Fixed Point Theory, 23(2022), No. 1, 35-44
DOI: 10.24193/fpt-ro.2022.1.03
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

GENERALIZED METRIC SPACES AND SOME RELATED FIXED POINT THEOREMS

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Abstract. In this paper, we study the existence of fixed point for relational endomorphisms. This class of mappings generalizes that of order and edge preserving mappings on posets and graphs respectively. As an application, we give a DeMarr-Type result for a family of Banach operator pairs in a binary relational system.
Key Words and Phrases: Binary relation, fixed point, generalized metric spaces, nonexpansive mapping, normal structure, relational endomorphisms.
2020 Mathematics Subject Classification: 37C25, 06A75, 47H09, 47H10, 08A02.

1. Introduction and preliminaries

The study of metric fixed point theory is a central theme in modern mathematics. The research in this theory often goes through a generalization of the space. Some generalizations of the metric space have been considered by modifying the metric axioms (see [4, 10, 16]).

Another way to generalize the metric space is to consider the positive cone as a set of values of the distance. However, it showed that, in this framework, most of fixed point results can be derived directly from the standard metric space. For more details one can see [10].

We consider generalized metric spaces which have their origins in the study of discrete structures, namely posets, graphs or binary relations. In this setting, instead of real numbers the distance takes values in an ordered monoid equipped with an involution (see [6, 7, 8, 9, 11] for more details). In this work, the ordered monoid considered is made of binary relations. The basic concepts about binary relations are the followings.

Definition 1.1. Let E be a set. A binary relation on E is any subset r of $E \times E$. The inverse of r is the binary relation $r^{-1} := \{(x, y) : (y, x) \in r\}$. The diagonal is $\Delta := \{(x, x) : x \in E\}$. The product $r \cdot s$ of two binary relations r and s is the binary relation made of pairs (x, y) such that $(x, z) \in r$ and $(z, y) \in s$ for some $z \in E$.

Let $\mathcal{R}$ be the set of binary relations on a nonempty set E satisfying the five following items:

- (i) Δ and $E \times E$ belong to $\mathcal{R}$;
- (ii) Each relation $r \in \mathcal{R}$ contains Δ;
- (iii) $\mathcal{R}$ is closed under arbitrary intersection;
- (iv) For all $r, s \in \mathcal{R}$, $r \cdot s \in \mathcal{R}$;
- (v) $r^{-1} \in \mathcal{R}$ for every $r \in \mathcal{R}$.

Denote by 0 the set Δ and $r \oplus s := r \cdot s$. Then $\mathcal{R}$ becomes a semigroup. Consider the inclusion order, i.e., for each $r, s \in \mathcal{R}$, $r \preceq s \Leftrightarrow r \subseteq s$. Then $\mathcal{R}$ is an ordered semigroup. If we set $\overline{r} := r^{-1}$, this defines an involution mapping on $\mathcal{R}$ which preserves the order and reverses the semigroup operation, that is, $r \preceq s \Rightarrow \overline{r} \preceq \overline{s}$ and $\overline{r \oplus s} = \overline{s} \oplus \overline{r}$. Note that the neutral element 0 of the semigroup $\mathcal{R}$ is the least element for the ordering. With this construction, the set $\mathcal{V} = (\mathcal{R}, \oplus, 0, ^-, \preceq)$ is an involutive algebra.

Let $x, y \in E$ and set

$$d(x, y) := \bigcap \{r \in \mathcal{V} : (x, y) \in r\}.$$

Note that for all $x, y \in E$ and $r \in \mathcal{R}$ we have

- $(x, y) \in d(x, y)$;
- $(x, y) \in r \Leftrightarrow d(x, y) \subseteq r$.

Let $x, y, z \in E$.
From $(x, z) \in d(x, z)$ and $(z, y) \in d(z, y)$, we have $(x, y) \in d(x, z) \cdot d(z, y)$.
Consequently,

$$d(x, y) \subseteq d(x, z) \cdot d(z, y).$$

Thus, we obtain the following result.

Lemma 1.2. *The mapping $d : E \times E \to \mathcal{V}$ defined by $(x, y) \mapsto d(x, y) = \bigcap \{r \in \mathcal{V} : (x, y) \in r\}$ satisfies the following conditions:*

- (d_1) *For all $x, y \in E$, $d(x, y) = 0 \Leftrightarrow x = y$;*
- (d_2) *For all $x, y \in E$, $d(x, y) = \overline{d(y, x)}$;*
- (d_3) *For all $x, y, z \in E$, $d(x, y) \preceq d(x, z) \oplus d(z, y)$.*

The pair (E, d) is called *generalized metric space* and the mapping d is called *generalized distance* (cf. [4, p. 75]).

Example 1.3. (Discrete distance) Let E be a nonempty set endowed with the binary relational system $\mathcal{R} = \{\Delta, E \times E\}$. The generalized metric d is defined as follows:

$$\begin{cases} d(x, y) = E \times E, & \text{if } x \neq y \\ d(x, y) = \Delta, & \text{if } x = y. \end{cases}$$

Conversely, if d is the ordinary discrete metric in E, then define the two binary relations on E by:

$$r_0 = \{(x,y) : d(x,y) = 0\} \text{ and } r_1 = \{(x,y) : d(x,y) \leq 1\}.$$

Then by definition of the discrete metric $r_0 = \Delta$ and $r_1 = E \times E$. The set $\mathcal{R} := \{r_0, r_1\}$ satisfies the five conditions above $(i) - (v)$. Thus, the discrete metric can be viewed as a generalized metric.

The following example shows that every ordinary convex metric space can be viewed as a generalized metric space. Recall that a metric space (E, δ) is a convex if for all $x, y \in E$ and $r_1, r_2 \in \mathbb{R}_+$ such that $\delta(x,y) \leq r_1 + r_2$, there exists $z \in E$ such that $\delta(x,z) \leq r_1$ and $\delta(z,y) \leq r_2$.

Example 1.4. Let (E, δ) be a real valued convex metric space. Consider the relational system $\mathcal{R} = \{\delta_r : r \in \mathbb{R}_+ \bigcup \{\infty\}\}$ such that: $\delta_r := \{(x,y) : \delta(x,y) \leq r\}$.

(1) For each $r \in \mathbb{R}_+ \bigcup \{\infty\}$, the relation δ_r is symmetric.

(2) The net $(\delta_r)_{r \in \mathbb{R}_+}$ is nondecreasing and

$$\bigcup_{r \in \mathbb{R}_+} \delta_r = \delta_\infty = E \times E \text{ and } \bigcap_{r \in \mathbb{R}_+} \delta_r = \delta_0 = \Delta.$$

(3) One can verify that $\delta_r \cdot \delta_{r'} = \delta_{r+r'}$, for all $r, r' \in \mathbb{R}_+ \bigcup \{\infty\}$.

The binary relational system $\mathcal{R}$ satisfies the five conditions above $(i) - (v)$ and the generalized metric d is defined as follows:

$$d(x,y) = \delta_r, \text{ where } r = \inf\{s \in \mathbb{R}_+ \bigcup \{\infty\} : \delta(x,y) \leq s\}, \text{ for all } x, y \in E.$$

Example 1.5. (Reflexive symmetric graph) Let E be a nonempty set endowed with a reflexive and symmetric binary relation ρ. We consider the involutive algebra $\mathcal{V} = (\mathcal{R}, \oplus, 0, ^-, \preceq)$ where the binary relational system $\mathcal{R} = \{\rho^n : n \in \mathbb{N} \cup \{\infty\}\}$ with $\rho^0 = \Delta$ and $\rho^\infty = E \times E$.

The symmetric reflexive graph $G = (E, \rho)$ is a generalized metric space over $\mathcal{V}$. The generalized metric d is defined by

$$d(x,y) := \rho^{\mu(x,y)}, \text{ for all } x, y \in E,$$

where $\mu(x,y) = \bigwedge\{\ n \in \mathbb{N} :\ (x,y) \in \rho^n\ \}$.

Conversely, every metric space (E, d) over $\mathcal{V}$ can be viewed as a reflexive symmetric graph; the vertices are the elements of E and the set of edges $\mathcal{E}$ (including the loops) is defined as follows:

$$(x,y) \in \mathcal{E} \Leftrightarrow d(x,y) \preceq \rho.$$

Definition 1.6. Let (E, d) be a generalized metric space and $T : E \to E$ a mapping. T is said to be

- *nonexpansive* if

$$d(Tx, Ty) \leq d(x,y), \quad \text{for all } \ x, y \in E;$$

- *endomorphism* if T preserves the relations of $\mathcal{R}$, that is, for every $r \in \mathcal{R}$,

$$(x,y) \in r \Rightarrow (Tx, Ty) \in r.$$

Example 1.7. Let n be a positive integer ≥ 2. Let $E = \mathbb{Z}$ endowed with the binary relational system $\mathcal{R} := \{\Delta, \rho, E \times E\}$, where

$$(x, y) \in \rho \Leftrightarrow x \equiv y \; mod \; n, \quad \text{for all} \quad x, y \in E.$$

Note that $\mathcal{R}$ verifies the conditions $(i) - (v)$ and so, E can be viewed as a generalized metric space.

Let $T : E \to E$ be the mapping defined by $Tx = 2x$. It is clear that T is an endomorphism but not a nonexpansive mapping with respect to the usual metric.

Remark 1.8. It is easy to check that on a generalized metric space (E, d), the nonexpansive mappings are exactly the endomorphisms on E.

The study of fixed point theorems for nonexpansive mappings finds its root in the work of Kirk [13] in Banach spaces. He showed that if A is a nonempty weakly compact convex subset in Banach space with the *normal structure*, then every nonexpansive mapping $T : A \to A$ has a fixed point. This result has been generalized and extended by several authors, see for instance [1, 12, 14, 15] and the references therein. In [17], Penot extended the notions of compactness, convexity and normal structure to metric spaces and gave some fixed point results in this framework.

Let $r \in \mathcal{V}$ be a binary relation and let $x \in E$. The *ball* of center x and radius r is the set

$$B(x, r) = \{y \in E \; : \; (x, y) \in r\}.$$

For a subset A of the generalized metric space (E, d), set

$$cov\,(A) = \bigcap \{B\,(x, r) \; : \; x \in A, \; A \subseteq B\,(x, r)\}\,.$$

We will say that A is an *admissible* set if $A = cov\,(A)$. The family of all admissible subsets of E will be denoted by $\mathcal{A}\,(E)$.

The notion of compactness in generalized metric spaces is defined as follows:

Definition 1.9. The generalized metric space (E, d) has *compact structure* if for every family $\mathcal{F}$ of members of $\mathcal{A}(E)$, the intersection of $\mathcal{F}$ is nonempty provided that the intersection of each finite subfamily of $\mathcal{F}$ is nonempty.

We have the following characterization of the compactness.

Lemma 1.10. [11, Lemma 2.5] *The generalized metric space (E, d) has compact structure if each descending chain of nonempty sets in $\mathcal{A}(E)$ has a nonempty intersection.*

Using the analogues of the metric Chebyshev center and radius, the authors in [11] introduced the notion of normal structure in generalized metric spaces.

Definition 1.11. Let A be a subset of E.

(1) The *relative radius* of A is the set
$r_x(A) := \bigcap\{r \in \mathcal{V} : A \subseteq B(x, r)\} = \bigcap\{r \in \mathcal{V} : \{x\} \times A \subseteq r\}$, for all $x \in A$.

(2) The *diameter* of A is the set

$$\delta(A) := \bigcap\{r \in \mathcal{V} : A \times A \subseteq r\}.$$

Claim 1.12. *For all $A \subseteq E$, for each $x \in A$, we have $r_x(A) \preceq \delta(A)$.*

Proof. Let $x \in A$ and let $r \in \mathcal{V}$ such that $A \times A \subseteq r$. Then, it is clear that

$$r_x(A) \subseteq r.$$

Thus, $r_x(A) \subseteq \bigcap\{r \in \mathcal{V} : A \times A \subseteq r\}$. Hence, $r_x(A) \preceq \delta(A)$.

Definition 1.13. We say that E has *normal structure* if for each admissible subset A of E such that $|A| > 1$, there exists $x \in A$ such that $r_x(A) \neq \delta(A)$.

In this paper, we generalize the constructive lemma due to Gillespie and Williams [5]. We prove that if a generalized metric space (E, d) has compact and normal structure, then every nonexpansive mapping has a fixed point. From this result, we obtain Tarski's fixed point theorem as a corollary. We conclude by proving DeMarr-type fixed point theorem for an arbitrary family of symmetric Banach operator pairs.

2. Fixed point property in generalized metric space

We start by investigating the validity of the Gillespie-Williams lemma in generalized metric spaces.

Lemma 2.1. *Let (E, d) be a generalized metric space and $T : E \to E$ be a non-expansive mapping. Let $A \in \mathcal{A}(E)$ such that $0 \prec \delta(A)$ and A is $T-$invariant, i.e., $T(A) \subseteq A$. If we suppose that E has normal structure, there exists $A_0 \in \mathcal{A}(E)$ such that $A_0 \subsetneq A$, $\delta(A_0) \prec \delta(A)$ and A_0 is $T-$invariant.*

Proof. Since E has the normal structure, there exists $u \in A$ such that $r_u(A) \prec \delta(A)$. Set $\alpha = r_u(A) \oplus (r_u(A))^{-1}$ and consider the set $D = \{x \in A : A \subseteq B(x, \alpha)\}$. From $u \in D$, we have $D \neq \emptyset$ and since $D = \bigcap\limits_{y \in A} B(y, \alpha) \bigcap A$, we get $D \in \mathcal{A}(E)$.

Now, consider the set

$$\mathcal{F} = \{M \in \mathcal{A}(E) : D \subseteq M \ \text{ and } \ T(M) \subseteq M\}.$$

Since $A \in \mathcal{F}$, we have $\mathcal{F} \neq \emptyset$. Set $L = \bigcap\limits_{M \in \mathcal{F}} M$. It is easy to see that $L \in \mathcal{F}$. Let us consider the set $B = D \bigcup T(L)$.

Claim 1. $cov(B) = L$. Indeed, since $D \subseteq L$ and $T(L) \subseteq L$, then $B \subseteq L$. Thus, $cov(B) \subseteq cov(L) = L$ and so $T(cov(B)) \subseteq T(L) \subseteq B \subseteq cov(B)$. From $cov(B) \in \mathcal{A}(E)$ we have $cov(B) \in \mathcal{F}$. It follows that that $L \subseteq cov(B)$. Therefore, $cov(B) = L$.

Define $A_0 = \{x \in L : L \subseteq B(x, \alpha)\}$. Then, $A_0 = \bigcap\limits_{y \in L} B(y, \alpha) \bigcap L$. Thus, $A_0 \in \mathcal{A}(E)$.

Claim 2. $T(A_0) \subseteq A_0$.

Indeed, let $x \in A_0$. Then, $L \subseteq B(x, \alpha)$. Since T is nonexpansive, $T(L) \subseteq B(Tx, \alpha)$. On the other hand,

$$z \in D \Rightarrow A \subseteq B(z, \alpha) \Rightarrow L \subseteq B(z, \alpha) \Rightarrow Tx \in B(z, \alpha) \Rightarrow z \in B(Tx, \alpha).$$

Then, $D \subseteq B(Tx, \alpha)$. Hence, $B \subseteq B(Tx, \alpha)$ and so $L = cov(B) \subseteq B(Tx, \alpha)$. According to the definition of A_0, we have $Tx \in A_0$. This proves the claim.

Claim 3. *$A_0 \subsetneq A$ and $\delta(A_0) \prec \delta(A)$.*

Indeed, according to the above construction, $D \subseteq A_0 \subseteq L \subseteq A$. Since for every $x \in A_0$, we have $A_0 \subseteq B(x, \alpha)$ amounting to $A_0 \times A_0 \subseteq \alpha$. It follows that $\delta(A_0) \preceq r_u(A) \prec \delta(A)$. Therefore, $A_0 \subsetneq A$.

The proof of Lemma 2.1 is therefore complete.

Theorem 2.2. *Let (E, d) be a generalized metric space and C a nonempty admissible subset of E. Let $T : C \to C$ be a nonexpansive mapping. If we suppose that E has normal and compact structure, then T admits a fixed point in C.*

Proof. If $\delta(C) = 0$, then C is reduced to a singleton which is a fixed point of T. So, we suppose that $0 \prec \delta(C)$. According to Lemma 2.1, there exists a nonempty admissible subset $C_0 \subsetneq C$ such that $T(C_0) \subseteq C_0$. If $\delta(C_0) = 0$, then, as above, T has a fixed point. If not, again by applying Lemma 2.1, there exists a nonempty admissible subset $C_1 \subsetneq C_0$ such that $T(C_1) \subseteq C_1$. If this process isn't over, we obtain a strictly decreasing chain $(C_i)_{i<\omega} \subseteq \mathcal{A}(E)$. Using the compactness of E, the intersection $\bigcap\limits_{i<\omega} C_i$ is nonempty admissible subset of C. Set $C_\omega := \bigcap\limits_{i<\omega} C_i$. Let α be an ordinal number. Suppose that the orbit $(C_\beta)_{\beta<\alpha}$ satisfying the following conditions is well constructed:

(i) C_β is a nonempty admissible $T-$invariant subset of C;
(ii) if β is a successor ordinal, then $C_\beta \subsetneq C_{\beta-1}$ if $0 < \delta(C_{\beta-1})$ and $C_\beta = C_{\beta-1}$ if $\delta(C_{\beta-1}) = 0$;
(iii) if β is a limit ordinal, $C_\beta = \bigcap\limits_{i<\beta} C_i$.

To define C_α, we have to distinguish two cases:

Case 1. If α is a successor ordinal , $C_\alpha = C_{\alpha-1}$ if $\delta(C_{\alpha-1}) = 0$. Otherwise, C_α will be defined by Lemma 2.1.

Case 2. If α is a limit ordinal, the compactness of E ensures that $C_\alpha := \bigcap\limits_{\beta<\alpha} C_\beta$ is a nonempty set, and it is easy to see that it is an admissible $T-$invariant subset.

Let Γ be an ordinal whose cardinality exceeds that of $\mathcal{P}(E)$. The orbit $(C_\alpha)_{\alpha\leq\Gamma}$ must be stationary, i.e., there exists an ordinal $\beta \leq \Gamma$ such that $TC_\beta \subseteq C_\beta$ and $\delta(C_\beta) = 0$. Hence, C_β is a fixed point of T.

As a corollary, we prove Tarski's fixed point theorem using our main result.

Corollary 2.3.[18, Tarski 1955] *Let $(E, \leq)$ be a complete lattice. If $T : E \to E$ is an order-preserving mapping, then T admits a fixed point in E.*

Proof. Consider on E the binary relational system $\mathcal{R} := \{\Delta, \leq, \geq, E \times E\}$ which is represented in the following figure:

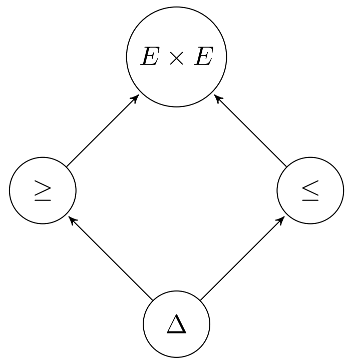

FIGURE 1. The order structure of the monoid $\mathcal{V}$.

Let $d : E \times E \to \mathcal{V}$ be the generalized metric defined as follows:

$$d(x,y) := \begin{cases} \Delta & \text{if} \quad x = y; \\ \leq & \text{if} \quad x \leq y; \\ \geq & \text{if} \quad x \geq y; \\ E \times E & \text{if} \quad x \text{ and } y \text{ are not comparable.} \end{cases}$$

In this case, the ball of center x and radius $\alpha \in \mathcal{V}$ is one of the following:

(i) $B(x,\Delta) = \{x\}$ (singleton).
(ii) $B(x,\leq) = \uparrow x := \{y \in E : x \leq y\}$ (principal final segment).
(iii) $B(x,\geq) = \downarrow x := \{y \in E : y \leq x\}$ (principal initial segment).
(iv) $B(x, E \times E) = E$.

Let A be an intersection of balls of E such that $|A| > 1$ and let $x \in A$. If $r_x(A) = \leq$ (resp. $r_x(A) = \geq$), then $A = B(x,\leq)$ (resp. $A = B(x,\geq)$). Thus in both cases $\delta(A) = E \times E$. On the other hand, if A contains at least two incomparable elements then $\delta(A) = E \times E$. Choosing $x = \bigvee A$, we get $r_x(A) = \geq$. Thus, $r_x(A) \prec \delta(A)$, which implies that E has the normal structure.

Now, let us show that E has the compact structure. For this, let $(B_i)_{i\in I}$ be a decreasing family of balls, where I is a totally ordered index set. We have to distinguish three cases:

(1) For each $i \in I$, B_i is a principal final segment generated by some element $x_i \in E$, that is $B_i = \uparrow x_i$. In this case, $\bigcap_{i\in I} B_i = \uparrow \vee x_i$. Hence, $\bigcap_{i\in I} B_i \neq \emptyset$.
(2) For each $i \in I$, B_i is a principal initial segment generated by some element $x_i \in E$, that is $B_i = \downarrow x_i$. In this case, $\bigcap_{i\in I} B_i = \downarrow \wedge x_i$. Hence, $\bigcap_{i\in I} B_i \neq \emptyset$.
(3) There exists $i_0 \in I$ such that for all $i \geq i_0$ in I we have $B_i = [x_i, y_i]$. Since the family $(B_i)_{i\in I}$ is decreasing we have

$$x_i \leq y_j, \text{ for all } i \geq i_0 \text{ and } j \geq i_0.$$

Then, $x_i \leq \wedge y_j$, for each $i \geq i_0$. And so, $\vee x_i \leq \wedge y_i$. Hence,

$$\emptyset \neq [\vee x_i, \wedge y_i] \subseteq \bigcap_{i \geq i_0} [x_i, y_i] = \bigcap_{i \geq i_0} B_i = \bigcap_{i \in I} B_i.$$

Consequently E has compact structure. As $T : E \to E$ is an endomorphism, from Remark 1.8, it is a nonexpansive mapping. Since E is an admissible subset, according to Theorem 2.2, T has a fixed point.

3. Application

Common fixed point results for a family of commuting nonexpansive mappings find their root in the work of Tarski [18] for complete lattices and that of DeMarr [3] for Banach spaces. In this section we give an application of Theorem 2.2 to obtain a common fixed point result for a family of nonexpansive mappings on generalized metric spaces. This will be done by dropping the commutativity condition and using the notion of Banach operator pair already defined in [2].

Definition 3.1. Let T and S be two self-mappings on a generalized metric space (E, d). The pair (S, T) is called *symmetric Banach operator pair* if

$$T(Fix(S)) \subseteq Fix(S) \text{ and } S(Fix(T)) \subseteq Fix(T).$$

Next, we recall the notion of one-local retract.

Definition 3.2. A subset C of E is a *one-local retract* if it is a retract of $C \cup \{x\}$ via the identity mapping for every $x \in E$, i.e., if there exists a nonexpansive mapping $g : C \cup \{x\} \to C$ such that $g(c) = c$ for every $c \in C$.

The following lemma shows that a one-local retract enjoys the same properties as the larger set.

Lemma 3.3.[11] *Let C be a one-local retract of E. If E has compact and normal structure, then C has compact and normal structure as well.*

Proposition 3.4.[11] *Let (E, d) be a generalized metric space and C a nonempty admissible subset of E. If (E, d) has compact and normal structure, then*

1. *for every nonexpansive mapping $T : C \to C$, the set $Fix(T)$ of fixed points of T is a one-local retract of C;*
2. *the intersection of every down-directed family of one-local retracts is a nonempty one-local retract.*

Theorem 3.5. *Let (E, d) be a generalized metric space and C a nonempty admissible subset of E. If (E, d) has compact and normal structure, then every family $\mathcal{F}$ of nonexpansive self-mappings on C has a common fixed point provided that any two mappings form a symmetric Banach operator pair. Moreover, the common fixed point set $Fix(\mathcal{F})$ is nonempty one-local retract of E.*

Proof. Let T_1 in $\mathcal{F}$. Using Theorem 2.2, we get $C_1 = Fix(T_1) \neq \emptyset$. Since any two mappings from $\mathcal{F}$ form a symmetric Banach operator pair, we have $T_2(C_1) \subseteq C_1$ for every $T_2 \in \mathcal{F}$. By item (1) of Proposition 3.4, C_1 is a one-local retract of C.

Using Lemma 3.3 and Theorem 2.2, T_2 has a fixed point in C_1. Using again item (1) and the fact that T_2 induces a nonexpansive mapping $T_{2\upharpoonright C_1}$ from C_1 into itself, $C_2 = C_1 \cap Fix(T_2)$ is a one-local retract of C_1.

Let $T_3 \in \mathcal{F}$, then $T_3(C_2) \subseteq C_2$. Since C_2 is a one-local retract of E, $C_3 = C_2 \cap Fix(T_3)$ is a one-local retract of E. By induction, one can check that any finite subset $\mathcal{F}'$ of $\mathcal{F}$ has a nonempty common fixed point set $Fix(\mathcal{F}')$ which is a one-local retract of C.

Let $\mathcal{P} = \{Fix(\mathcal{F}') \ : \ |\mathcal{F}'| < \aleph_0\}$. For any $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{P}$, we get

$$\mathcal{P}_1 \bigcap \mathcal{P}_2 \subseteq \mathcal{P}_1 \text{ and } \mathcal{P}_1 \bigcap \mathcal{P}_2 \subseteq \mathcal{P}_2.$$

Hence, $\mathcal{P}$ is a down-directed family of one-local retracts. According to item (2) of Proposition 3.4, $\bigcap \mathcal{P} = Fix(\mathcal{F})$ is a nonempty one-local retract of E.

Acknowledgement. The authors would like to thank Maurice Pouzet with whom we had fruitful discussions regarding this work. Also, we are pleased to thank the referees of this paper for thoughtful suggestions and numerous corrections.

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Received: April 22, 2020; Accepted: April 2, 2021.