_n+\gamma_n = 1$;

(B4) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, $0 < c_1 \leq \xi_n$, $0 < c_2 \leq \gamma_n$, $0 < a \leq \tau_n \leq b < 1$;
(B5) $\{r_{n,i}\}, \{s_{n,i}\}$ are positive real sequences such that $\liminf_{n\to\infty} r_{n,i} > 0$ and $\liminf_{n\to\infty} s_{n,i} > 0$;
(B6) Let $\theta \geq 3$ and let $\{\epsilon_n\}$ be nonnegative sequence such that $0 < d \leq \epsilon_n$;
(B7) $\epsilon_n = o(\alpha_n)$, i.e., $\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n} = 0$, $\big($e.g. $\epsilon_n = \frac{1}{(n+1)^2}, \alpha_n = \frac{1}{n+1}\big)$.

Now, the algorithm is presented as follows:

Algorithm 3.1.

Step 0. Select initial data $x_0, x_1 \in C$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \ \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.2}$$

Step 3. Compute

$$z_{n,i} = w_n - \lambda_{n,i}A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n), \tag{3.3}$$

where

$$\lambda_{n,i} := \begin{cases} \tau_n \frac{||(I - T_{s_{n,i}}^{g_i})A_i w_n||^2}{||A_i^*(I - T_{s_{n,i}}^{g_i})A_i w_n||^2}, & \text{if } A_i w_n \neq T_{s_{n,i}}^{g_i} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.4}$$

Step 4. Compute

$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} T_{r_{n,i}}^{f_i} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^{m} \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \tag{3.5}$$

where $v_{n,i} \in S_i u_n$. Set $n := n + 1$ and return to **Step 1**.

Remark 3.2. Observe that from (3.1) and Assumption (A6), we have

$$\lim_{n\to\infty}\theta_n||x_n - x_{n-1}|| = 0 \text{ and } \lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0. \tag{3.6}$$

We also point out that **Step 1** in our Algorithm 3.1 is easily implemented in numerical computation since the value of $||x_n - x_{n-1}||$ is known a priori before choosing θ_n.

Remark 3.3. Also, note that in (3.4), the choice of $\lambda_{n,i}$ is independent of the norm of the operator $||A_i||$, for each $i = 1, 2, \ldots, m$. The value of λ does not influence the considered algorithm but was introduced for clarity.

Now, we state the strong convergence theorem for the proposed algorithm.

Theorem 3.4. *Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, for $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be the sequence generated by Algorithm 3.1 such that Assumptions (A1)-(A4) and (B1) - (B7) are satisfied. Then $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega \cap \bigcap_{i=1}^{m} F(S_i)$, where $\hat{x} = P_{\Omega\cap\bigcap_{i=1}^{m} F(S_i)} h(\hat{x})$.*

In order to prove Theorem 3.4, we first establish the following lemmas which will be employed in the proof.

Lemma 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1, then $\{x_n\}$ is bounded.*

Proof. Let

$$p = P_{\Omega\cap\bigcap_{i=1}^m F(S_i)} h(p),$$

then $p \in \Omega \cap \bigcap_{i=1}^m F(S_i)$. From Algorithm 3.1, it follows that

$$\begin{aligned}
||z_{n,i} - p||^2 &= ||w_n - \lambda_{n,i} A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n) - p|| \\
&\leq ||w_n - p||^2 - 2\lambda_{n,i}\langle w_n - p, A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n)\rangle \\
&\quad + \lambda_{n,i}^2 ||A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n)||^2 \\
&= ||w_n - p||^2 - 2\lambda_{n,i}\langle A_i(w_n - p), A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n\rangle \\
&\quad + \lambda_{n,i}^2 ||A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n)||^2. \qquad (3.7)
\end{aligned}$$

By Lemma 2.4(iii), we have

$$\begin{aligned}
&-2\lambda_{n,i}\langle A_i(w_n - p),\ A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n\rangle \\
&= 2\lambda_{n,i}\langle A_i(w_n - p), (T_{s_{n,i}}^{g_i} - I)A_i w_n\rangle \\
&= 2\lambda_{n,i}\langle T_{s_{n,i}}^{g_i} A_i w_n - A_i p - (T_{s_{n,i}}^{g_i} - I)A_i w_n,\ (T_{s_{n,i}}^{g_i} - I)A_i w_n\rangle \\
&= 2\lambda_{n,i}\big[\langle T_{s_{n,i}}^{g_i} A_i w_n - A_i p,\ (T_{s_{n,i}}^{g_i} - I)A_i w_n\rangle \\
&- \langle (T_{s_{n,i}}^{g_i} - I)A_i w_n,\ (T_{s_{n,i}}^{g_i} - I)A_i w_n\rangle\big] \\
&= 2\lambda_{n,i}\big[\langle T_{s_{n,i}}^{g_i} A_i w_n - A_i p,\ (T_{s_{n,i}}^{g_i} - I)A_i w_n\rangle - ||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2\big] \\
&= \lambda_{n,i}\big[||T_{s_{n,i}}^{g_i} A_i w_n - A_i p||^2 + ||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2 \\
&- ||T_{s_{n,i}}^{g_i} A_i w_n - A_i p - (T_{s_{n,i}}^{g_i} - I)A_i w_n||^2 - 2||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2\big] \\
&= \lambda_{n,i}\big[||T_{s_{n,i}}^{g_i} A_i w_n - A_i p||^2 + ||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2 \\
&- ||A_i w_n - A_i p||^2 - 2||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2\big] \\
&= \lambda_{n,i}\big[||T_{s_{n,i}}^{g_i} A_i w_n - A_i p||^2 - ||A_i w_n - A_i p||^2 - ||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2\big] \\
&\leq \lambda_{n,i}\big[||A_i w_n - A_i p||^2 - ||A_i w_n - A_i p||^2 - ||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2\big] \\
&= -\lambda_{n,i}||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2. \qquad (3.8)
\end{aligned}$$

Hence, from (3.7), (3.8) and the definition of $\lambda_{n,i}$ we get

$$\begin{aligned}
||z_{n,i} - p||^2 &\leq ||w_n - p||^2 - \lambda_{n,i}||(T_{s_{n,i}}^{g_i} - I)A_i w_n||^2 + \lambda_{n,i}^2||A_i^*(A_i w_n - T_{s_{n,i}}^{g_i} A_i w_n)||^2 \\
&= ||w_n - p||^2 - \lambda_{n,i}(1 - \tau_n)||(I - T_{s_{n,i}}^{g_i})A_i w_n||^2 \\
&= ||w_n - p||^2 - \frac{\tau_n(1 - \tau_n)||(I - T_{s_{n,i}}^{g_i})A_i w_n||^4}{||A_i^*(I - T_{s_{n,i}}^{g_i})A_i w_n||^2}. \qquad (3.9)
\end{aligned}$$

Since $0 < \tau_n < 1$, then we obtain

$$||z_{n,i} - p||^2 \leq ||w_n - p||^2. \qquad (3.10)$$

Applying Lemma 2.5 together with (3.10), we have

$$\begin{aligned}
||u_n - p||^2 &= ||\beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} T^{f_i}_{r_{n,i}} z_{n,i} - p||^2 \\
&\leq \beta_{n,0}||w_n - p||^2 + \sum_{i=1}^{m} \beta_{n,i}||T^{f_i}_{r_{n,i}} z_{n,i} - p||^2 \\
&- \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}} z_{n,i} - w_n||^2 \\
&\leq \beta_{n,0}||w_n - p||^2 + \sum_{i=1}^{m} \beta_{n,i}||z_{n,i} - p||^2 - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}} z_{n,i} - w_n||^2 \qquad (3.11) \\
&\leq \beta_{n,0}||w_n - p||^2 + \sum_{i=1}^{m} \beta_{n,i}||w_n - p||^2 - \sum_{i=1}^{m} \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}} z_{n,i} - w_n||^2 \\
&\leq ||w_n - p||^2. \qquad (3.12)
\end{aligned}$$

Again, applying Lemma 2.5, we obtain

$$\begin{aligned}
||y_n - p||^2 &= ||\delta_{n,0} u_n + \sum_{i=1}^{m} \delta_{n,i} v_{n,i} - p||^2 \\
&\leq \delta_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \delta_{n,i}||v_{n,i} - p||^2 - \delta_{n,0}\sum_{i=1}^{m} \delta_{n,i}||v_{n,i} - u_n||^2 \\
&\leq \delta_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \delta_{n,i} H(S_i u_n, S_i p) - \delta_{n,0}\sum_{i=1}^{m} \delta_{n,i}||v_{n,i} - u_n||^2 \\
&\leq \delta_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \delta_{n,i}(||u_n - p||^2 + k_i d(u_n, S_i u_n)) \\
&- \delta_{n,0}\sum_{i=1}^{m} \delta_{n,i}||v_{n,i} - u_n||^2 \\
&\leq \delta_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \delta_{n,i}(||u_n - p||^2 + k_i||u_n - v_{n,i}||^2) \\
&- \delta_{n,0}\sum_{i=1}^{m} \delta_{n,i}||v_{n,i} - u_n||^2 \\
&= ||u_n - p||^2 - \sum_{i=1}^{m} \delta_{n,i}(\delta_{n,0} - k_i)||u_n - v_{n,i}||^2 \qquad (3.13) \\
&\leq ||u_n - p||^2. \qquad (3.14)
\end{aligned}$$

Applying the triangle inequality, we get

$$\begin{aligned}||w_n - p|| &= ||x_n + \theta_n(x_n - x_{n-1}) - p|| \\ &\leq ||x_n - p|| + \theta_n||x_n - x_{n-1}|| \\ &= ||x_n - p|| + \alpha_n \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||. \end{aligned} \tag{3.15}$$

Since by Remark 3.2,

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0,$$

it follows that there exists a constant $M_1 > 0$ such that $\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \leq M_1$, for all $n \geq 1$. Hence from (3.15), we obtain

$$||w_n - p|| \leq ||x_n - p|| + \alpha_n M_1. \tag{3.16}$$

By applying (3.12), (3.14) and (3.16), we get

$$\begin{aligned}||x_{n+1} - p|| &= ||\alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n - p|| \\ &= ||\alpha_n(h_n(x_n) - h_n(p)) + \alpha_n(h_n(p) - p) + \xi_n(x_n - p) + \gamma_n(y_n - p)|| \\ &\leq \alpha_n\mu_n||x_n - p|| + \alpha_n||h_n(p) - p|| + \xi_n||x_n - p|| + \gamma_n||y_n - p|| \\ &\leq \alpha_n\mu_n||x_n - p|| + \alpha_n||h_n(p) - p|| + \xi_n||x_n - p|| \\ &+ \gamma_n(||x_n - p|| + \alpha_n M_1) \\ &= \alpha_n\mu_n||x_n - p|| + \alpha_n||h_n(p) - p|| + (1 - \alpha_n)||x_n - p|| + \alpha_n\gamma_n M_1 \\ &\leq \alpha_n\mu^*||x_n - p|| + \alpha_n||h_n(p) - p|| + (1 - \alpha_n)||x_n - p|| + \alpha_n\gamma_n M_1 \\ &= (\alpha_n\mu^* + (1 - \alpha_n))||x_n - p|| + \alpha_n||h_n(p) - p|| + \alpha_n\gamma_n M_1 \\ &= (1 - \alpha_n(1 - \mu^*))||x_n - p|| + \alpha_n(1 - \mu^*)\Big\{\frac{||h_n(p) - p||}{1 - \mu^*} + \frac{\gamma_n M_1}{1 - \mu^*}\Big\}\end{aligned}$$

By the uniform convergence of $\{h_n\}$ on K, there exists $M_2 > 0$ such that

$$||h_n(p) - p|| \leq M_2,$$

for all $n \geq 1$. Hence, we have

$$\begin{aligned}||x_{n+1} - p|| \leq &= (1 - \alpha_n(1 - \mu^*))||x_n - p|| + \alpha_n(1 - \mu^*)\Big\{\frac{M_2}{1 - \mu^*} + \frac{\gamma_n M_1}{1 - \mu^*}\Big\} \\ &\leq (1 - \alpha_n(1 - \mu^*))||x_n - p|| + \alpha_n(1 - \mu^*)M^*,\end{aligned}$$

where

$$M^* := \sup_{n\in\mathbb{N}}\Big\{\frac{M_2}{1 - \mu^*} + \frac{\gamma_n M_1}{1 - \mu^*}\Big\}.$$

Setting

$$a_n := ||x_n - p||,\ b_n := \alpha_n(1 - \mu^*)M^*,\ c_n := 0,$$

and

$$\sigma_n := \alpha_n(1 - \mu^*).$$

By Lemma 2.7(1) and our assumptions, it follows that $\{||x_n - p||\}$ is bounded and thus $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{z_{n,i}\}, \{u_n\}$ and $\{y_n\}$ are all bounded.

Lemma 3.6. *The following inequality holds for all* $p \in \Omega \cap \bigcap_{i=1}^m F(S_i)$ *and* $n \in \mathbb{N}$*:*

$$\begin{aligned}
||x_{n+1} - p||^2 \leq & \Big(1 - \frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big)||x_n - p||^2 \\
&+ \frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big\{\frac{\alpha_n}{2(1-\mu^*)}M_3 + \frac{3M_2\gamma_n(1-\alpha_n)}{2(1-\mu^*)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\
&+ \frac{1}{(1-\mu^*)}\langle h_n(p) - p, x_{n+1} - p\rangle\Big\} \\
&- \frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\mu^*)}\Big\{\sum_{i=1}^m \beta_{n,i}\frac{\tau_n(1-\tau_n)||(I - T^{g_i}_{s_{n,i}})A_i w_n||^4}{||A_i^*(I - T^{g_i}_{s_{n,i}})A_i w_n||^2} \\
&+ \sum_{i=1}^m \delta_{n,i}(\delta_{n,0} - k_i)||u_n - v_{n,i}||^2 + \sum_{i=1}^m \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}} z_{n,i} - w_n||^2\Big\}.
\end{aligned}$$

Proof. Let $p \in \Omega \cap \bigcap_{i=1}^m F(S_i)$, then by the Cauchy-Schwartz inequality and Lemma 2.4(ii), we get

$$\begin{aligned}
||w_n - p||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - p||^2 \\
&= ||x_n - p||^2 + \theta_n^2||x_n - x_{n-1}||^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq ||x_n - p||^2 + \theta_n^2||x_n - x_{n-1}||^2 + 2\theta_n||x_n - x_{n-1}||||x_n - p|| \\
&= ||x_n - p||^2 + \theta_n||x_n - x_{n-1}||(\theta_n||x_n - x_{n-1}|| + 2||x_n - p||) \\
&\leq ||x_n - p||^2 + 3M_2\theta_n||x_n - x_{n-1}|| \\
&= ||x_n - p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||, \qquad (3.17)
\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n - p||, \theta_n||x_n - x_{n-1}||\} > 0$.
By applying Lemma 2.4, (3.13), (3.11), (3.9) and (3.17), we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n - p||^2 \\
&= ||\alpha_n(h_n(x_n) - p) + \xi_n(x_n - p) + \gamma_n(y_n - p)||^2 \\
&\leq ||\xi_n(x_n - p) + \gamma_n(y_n - p)||^2 + 2\alpha_n\langle h_n(x_n) - p, x_{n+1} - p\rangle \\
&\leq \xi_n^2||x_n - p||^2 + \gamma_n^2||y_n - p||^2 + 2\xi_n\gamma_n||x_n - p||||y_n - p|| \\
&+ 2\alpha_n\langle h_n(x_n) - p, x_{n+1} - p\rangle \\
&\leq \xi_n^2||x_n - p||^2 + \gamma_n^2||y_n - p||^2 + \xi_n\gamma_n(||x_n - p||^2 + ||y_n - p||^2) \\
&+ 2\alpha_n\langle h_n(x_n) - p, x_{n+1} - p\rangle \\
&= \xi_n(\xi_n + \gamma_n)||x_n - p||^2 + \gamma_n(\gamma_n + \xi_n)||y_n - p||^2 \\
&+ 2\alpha_n\langle h_n(x_n) - p, x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \xi_n(1-\alpha_n)||x_n-p||^2 + \gamma_n(1-\alpha_n)\Big\{\beta_{n,0}||w_n-p||^2 \\
&+ \sum_{i=1}^m \beta_{n,i}\Big(||w_n-p||^2 \\
&- \frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2}\Big) - \sum_{i=1}^m \delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2 \\
&- \sum_{i=1}^m \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\} \\
&+ 2\alpha_n\langle h_n(x_n)-h_n(p), x_{n+1}-p\rangle + 2\alpha_n\langle h_n(p)-p, x_{n+1}-p\rangle \\
&\leq \xi_n(1-\alpha_n)||x_n-p||^2 \\
&+ \gamma_n(1-\alpha_n)\Big\{||w_n-p||^2 - \sum_{i=1}^m \beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2} \\
&- \sum_{i=1}^m \delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2 - \sum_{i=1}^m \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\} \\
&+ 2\alpha_n\mu_n||x_n-p||||x_{n+1}-p|| \\
&+ 2\alpha_n\langle h_n(p)-p, x_{n+1}-p\rangle \\
&\leq \xi_n(1-\alpha_n)||x_n-p||^2 \\
&+ \gamma_n(1-\alpha_n)\Big\{||x_n-p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| \\
&- \sum_{i=1}^m \beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2} \\
&- \sum_{i=1}^m \delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2 \\
&- \sum_{i=1}^m \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\} + \alpha_n\mu^*(||x_n-p||^2 + ||x_{n+1}-p||^2) \\
&+ 2\alpha_n\langle h_n(p)-p, x_{n+1}-p\rangle \\
&= \big((1-\alpha_n)^2 + \alpha_n\mu^*\big)||x_n-p||^2 + \alpha_n\mu^*||x_{n+1}-p||^2 \\
&+ 3\alpha_n\gamma_n(1-\alpha_n)M_2\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| \\
&+ 2\alpha_n\langle h_n(p)-p, x_{n+1}-p\rangle - \gamma_n(1-\alpha_n)\Big\{\sum_{i=1}^m \beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2} \\
&+ \sum_{i=1}^m \delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2 + \sum_{i=1}^m \beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&||x_{n+1}-p||^2\\
\leq\ & \frac{(1-2\alpha_n+\alpha_n^2+\alpha_n\mu^*)}{(1-\alpha_n\mu^*)}||x_n-p||^2\\
+\ & \frac{\alpha_n}{(1-\alpha_n\mu^*)}\Big\{3\gamma_n(1-\alpha_n)M_2\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+2\langle h_n(p)-p,x_{n+1}-p\rangle\Big\}\\
-\ & \frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\mu^*)}\Big\{\sum_{i=1}^m\beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2}\\
+\ & \sum_{i=1}^m\delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2+\sum_{i=1}^m\beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\}\\
=\ & \frac{(1-2\alpha_n+\alpha_n\mu^*)}{(1-\alpha_n\mu^*)}||x_n-p||^2+\frac{\alpha_n^2}{(1-\alpha_n\mu^*)}||x_n-p||^2\\
+\ & \frac{\alpha_n}{(1-\alpha_n\mu^*)}\Big\{3\gamma_n(1-\alpha_n)M_2\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+2\langle h_n(p)-p,x_{n+1}-p\rangle\Big\}\\
-\ & \frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\mu^*)}\Big\{\sum_{i=1}^m\beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2}\\
+\ & \sum_{i=1}^m\delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2\\
+\ & \sum_{i=1}^m\beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\}\\
\leq\ & \Big(1-\frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big)||x_n-p||^2\\
+\ & \frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big\{\frac{\alpha_n}{2(1-\mu^*)}M_3+\frac{3M_2\gamma_n(1-\alpha_n)}{2(1-\mu^*)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
+\ & \frac{1}{(1-\mu^*)}\langle h_n(p)-p,x_{n+1}-p\rangle\Big\}\\
-\ & \frac{\gamma_n(1-\alpha_n)}{(1-\alpha_n\mu^*)}\Big\{\sum_{i=1}^m\beta_{n,i}\frac{\tau_n(1-\tau_n)||(I-T^{g_i}_{s_{n,i}})A_iw_n||^4}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_iw_n||^2}\\
+\ & \sum_{i=1}^m\delta_{n,i}(\delta_{n,0}-k_i)||u_n-v_{n,i}||^2+\sum_{i=1}^m\beta_{n,0}\beta_{n,i}||T^{f_i}_{r_{n,i}}z_{n,i}-w_n||^2\Big\},
\end{aligned}
$$

where

$$M_3:=\sup\{||x_n-p||^2:n\in\mathbb{N}\}.$$

Hence, the proof is complete.

Lemma 3.7. *Let $p\in\Omega\cap\bigcap_{i=1}^m F(S_i)$. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\liminf_{k\to\infty}(||x_{n_k+1}-p||-||x_{n_k}-p||)\geq 0$, then $x_{n_k}\rightharpoonup x^*\in\Omega\cap\bigcap_{i=1}^m F(S_i)$, i.e. $w_\omega(x_n)\subset\Omega\cap\bigcap_{i=1}^m F(S_i)$.*

Proof. From Lemma 3.6, it follows that

$$\begin{aligned}\frac{\gamma_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\mu^*)}\sum_{i=1}^{m}\beta_{n_k,i}&\frac{\tau_{n_k}(1-\tau_{n_k})||(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||^4}{||A_i^*(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||^2}\\ &\leq\Big(1-\frac{2\alpha_{n_k}(1-\mu^*)}{(1-\alpha_{n_k}\mu^*)}\Big)||x_{n_k}-p||^2\\ &-||x_{n_k+1}-p||^2+\frac{2\alpha_{n_k}(1-\mu^*)}{(1-\alpha_{n_k}\mu^*)}\Big\{\frac{\alpha_{n_k}}{2(1-\mu^*)}M_3\\ &+\frac{3M_2\gamma_{n_k}(1-\alpha_{n_k})}{2(1-\mu^*)}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||\\ &+\frac{1}{(1-\mu^*)}\langle h_{n_k}(p)-p,x_{n_k+1}-p\rangle\Big\}.\end{aligned}$$

By the hypothesis of Lemma 3.7 and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we obtain

$$\frac{\gamma_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\mu^*)}\sum_{i=1}^{m}\beta_{n_k,i}\frac{\tau_{n_k}(1-\tau_{n_k})||(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||^4}{||A_i^*(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||^2}\to 0 \text{ as } k\to\infty.$$

Since $0<a\leq\tau_{n_k}\leq b<1$,

$$\lim_{k\to\infty}\alpha_{n_k}=0,\ \beta_{n_k,i}>0$$

for all $i=1,2,\dots,m$ and $||A_i^*(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||$ is bounded, it follows that

$$\lim_{k\to\infty}||(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||=0 \quad \text{for all } i=1,2,\dots,m, \tag{3.18}$$

and this implies that

$$\lim_{k\to\infty}||A_i^*(I-T^{g_i}_{s_{n_k},i})A_iw_{n_k}||=0. \tag{3.19}$$

By similar argument, it follows from Lemma 3.6 that

$$\lim_{k\to\infty}||T^{f_i}_{r_{n_k},i}z_{n_k,i}-w_{n_k}||=0 \quad\text{and}\quad \lim_{k\to\infty}||u_{n_k}-v_{n_k,i}||=0. \tag{3.20}$$

Also, it follows from (3.19), (3.20) and Algorithm 3.1 that

$$||z_{n_k,i}-w_{n_k}||=||\lambda_{n_k,i}A_i^*(A_iw_{n_k}-T^{g_i}_{s_{n_k},i}A_iw_{n_k}||\to 0 \text{ as } k\to\infty, \tag{3.21}$$

$$||u_{n_k}-w_{n_k}||\leq\beta_{n_k,0}||w_{n_k}-w_{n_k}||+\sum_{i=1}^{m}\beta_{n_k,i}||T^{f_i}_{r_{n_k},i}z_{n_k,i}-w_{n_k}||\to 0 \text{ as } k\to\infty, \tag{3.22}$$

$$||y_{n_k}-u_{n_k}||\leq\delta_{n_k,0}||u_{n_k}-u_{n_k}||+\sum_{i=1}^{m}\delta_{n_k,i}||v_{n_k,i}-u_{n_k}||\to 0 \text{ as } k\to\infty. \tag{3.23}$$

By Remark 3.2, we get

$$||w_{n_k}-x_{n_k}||\leq||x_{n_k}-x_{n_k}||+\theta_{n_k}||x_{n_k}-x_{n_k-1}||\to 0 \text{ as } k\to\infty. \tag{3.24}$$

Also, from (3.21) - (3.24), we obtain

$$||y_{n_k}-x_{n_k}||\to 0,\quad ||u_{n_k}-x_{n_k}||\to 0\quad\text{and}\quad ||y_{n_k}-w_{n_k}||\to 0. \tag{3.25}$$

Thus,

$$\begin{aligned}||x_{n_k+1} - x_{n_k}|| &= ||\alpha_{n_k} h_n(x_{n_k}) + \xi_{n_k} x_{n_k} + \gamma_{n_k} y_{n_k} - x_{n_k}|| \\ &\le \alpha_{n_k}||h_n(x_{n_k}) - x_{n_k}|| + \xi_{n_k}||x_{n_k} - x_{n_k}|| + \gamma_{n_k}||y_{n_k} - x_{n_k}|| \to 0.\end{aligned} \tag{3.26}$$

Let $x^* \in \omega_\omega(x_n)$, then by (3.25) $u_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By the demiclosedness of $I - S_i$, it follows from (3.20) that $x^* \in F(S_i)$ for all $i = 1, 2, \dots, m$. This implies that $x^* \in \bigcap_{i=1}^m F(S_i)$. Next, we show that $x^* \in \bigcap_{i=1}^m GMEP(f_i, P_i, \phi_i)$. We first show that $x^* \in GMEP(f_1, P_1, \phi_1)$, where $f_1 = f_{n_k}, P_1 = P_{n_k}$ and $\phi_1 = \phi_{n_k}$ for some $k \ge 1$. Note that, for a finite family of generalized mixed equilibrium problems, the indexing $f_1 = f_{n_k}, P_1 = P_{n_k}$ and $\phi_1 = \phi_{n_k}$ results from the modulo function $k \equiv 1$ (mod m) and the corresponding term of the infinite sequence $\{x_n\}$ would then be $\{x_{n_k}\}$. Similarly, we can have $f_2 = f_{n_j}, P_2 = P_{n_j}$ and $\phi_2 = \phi_{n_j}$ for some $j \ge 1$. Let $t_{n_k,i} = T^{f_1}_{r_{n_k,i}} z_{n_k,i}$, then by (3.20) and (3.24) it follows that $w_\omega(t_{n_k,i}) = w_\omega(x_{n_k})$. By the definition of $T^{f_1}_{r_{n_k,i}}$ we have that

$$f_1(t_{n_k,i}, y) + \langle P_1 t_{n_k,i}, y - t_{n_k,i}\rangle + \phi_1(y) - \phi_1(t_{n_k,i}) + \frac{1}{r_{n_k,i}}\langle y - t_{n_k,i}, t_{n_k,i} - z_{n_k,i}\rangle \ge 0.$$

By the monotonicity of $F_1(x, y) := f_1(x, y) + \langle P_1 x, y - x\rangle + \phi_1(y) - \phi_1(x)$, we have

$$\frac{1}{r_{n_k,i}}\langle y - t_{n_k,i}, t_{n_k,i} - z_{n_k,i}\rangle \ge f_1(y, t_{n_k,i}) + \langle P_1 y, t_{n_k,i} - y\rangle + \phi_1(t_{n_k,i}) - \phi_1(y).$$

Since $x^* \in w_\omega(t_{n,i})$, then it follows from (3.20), (3.21), $\liminf_{k\to\infty} r_{n_k,i} > 0$ and Condition (A4) that

$$f_1(y, x^*) + \langle P_1 y, x^* - y\rangle + \phi_1(x^*) - \phi_1(y) \le 0 \quad \text{for all } y \in C_1.$$

Now, for fixed $y \in C_1$, let $y_t := ty + (1 - t)x^*$ for all $t \in (0, 1)$. This implies that $y_t \in C_1$. Then by Conditions (A1) and (A4), we have

$$\begin{aligned}0 &= f_1(y_t, y_t) + \langle P_1 y_t, y_t - y_t\rangle + \phi_1(y_t) - \phi_1(y_t) \\ &\le t\big\{f_1(y_t, y) + \langle P_1 y_t, y - y_t\rangle + \phi_1(y) - \phi_1(y_t)\big\} \\ &+ (1 - t)\big\{f_1(y_t, x^*) + \langle P_1 y_t, x^* - y_t\rangle + \phi_1(x^*) - \phi_1(y_t)\big\} \\ &\le t\big\{f_1(y_t, y) + \langle P_1 y_t, y - y_t\rangle + \phi_1(y) - \phi_1(y_t)\big\}.\end{aligned}$$

Hence,

$$f_1(y_t, y) + \langle P_1 y_t, y - y_t\rangle + \phi_1(y) - \phi_1(y_t) \ge 0.$$

Moreover, letting $t \to 0$, by Condition (A3) we get

$$f_1(x^*, y) + \langle P_1 x^*, y - x^*\rangle + \phi_1(y) - \phi_1(x^*) \ge 0 \quad \text{for all } y \in C_1,$$

which implies that $x^* \in GMEP(f_1, P_1, \phi_1)$. Following similar argument, we can show that $x^* \in GMEP(f_2, P_2, \phi_2)$ where $f_2 = f_{n_j}, P_2 = P_{n_j}$ and $\phi_2 = \phi_{n_j}$ for some $j \ge 1$. Hence, $x^* \in \bigcap_{i=1}^m GMEP(f_i, P_i, \phi_i)$. Next, we show that $A_i x^* \in \bigcap_{i=1}^m GMEP(g_i, Q_i, \varphi_i)$. Reasoning as above, we first show that $A_i x^* \in GMEP(g_1, Q_1, \varphi_1)$, where $g_1 = g_{n_l}, Q_1 = Q_{n_l}$ and $\varphi_1 = \varphi_{n_l}$ for some $l \ge 1$. By

(3.24), we have that $w_\omega(w_{n_l}) = w_\omega(x_{n_l})$ and since A_i is a bounded linear operator for all $i = 1, 2, \ldots, m$, we have

$$A_i w_{n_l} \rightharpoonup A_i x^*. \tag{3.27}$$

Set $q_{n_l,i} = A_i w_{n_l} - T^{g_1}_{s_{n_l},i} A_i w_{n_l}$. Then it follows that $A_i w_{n_l} - q_{n_l,i} = T^{g_1}_{s_{n_l},i} A_i w_{n_l}$, and from (3.18) we have

$$\lim_{l\to\infty} q_{n_l,i} = 0 \quad \text{for each } i = 1, 2, \ldots, m. \tag{3.28}$$

From the definition of $T^{g_1}_{s_{n_l},i}$, we obtain

$$\begin{aligned} & g_1(A_i w_{n_l} - q_{n_l,i}, y) + \langle Q_1(A_i w_{n_l} - q_{n_l,i}), y - A_i w_{n_l} + q_{n_l,i}\rangle \\ & \quad + \varphi_1(y) - \varphi_1(A_i w_{n_l} - q_{n_l,i}) \\ & \quad + \frac{1}{s_{n_l,i}}\langle y - A_i w_{n_l} + q_{n_l,i},\ A_i w_{n_l} - q_{n_l,i} - A_i w_{n_l}\rangle \geq 0 \quad \text{for all } y \in D_1. \end{aligned} \tag{3.29}$$

Since g_1 is upper semi-continuous in the first argument, then G_1 defined by

$$G_1(x, y) := g_1(x, y) + \langle Q_1 x, y - x\rangle + \varphi_1(y) - \varphi_1(x)$$

is also upper semi-continuous in the first argument. Hence, taking lim sup of inequality (3.29) as $l \to \infty$ and using (3.27) (3.28), we obtain

$$g_1(A_i x^*, y) + \langle Q_1 A_i x^*, y - A_i x^*\rangle + \varphi_1(y) - \varphi_1(A_i x^*) \geq 0 \quad \text{for all } y \in D_1,$$

which implies that $A_i x^* \in GMEP(g_1, Q_1, \varphi_1)$.
Similarly, we can show that $A_i x^* \in GMEP(g_i, Q_i, \varphi_i)$ for each $i = 1, 2, \ldots, m$.
Consequently, we have that $A_i x^* \in \bigcap_{i=1}^m GMEP(g_i, Q_i, \varphi_i)$.
Hence, $x^* \in \Omega \cap \bigcap_{i=1}^m F(S_i)$ as required.
Now, we present the proof of Theorem 3 as follows.
Proof. (*Proof of Theorem 3.4*) Let $\hat{x} = P_{\Omega\cap\bigcap_{i=1}^m F(S_i)} h(\hat{x})$, then it follows from Lemma 3.6 that

$$\begin{aligned} ||x_{n+1} - \hat{x}||^2 \leq & \Big(1 - \frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big)||x_n - \hat{x}||^2 \\ & + \frac{2\alpha_n(1-\mu^*)}{(1-\alpha_n\mu^*)}\Big\{\frac{\alpha_n}{2(1-\mu^*)}M_3 + \frac{3M_2\gamma_n(1-\alpha_n)}{2(1-\mu^*)}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \\ & + \frac{1}{(1-\mu^*)}\langle h_n(\hat{x}) - \hat{x}, x_{n+1} - \hat{x}\rangle\Big\}. \end{aligned} \tag{3.30}$$

Now, we claim that the sequence $\{||x_n - \hat{x}||^2\}$ converges to zero. To establish this, by Lemma 2.6, it suffices to show that $\limsup_{k\to\infty}\langle h_{n_k}(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle \leq 0$ for every subsequence $\{||x_{n_k} - \hat{x}||\}$ of $\{||x_n - \hat{x}||\}$ satisfying

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0.$$

Now, suppose that $\{||x_{n_k} - \hat{x}||\}$ is a subsequence of $\{||x_n - \hat{x}||\}$ such that

$$\liminf_{k\to\infty}(||x_{n_k+1} - \hat{x}|| - ||x_{n_k} - \hat{x}||) \geq 0.$$

Then, by Lemma 3.7, we have that $w_\omega\{x_n\} \subset \Omega \cap \bigcap_{i=1}^m F(S_i)$. It also follows from (3.25) that $w_\omega\{u_n\} = w_\omega\{x_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle h_{n_{k_j}}(\hat{x})-\hat{x}, x_{n_{k_j}}-\hat{x}\rangle = \limsup_{k\to\infty}\langle h_{n_k}(\hat{x})-\hat{x}, x_{n_k}-\hat{x}\rangle = \limsup_{k\to\infty}\langle h_{n_k}(\hat{x})-\hat{x}, u_{n_k}-\hat{x}\rangle.$$

Since $\hat{x} = P_{\Omega\cap\bigcap_{i=1}^m F(S_i)}h(\hat{x})$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ on K, then it follows that

$$\limsup_{k\to\infty}\langle h_{n_k}(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle = \lim_{j\to\infty}\langle h_{n_{k_j}}(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \langle h(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \tag{3.31}$$

Combining (3.26) and (3.31), we have

$$\limsup_{k\to\infty}\langle h_{n_k}(\hat{x})-\hat{x}, x_{n_k+1}-\hat{x}\rangle \leq \limsup_{k\to\infty}\langle h_{n_k}(\hat{x})-\hat{x}, x_{n_k}-\hat{x}\rangle = \langle h(\hat{x})-\hat{x}, x^\dagger-\hat{x}\rangle \leq 0. \tag{3.32}$$

Applying Lemma 2.6 to (3.30), and using (3.32) together with the fact that $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we deduce that $\lim_{n\to\infty}||x_n-\hat{x}|| = 0$ as required.

By the properties of the best approximation operator, we have the following consequent result.

Corollary 3.8. Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_i \pmod m : H_1 \to H_2$ be a finite family of bounded linear operators with adjoint A_i^*, $f_i \pmod m : C_i \times C_i \to \mathbb{R}$ and $g_i \pmod m : D_i \times D_i \to \mathbb{R}$ are two finite families of bifunctions satisfying conditions *(A1) - (A4)* and g_i is upper semicontinuous in the first argument for each $i = 1, 2, \dots, m$. Let $\phi_i \pmod m : C_i \to \mathbb{R}\cup +\infty$ and $\varphi_i \pmod m : D_i \to \mathbb{R}\cup +\infty$ be proper lower semicontinuous and convex functions, and $P_i \pmod m : C_i \to H_1$ and $Q_i \pmod m : D_i \to H_2$ are continuous and monotone mappings. Let $S_i \pmod m : C_i \to P(C_i)$ be a finite family of multivalued mappings such that P_{S_i} is k_i-demicontractive with $k = \max\{k_i\}$ and suppose $I - P_{S_i}$ is demiclosed at zero for each $i = 1, 2, \dots, m$, and let $\{h_n\}$ $(h_n : H_1 \to H_1)$ be a sequence of μ_n-contractive mappings with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ for any $x \in K$, where K is any bounded subset of H_1. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.9.

Step 0. Select initial data $x_0, x_1 \in C$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \ \frac{\epsilon_n}{||x_n-x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.}\end{cases} \tag{3.33}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.34}$$

Step 3. Compute

$$z_{n,i} = w_n - \lambda_{n,i}A_i^*(A_iw_n - T_{s_{n,i}}^{g_i}A_iw_n), \tag{3.35}$$

where

$$\lambda_{n,i} := \begin{cases} \tau_n \frac{||(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}, & \text{if } A_i w_n \neq T^{g_i}_{s_{n,i}} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.36}$$

Step 4. Compute

$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} T^{f_i}_{r_{n,i}} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \tag{3.37}$$

where $v_{n,i} \in P_{S_i}(u_n)$. Set $n := n+1$ and return to **Step 1**.

Suppose that the solution set $\Omega \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$, and suppose Assumptions (A1)-(A4) and (B1) - (B7) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.9 converges strongly to a point $\hat{x} \in \Omega \cap \bigcap_{i=1}^m F(S_i)$, where $\hat{x} = P_{\Omega \cap \bigcap_{i=1}^m F(S_i)} h(\hat{x})$.

Proof. Since P_{S_i} satisfies the common endpoint condition and $F(S_i) = F(P_{S_i})$ for each $i = 1, 2, \ldots, m$, then the result follows from Theorem 3.4.

If we set $P_i = Q_i = 0$ in (1.16)-(1.17), we obtain the SSMEP (1.18)-(1.19). In [28], the author proved a weak convergence theorem for solving (1.18)-(1.19) and fixed point problem for a nonexpansive mapping. However, setting $P_i = Q_i = 0$ in Theorem 3.4 we obtain a strong convergence result for approximating a common solution of the *SSMEP* (1.18)-(1.19) and fixed point of finite family of multivalued demicontractive mappings. Hence, the following result complements the result in [28].

Corollary 3.10. Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_i \pmod m : H_1 \to H_2$ be a finite family of bounded linear operators with adjoint A_i^*, $f_i \pmod m : C_i \times C_i \to \mathbb{R}$ and $g_i \pmod m : D_i \times D_i \to \mathbb{R}$ are two finite families of bifunctions satisfying conditions *(A1) - (A4)* and g_i is upper semicontinuous in the first argument for each $i = 1, 2, \ldots, m$. Let $\phi_i \pmod m : C_i \to \mathbb{R} \cup +\infty$ and $\varphi_i \pmod m : D_i \to \mathbb{R} \cup +\infty$ be proper lower semicontinuous and convex functions and let $S_i \pmod m : C_i \to CB(C_i)$ be a finite family of multivalued demicontractive mappings with constant k_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, and $k = \max\{k_i\}$. Let $\{h_n\}$ ($h_n : H_1 \to H_1$) be a sequence of μ_n-contractive mappings with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ for any $x \in K$, where K is any bounded subset of H_1. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.11.

Step 0. Select initial data $x_0, x_1 \in C$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.38}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.39}$$

Step 3. Compute

$$z_{n,i} = w_n - \lambda_{n,i} A_i^*(A_i w_n - T^{g_i}_{s_{n,i}} A_i w_n), \tag{3.40}$$

where

$$\lambda_{n,i} := \begin{cases} \tau_n \frac{||(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}, & \text{if } A_i w_n \neq T^{g_i}_{s_{n,i}} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.41}$$

Step 4. Compute

$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} T^{f_i}_{r_{n,i}} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \tag{3.42}$$

where $v_{n,i} \in S_i u_n$. Set $n := n+1$ and return to **Step 1**.

Suppose that the solution set $\Omega_{\phi,\varphi} \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$, and suppose Assumptions (A1)-(A4) and (B1) - (B7) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.11 converges strongly to a point $\hat{x} \in \Omega_{\phi,\varphi} \cap \bigcap_{i=1}^m F(S_i)$, where

$$\hat{x} = P_{\Omega_{\phi,\varphi} \cap \bigcap_{i=1}^m F(S_i)} h(\hat{x}).$$

Setting $\phi_i = \varphi_i = 0$ in Theorem 3.4, we obtain the following consequent result for approximating a common solution of the SSGEP and fixed point problem for multivalued demicontractive mappings. The result generalizes and complements the results in [13].

Corollary 3.12. Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_i \pmod m : H_1 \to H_2$ be a finite family of bounded linear operators with adjoint A_i^*, $f_i \pmod m : C_i \times C_i \to \mathbb{R}$ and $g_i \pmod m : D_i \times D_i \to \mathbb{R}$ are two finite families of bifunctions satisfying conditions *(A1) - (A4)* and g_i is upper semicontinuous in the first argument for each $i = 1, 2, \ldots, m$. Let $P_i \pmod m : C_i \to H_1$ and $Q_i \pmod m : D_i \to H_2$ be continuous and monotone mappings and let $S_i \pmod m : C_i \to CB(C_i)$ be a finite family of multivalued demicontractive mappings with constant k_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, and $k = \max\{k_i\}$. Let $\{h_n\}$ $(h_n : H_1 \to H_1)$ be a sequence of μ_n-contractive mappings with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ for any $x \in K$, where K is any bounded subset of H_1. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.13.

Step 0. Select initial data $x_0, x_1 \in C$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \ \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.43}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.44}$$

Step 3. Compute

$$z_{n,i} = w_n - \lambda_{n,i} A_i^*(A_i w_n - T^{g_i}_{s_{n,i}} A_i w_n), \tag{3.45}$$

where

$$\lambda_{n,i} := \begin{cases} \tau_n \frac{||(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}{||A_i^*(I-T^{g_i}_{s_{n,i}})A_i w_n||^2}, & \text{if } A_i w_n \neq T^{g_i}_{s_{n,i}} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.46}$$

Step 4. Compute

$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} T^{f_i}_{r_{n,i}} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \tag{3.47}$$

where $v_{n,i} \in S_i u_n$. Set $n := n+1$ and return to **Step 1**.

Suppose that the solution set $\Omega_{P,Q} \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$, and suppose Assumptions (A1)-(A4) and (B1) - (B7) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.13 converges strongly to a point $\hat{x} \in \Omega_{P,Q} \cap \bigcap_{i=1}^m F(S_i)$, where

$$\hat{x} = P_{\Omega_{P,Q} \cap \bigcap_{i=1}^m F(S_i)} h(\hat{x}).$$

Putting $\phi_i = \varphi_i = 0$ and $P_i = Q_i = 0$ in Theorem 3.4, we obtain the following consequent result for approximating a common solution of the SSEP and fixed point problem for multivalued demicontractive mappings. The result complements the results in [31, 51] and generalises as well as improves the results in [38, 40].

Corollary. Let C_i and D_i be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_i \pmod m : H_1 \to H_2$ be a finite family of bounded linear operators with adjoint A_i^*, $f_i \pmod m : C_i \times C_i \to \mathbb{R}$ and $g_i \pmod m : D_i \times D_i \to \mathbb{R}$ are two finite families of bifunctions satisfying conditions *(A1) - (A4)* and g_i is upper semicontinuous in the first argument for each $i = 1, 2, \ldots, m$. Let $S_i \pmod m : C_i \to CB(C_i)$ be a finite family of multivalued demicontractive mappings with constant k_i such that each $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in F(S_i)$, and $k = \max\{k_i\}$. Let $\{h_n\}$ ($h_n : H_1 \to H_1$) be a sequence of μ_n-contractive mappings with $0 < \mu_* \leq \mu_n \leq \mu^* < 1$ and $\{h_n(x)\}$ is uniformly convergent to $h(x)$ for any $x \in K$, where K is any bounded subset of H_1. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.15.

Step 0. Select initial data $x_0, x_1 \in C$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.48}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}). \tag{3.49}$$

Step 3. Compute
$$z_{n,i} = w_n - \lambda_{n,i} A_i^*(A_i w_n - T^{g_i}_{s_{n,i}} A_i w_n), \tag{3.50}$$
where
$$\lambda_{n,i} := \begin{cases} \tau_n \frac{||(I - T^{g_i}_{s_{n,i}})A_i w_n||^2}{||A_i^*(I - T^{g_i}_{s_{n,i}})A_i w_n||^2}, & \text{if } A_i w_n \neq T^{g_i}_{s_{n,i}} A_i w_n, \\ \lambda, & \text{otherwise } (\lambda \text{ being any nonnegative real number}). \end{cases} \tag{3.51}$$

Step 4. Compute
$$\begin{cases} u_n = \beta_{n,0} w_n + \sum_{i=1}^m \beta_{n,i} T^{f_i}_{r_{n,i}} z_{n,i} \\ y_n = \delta_{n,0} u_n + \sum_{i=1}^m \delta_{n,i} v_{n,i} \\ x_{n+1} = \alpha_n h_n(x_n) + \xi_n x_n + \gamma_n y_n, \end{cases} \tag{3.52}$$
where $v_{n,i} \in S_i u_n$. Set $n := n + 1$ and return to **Step 1**.

Suppose that the solution set $\Omega_{0,0} \cap \bigcap_{i=1}^m F(S_i) \neq \emptyset$, and suppose Assumptions (A1)-(A4) and (B1) - (B7) are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 3.15 converges strongly to a point $\hat{x} \in \Omega_{0,0} \cap \bigcap_{i=1}^m F(S_i)$, where
$$\hat{x} = P_{\Omega_{0,0} \cap \bigcap_{i=1}^m F(S_i)} h(\hat{x}).$$

4. Application and numerical example

4.1 **Split Convex Minimization Problem.** In this subsection, we apply our result to study the following system of split convex minimisation problem: Find
$$\hat{x} \in \bigcap_{i=1}^m F(S_i) \text{ such that } \hat{x} = \arg\min_{x \in C_i} (F_i(x) + \Theta_i(x) + \Phi_i(x)), \tag{4.1}$$
such that
$$A_i \hat{x} = \arg\min_{y \in D_i} (G_i(y) + \Psi_i(y) + \Pi_i(y)), \tag{4.2}$$
where C_i and D_i are nonempty closed and convex subset of H_1 and H_2 respectively. Moreover, $F_i, \Phi_i : C_i \to \mathbb{R}$ and $G_i, \Pi_i : D_i \to \mathbb{R}$ are four convex and lower semicontinuous functionals, $\Theta_i : C_i \to \mathbb{R}$ and $\Psi_i : D_i \to \mathbb{R}$ are convex continuously differentiable functions and $A_i : H_1 \to H_2$ is a bounded linear operator. Let $f_i(x,y) = F_i(y) - F_i(x), g_i(x,y) = G_i(y) - G_i(x), P_i = \nabla\Theta_i, Q_i = \nabla\Psi_i$, where $\nabla\Theta_i$ and $\nabla\Psi_i$ denote the gradient of Θ_i and Ψ_i respectively, and let $\phi_i = \Phi_i$ and $\varphi_i = \Pi_i$. Then the system of split convex minimisation problem (4.1)-(4.2) can be formulated as the following system of split generalized mixed equilibrium problem:
find $\hat{x} \in \bigcap_{i=1}^m F(S_i)$, such that
$$F_i(x) - F_i(\hat{x}) + \langle \nabla\Theta_i \hat{x}, x - \hat{x} \rangle + \Phi_i(x) - \Phi_i(\hat{x}) \geq 0, \text{ for all } x \in C_i, \tag{4.3}$$
and $\hat{y} = A_i \hat{x} \in D = \cap_{i=1}^m D_i$ solves
$$G_i(y) - G_i(\hat{x}) + \langle \nabla\Psi_i \hat{x}, y - \hat{x} \rangle + \Pi_i(y) - \Pi_i(\hat{x}) \geq 0, \text{ for all } y \in D_i. \tag{4.4}$$
Hence, Theorem 3.4 provides a strong convergence result for solving a system of split convex minimisation problem (4.1)-(4.2).

4.2 **Numerical Example.** In this subsection, we provide a numerical example to compare the performance of our proposed Algorithm 3 with its non-inertial version.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}$ with the usual norm. For $i = 1, 2, \ldots, 5$, let $C_i = [-i, 0]$ and $D_i = [-10 - i, 0]$, then we have that $C = \bigcap_{i=1}^{5} C_i = [-1, 0]$ and $D = \bigcap_{i=1}^{5} D_i = [-11, 0]$. Define $P_i = Q_i = 0, \phi_i = \varphi_i = 0$, and for each $x, y \in C_i$ define $f_i : C_i \times C_i \to H_1$ by $f_i(x, y) = ix(y - x)$ and define $g_i : D_i \times D_i$ by $g_i(u, v) = (10 + i)u(v - u)$, for each $u, v \in D_i$. Define $A_i : H_1 \to H_2$ by $A_i x = \frac{ix}{2}$ for all $x \in H_1$, then $A_i^* y = \frac{y}{2}$ for all $y \in H_2$. It can be verified that

$$T^{g_i}_{s_{n,i}} A_i x = \frac{ix}{2 + 2(10 + i)s_{n,i}} \quad \text{for all } x \in H_1$$

and

$$T^{f_i}_{r_{n,i}} x = \frac{x}{1 + ir_{n,i}} \quad \text{for all } x \in H_1$$

Define $S_i : C_i \to CB(C_i)$ by

$$S_i x = \begin{cases} [-\frac{i|x|}{i|x|+1}, 0], & x \in [-5, -1); \\ \{0\}, & x \in [-1, 0]. \end{cases} \tag{4.5}$$

It is easy to see that S_i is quasi-nonexpansive and thus $0-$demicontractive with $F(S_i) = \{0\}$, for each $i = 1, 2, \ldots, 5$.

Next, we find a common solution $\hat{x} \in C$ for the following system of generalized mixed equilibrium problems:

$$f_i(\hat{x}, x) + \langle P_i \hat{x}, x - \hat{x} \rangle + \phi_i(x) - \phi_i(\hat{x}) \geq 0, \quad \text{for all } x \in C_i,\ i = 1, 2, \ldots, m.$$

Since $\phi_i = 0$ and $P_i = 0$, then we find a point $\hat{x}$ that has to be a solution of the inequality $i\hat{x}(x - \hat{x}) \geq 0$ for all $x \in C_i$. This problem has a unique solution $\hat{x} = 0$. Then it follows that the point $\hat{y} = A_i \hat{x} = 0$ will be a solution for the following system of generalized mixed equilibrium problems:

$$g_i(\hat{y}, y) + \langle Q_i \hat{y}, y - \hat{y} \rangle + \varphi_i(y) - \varphi_i(\hat{y}) \geq 0, \quad \text{for all } y \in D_i,\ i = 1, 2, \ldots, m.$$

That is, $\hat{y} = 0$ solves the inequality $(10 + i)\hat{y}(y - \hat{y}) \geq 0$ for all $y \in D_i$. Hence, we obtain that $\hat{x} = 0$ is a common solution for the system of split generalized mixed equilibrium problem and fixed point problem, that is, $0 \in \Omega \cap \bigcap_{i=1}^{m} F(S_i)$.

Let $h_n(x) = \frac{(n+1)x}{3n}$, $\beta_{n,0} = \frac{1}{n+2}$, $\beta_{n,i} = \frac{n+1}{5(n+2)}$, $\delta_{n,0} = \frac{1}{n+1}$, $\delta_{n,i} = \frac{n}{5(n+1)}$, $\alpha_n = \frac{1}{3n}$, $\xi_n = \gamma_n = \frac{1}{2}(1 - \frac{1}{3n})$, $\epsilon = \frac{1}{(n+1)^4}$ and $\theta = 4$, $r_{n,i} = s_{n,i} = \frac{n}{n+i}$ in Algorithm 3.1 for each $n \in \mathbb{N}$. It is easy to check that f_i, g_i and the control parameters satisfy all the conditions in Theorem 3.4.

We choose different initial values as follows:

Case Ia: $x_0 = 16, x_1 = 3$;

Case Ib: $x_0 = 55, x_1 = 6$;

Case Ic: $x_0 = 15, x_1 = 61$;

Case Id: $x_0 = -15, x_1 = 61$.

Using MATLAB 2017(b), we compare the performance of our Algorithm 3.1 with its non-inertial version. The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-3}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 1 and Table 1, respectively.

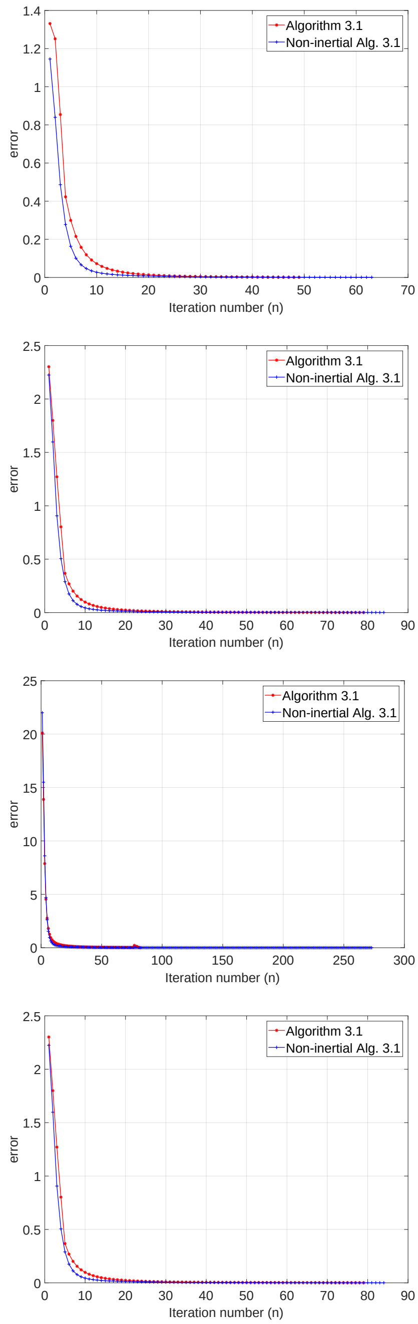

FIGURE 1. Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

Table 1. Numerical results

		Alg. 3	Non-inertial Alg. 3
Case Ia	CPU time (sec) No of Iter.	0.0013 49	0.0015 63
Case Ib	CPU time (sec) No. of Iter.	0.0017 79	0.0022 84
Case Ic	CPU time (sec) No of Iter.	0.0014 82	0.0023 273
Case Id	CPU time (sec) No of Iter.	0.0016 79	0.0020 84

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