Fixed Point Theory, 23(2022), No. 1, 3-20
DOI: 10.24193/fpt-ro.2022.1.01
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON SPLIT COMMON FIXED POINT AND MONOTONE INCLUSION PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we study split common fixed point problems of Bregman demigeneralized and Bregman quasi-nonexpansive mappings in reflexive Banach spaces. Using the Bregman technique together with a Halpern iterative algorithm, we approximate a solution of split common fixed point problem and sum of two monotone operators in reflexive Banach spaces. We establish a strong convergence result for approximating the solution of the aforementioned problems. It is worth mentioning that the iterative algorithm employ in this article is design in such a way that it does not require prior knowledge of operator norm and we do not employ Fejer monotinicity condition in the strategy of proving our convergence theorem. We apply our result to solve variational inequality and convex minimization problems. The result discuss in this paper extends and complements many related results in literature.
Key Words and Phrases: Bregman quasi-nonexpansive, Bregman demigeneralized mapping, monotone operators, fixed point, iterative scheme, split common fixed point problem.
2020 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25, 47H10.

1. Introduction

Let E be a reflexive Banach space with E^* its dual and Q be a nonempty closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Fenchel conjugate of f denoted as $f^*: E^* \to (-\infty, +\infty]$ is define as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of f be denoted as $dom f = \{x \in E : f(x) < +\infty\}$, hence for any $x \in int dom f$ and $y \in E$, we define the right-hand derivative of f at x in the direction of y by

$$f^0(x,y) = \lim_{t\to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be

(i) Gâteaux differentiable at x if $\lim\limits_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^0(x,y)$ coincides with $\bigtriangledown f(x)$ (the value of the gradient $\bigtriangledown f$ of f at x);
(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in int dom f$;
(iii) Fr*é*chet differentiable at x, if its limit is attained uniformly in $||y|| = 1$;
(iv) Uniformly Fr*é*chet differentiable on a subset Q of E, if the above limit is attained uniformly for $x \in Q$ and $||y|| = 1$.

Let $f : E \to (-\infty, +\infty]$ be a function, then f is said to be:

(i) essentially smooth, if the subdifferential of f denoted as ∂f is both locally bounded and single-valued on its domain, where

$$\partial f(x) = \{w \in E : f(x) - f(y) \geq \langle w, y - x\rangle,\ y \in E\};$$

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of *dom* ∂f;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [8, 9] for more details on Legendre functions.

Alternatively, a function f is said to be Legendre if it satisfies the following conditions:

(i) The $int dom f$ is nonempty, f is Gâteaux differentiable on $int dom f$ and $dom \bigtriangledown f = int dom f$;
(ii) The $int dom f^*$ is nonempty, f^* is Gâteaux differentiable on $int dom f^*$ and $dom \bigtriangledown f^* = int dom f$.

Let E be a Banach space and $B_s := \{z \in E : ||z|| \leq s\}$ for all $s > 0$. Then, a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [see pp. 203 and 221] [49] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y\in B_s, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha) f(y) - f(\alpha(x) + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of f. The function f is also said to be uniformly smooth on bounded subsets of E, [see pp. 221] [49], if $\lim\limits_{t\downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B, y\in S_E, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)ty) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$.

The function f is said to be uniformly convex if the function $\delta f : [0,+\infty) \to [0,+\infty)$ defined by

$$\delta f(t) := \sup\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f(\frac{x+y}{2}) : ||y-x|| = t\},$$

satisfies $\lim\limits_{t\downarrow 0} \frac{\delta f(t)}{t} = 0$.

Definition. [12] Let E be a Banach space. A function $g : E \to (-\infty,\infty]$ is said to be proper if the interior of its domain $dom(g)$ is nonempty. Let $g : E \to (-\infty,\infty]$ be a convex and Gâteaux differentiable function. Then the Bregman distance corresponding to g is the function $D_g : dom(g) \times intdom(g) \to \mathbb{R}$ defined by

$$D_g(x,y) := g(x) - g(y) - \langle x-y, \bigtriangledown_E^g(y)\rangle,\ \forall\ x,y \in E. \tag{1.1}$$

is called the Bregman distance with respect to g. It is clear that $D_g(x,y) \geq 0$ for all $x,y \in E$.

It is well-known that Bregman distance D_g does not satisfy the properties of a metric because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in domg$ and $y,z \in intdomg$,

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x-y, \bigtriangledown_E^g(y) - \bigtriangledown_E^g(z)\rangle. \tag{1.2}$$

In particular,

$$D_g(x,y) = -D_g(y,x) + \langle y-x, \bigtriangledown_E^g(y) - \bigtriangledown_E^g(x)\rangle,\ \forall\ x,y \in E.$$

Let $g : E \to \mathbb{R}$ be a strictly convex and Gâteaux differentiable function and $T : Q \to int(domg)$ be a mapping, a point $x \in Q$ is called a fixed point of T, if for all $x \in Q,\ Tx = x$. We denote by $Fix(T)$ the set of all fixed points of T. Furthermore, a point $p \in Q$ is called an asymptotic fixed point of T if Q contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim\limits_{n\to\infty} ||Tx_n - x_n|| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T.

Let Q be a nonempty closed and convex subset of int(dom g), then we define an operator $T : Q \to int(domg)$ to be :

(i) Bregman relatively nonexpansive, if $Fix(T) \neq \emptyset$, and

$$D_f(p,Tx) \leq D_f(p,x),\ \forall\ p \in Fix(T),\ x \in Q \text{ and } Fi\hat{x}(T) = Fix(T).$$

(ii) Bregman quasi-nonexpansive mapping if $Fix(T) \neq \emptyset$ and

$$D_f(p,Tx) \leq D_f(p,x), \forall\ x \in Q \text{ and } p \in Fix(T).$$

(iii) Bregman firmly nonexpansive (BFNE), if

$$\langle \bigtriangledown_E^g(Tx) - \bigtriangledown_E^g(Ty), Tx-Ty\rangle \leq \langle \bigtriangledown_E^g(x) - \bigtriangledown_E^g(y), Tx-Ty\rangle,\ \forall\ x,y \in E.$$

Definition. [22] Let C be a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty,+\infty]$ be a strongly coercive Bregman function. Let β and γ be real numbers with $\beta \in (-\infty,1)$ and $\gamma \in [0,\infty)$, respectively. Then a mapping

$T : C \to E$ with $F(T) \neq \emptyset$ is called Bregman (β, γ)-demigeneralized if for any $x \in C$ and $p \in F(T)$,

$$\langle x - p, \nabla_E^g(x) - \nabla_E^g(Tx)\rangle \geq (1-\beta)D_g(x, Tx) + \gamma D_g(Tx, x), \ \forall \ x \in E \text{ and } p \in F(T).$$

For modelling inverse problems which arises from phase retrievals and medical image reconstruction, (see [13]), Censor and Elfving [17] introduced the Split Feasibility Problem (SFP) in 1994, which is to find

$$u^* \in C \text{ such that } Fu^* \in Q; \tag{1.3}$$

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $F : E_1 \to E_2$ is a bounded linear operator. The SFP have been well studied in the framework of real Hilbert spaces, uniformly convex and uniformly smooth Banach spaces, see ([19, 26, 44] and other references contained in). Different optimization problems have been formulated in terms of SFP (1.3), for instance, If $Q = \{b\}$ in SFP (1.3) is a singleton, then we have the following convexly constrained linear inverse problem (in short, CCLIP) defined as follows:

$$\text{Find a point } u^* \in C \text{ such that } Fu^* = b.$$

Also, if $C = Fix(T)$ and $Q = Fix(S)$, then SFP (1.3) becomes split common fixed point problem (in short SCFPP) which is to find a point

$$u^* \in Fix(T) \text{ such that } Fu^* \in Fix(S). \tag{1.4}$$

Since the inception of SCFPP (1.4), several authors have considered solving this problem. For instance, Censor and Segal [18] introduced the following iterative algorithm for the solving SCFPP (1.4) in finite dimensional spaces. They defined their algorithm as follows:

$$x_{n+1} = T(x_n + \tau F^t(S - I)Fx_n),$$

for each $n \geq 1$, where $\tau \in (0, \frac{0}{\gamma})$ with γ being the largest eigenvalue of the matrix F^tF (F^t being the matrix transposition). Also, Moudafi [30] introduced a relaxed algorithm to approximate a solution of SCFPP (1.4) and proved some weak convergence result in Hilbert spaces with the mappings T and S being quasi-nonexpansive mappings.

Recently, Ansari and Rehan [5] introduced a generalized SFP (in short GSFP) as follows:

$$\text{Find } u^* \in Fix(T) \cap A^{-1}(0) \text{ such that } Fu^* \in Fix(S), \tag{1.5}$$

where E_1 and E_2 are uniformly convex and uniformly smooth Banach spaces, C is a nonempty closed and convex subset of E_1, $T : C \to C$ is a generalized nonspreading mapping such that $Fix(T) \neq \emptyset$, $S : E_2 \to E_2$ is a nonexpansive mapping, $A : E_1 \to 2^{E_1^*}$ is a maximal monotone operator and $F : E_1 \to E_2$ is a bounded linear operator. They proved a weak convergence to a solution of GSFP (1.5).

Very recently, Izuchukwu et al. [24] studied the following split monotone variational inclusion and fixed point problem between Hilbert space and a Banach space which is defined as follows:

$$\text{Find } x^* \in Fix(T) \cap (A+B)^{-1}(0) \text{ such that } Fu^* \in G^{-1}(0),$$

where H is a Hilbert space, E is a uniformly convex and uniformly smooth Banach space, $T : H \to CB(H)$ is multivalued quasi-nonexpansive mapping, $B : H \to 2^H$ and $G : E \to 2^E$ are maximal monotone operators, $F : H \to E$ is a bounded linear operator. They proposed a viscosity iterative scheme and under mild conditions and proved a strong convergence theorem.

Question. Can we generalize the results of [2, 5, 24, 34, 35] to a more general Banach spaces (reflexive Banach spaces) and employ an approach diferrent from theirs to prove a strong convergence result?
Let $B : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of B by $domB = \{x \in E : Bx \neq \emptyset\}$ and $ranB = \bigcup_{x\in E} Bx$, respectively. The graph of B denoted by $G(B) = \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. The mapping $B \subset E \times E^*$ is said to be monotone [40] if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in B$. It is also said to be maximal monotone [39] if its graph is not contained in the graph of any other monotone operator on E. If $B \subset E \times E^*$ is maximal monotone, then we can represent the set $B^{-1}(0) = \{z \in E : 0 \in Bz\}$ is closed and convex.
Let $A : E \to 2^{E^*}$ be a mapping, then the resolvent associated with A and λ for any $\lambda > 0$ is the mapping $Res^g_{\lambda A} : E \to 2^E$ defined by

$$Res^g_{\lambda A} := (\triangledown^g_E + \lambda A)^{-1} \circ \triangledown^g_E.$$

It is worth mentioning that a mapping $A : E \to 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) on the set C if

$$C \cap (domg) \cap (int\ dom\ g) \neq \emptyset,$$

and for any $x, y \in C \cap (int\ dom\ g)$, $\eta \in Ax$ and $\xi \in Ay$, we have

$$\langle \eta - \xi, (\triangledown^{g^*}_{E^*}(x) - \eta) - \triangledown^{g^*}_{E^*}(\triangledown^g_E(y) - \xi)\rangle \geq 0.$$

The anti-resolvent $A^g_\lambda : E \to 2^E$ associated with the mapping $A : E \to 2^{E^*}$ and $\lambda > 0$ is defined by

$$A^g_\lambda := \triangledown^g_E \circ (\triangledown^g_E - \lambda A). \tag{1.6}$$

Let $A : E \to E^*$ be a single-valued monotone mapping and $B : E \to 2^{E^*}$ be a multivalued monotone mapping. Then, the Monotone Variational Inclusion Problem (MVIP) (also known as the problem of finding a zero of sum of two monotone mappings) is to find $x \in E$ such that

$$0^* \in A(x) + B(x). \tag{1.7}$$

We denote by Ω, the solution set of problem (1.7).
It is well known that many interesting problems arising from mechanics, economics, economics, finance, nonlinear programming, applied sciences, optimization such as equilibrium and variational inequality problems can be solved using MVIP. Considerable efforts have been devoted to develop efficient iterative algorithms to approximate

solutions of MVIP in which the resolvent operator technique is one of the vital technique.
Many authors have considered approximating solutions of (1.7) together with fixed point problems in real Hilbert and Banach spaces, see [2, 1, 3, 35, 43].
For instance, Okeke and Izuchukwu [34] studied and analysed an iterative algorithm for approximating split feasibility problem and variational inclusion problem in p-uniformly Banach spaces which are uniformly smooth, Using their iterative scheme, they proved a strong convergence result for approximating the solution of the aforementioned problems. Applications and numerical example were displayed to show the behaviour of their result.
Suppose $A = 0$ in (1.7), we obtain the following Monotone Inclusion Problem (MIP), which is to find $x \in E$ such that

$$0^* \in B(x). \tag{1.8}$$

Many results on MIP have been extended by authors from real Hilbert spaces to more general Banach spaces. For instance, Reich and Sabach [38, 25] introduced some iterative algorithms and proved two strong convergence results for approximating a common solution of a finite family of MIP (1.8) in a reflexive Banach space. Recently, Timnak et. al. [47] introduced a new Halpern-type iterative scheme for finding a common zero of finitely many maximal monotone mappings in a reflexive Banach space and prove the following strong convergence theorem.
Theorem. Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subset of E. Let $A_i : E \to 2^{E^*}, i = 1, 2, ...,$ be N maximal monotone operators such that $Z := \cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $(0, 1)$ satisfying the following control conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E, \ x_1 \in E \text{ chosen arbitrarily}, \\ y_n = \triangledown f^*[\beta_n \triangledown f(x_n) + (1-\beta_n) \triangledown f(Res^f_{r_N A_N}) \cdots (Res^f_{r_1 A_1}(x_n))], \\ x_{n+1} = \triangledown f^*[\alpha_n \triangledown f(u) + (1-\alpha_n) \triangledown f(y_n)], \end{cases} \tag{1.9}$$

for $n \in \mathbb{N}$, where $\triangledown f$ is the gradient of f. If $r_i > 0$, for each $i = 1, 2, ..., N$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (1.9) converges strongly to $proj^f_Z u$ as $n \to \infty$.
Very recently, Ogbuisi and Izuchukwu [31] introduced an iterative algorithm and obtained a strong convergence result for approximating a zero of sum of two maximal monotone operators which is also a fixed point of a Bregman strongly nonexpansive mapping in the framework of a reflexive Banach space.
We observed that very few results have been carried out on split common fixed point problem and zeros of sum of two maximal monotone in reflexive Banach space. We will also like to emphasize that approximating a common solution of MVIP and SCFPP have some possible applications to mathematical models whose constraints can be

expressed as MVIP and SFP. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, to mention a few, (see [23]). It is worth mentioning that the problem considered in this article generalizes the ones in [5, 18, 30].

Inspired by the results discussed above, we introduce an iterative algorithm which does not require the prior knowledge of operator norm as this may give difficulty in computing, to approximate a common solution of split common fixed point problem of Bregman demigeneralized type mapping and zeros of sum of two maximal monotone operators which is also a solution of fixed point problem of Bregman quasi-nonexpansive mapping in reflexive Banach spaces. Using our iterative algorithm with our unique approach, we prove a strong convergence result for appproximating solutions of the aforementioned problems and apply our result to solve variational inequality and convex minimization problems. The result discussed in this paper complements and extends many related results in literature.

We state our contributions in this article as follows:

(1) The main result in this article generalizes the results in [10], [32] and [34] from p-uniformly Banach spaces which are also uniformly smooth to reflexive Banach spaces.
(2) The problem considered in [47] is a special case of the one considered in this article and generalizes the results in [3, 5, 18, 30, 33, 34, 47] from real Hilbert spaces to a reflexive Banach spaces.
(3) It is worth mentioning that the proof of convergence proposed in this paper is different from the ones in [2, 1, 5, 10, 32, 18, 33, 34, 35] in the sense that our approach does not distinguish between whether the sequence generated by our algorithm is Fejer-monotone or not. Our approach is simple and more elegant.
(4) We dispensed the sets $\{C_n, D_n, Q_n\}_{n\in\mathbb{N}}$ in our algorithm as this gives difficulties in computation. Lastly, our iterative algorithm is designed in such a way that it does not require prior knowledge of operator norm as this also gives difficulties in computation.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Definition. A function $g : E \to \mathbb{R}$ is said to be strongly coercive if

$$\lim_{||x_n||\to\infty} \frac{g(x_n)}{||x_n||} = \infty.$$

Lemma. [47] Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,

(i) For any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} & D_g\big(x, \triangledown_{E^*}^{g^*}[\alpha \triangledown_E^g \triangledown_E^g(y) + (1-\alpha) \triangledown_E^g (z)]\big) \\ & \le \alpha D_g(x,y) + (1-\alpha) D_g(x,z) - \alpha(1-\alpha)\rho_s(|| \triangledown_E^g (y) - \triangledown_E^g(z)||), \end{aligned}$$

(ii) For any $x, y \in B_s$,

$$\rho_s(||x-y||) \le D_g(x,y).$$

Here, $B_s := \{z \in E : ||z|| \le s\}$.

Lemma. [16] Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by

$$V(x,x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*),\ x \in E,\ x^* \in E^*.$$

The following assertions also hold:

$$D_g(x, \triangledown \triangledown_{E^*}^{g^*} (x^*)) = V(x,x^*),\ \text{for all } x \in E \text{ and } x^* \in E^*.$$

$$V(x,x^*) + \langle \triangledown_{E^*}^{g^*}(x^*) - x, y^* \rangle \le V(x, x^*+y^*) \text{ for all } x \in E \text{and } x^*, y^* \in E^*.$$

Lemma. [22] Let E_1 and E_2 be two Banach spaces. Let $F : E_1 \to E_2$ be a bounded linear operator and $T : E_2 \to E_2$ be a Bregman (ϕ, σ)-demigeneralized for some $\phi \in (-\infty, 1)$ and $\sigma \in [0,\infty)$. Suppose that $K = ran(A) \cap Fix(T) \neq \emptyset$ (where $ran(B)$ denotes the range of B). Then for any $(x,q) \in E_1 \times K$,

$$\begin{aligned} \langle x-q, F^*(\triangledown_{E_2}^{g_2}(T(Fx)))\rangle &\ge (1-\phi) D_{g_2}(Fx, T(Fx)) + \sigma D_{g_2}(T(Fx), Fx) \\ &\ge (1-\phi) D_{g_2}(Fx, T(Fx)). \end{aligned} \tag{2.1}$$

So, given any real numbers ξ_1 and ξ_2, the mapping $L_1 : E_1 \to [0,\infty)$ and $L_2 : E_2 \to [0.\infty)$ formulated for $x \in E_1$ as

$$L_1(x) = \begin{cases} \frac{D_{g_2}(Fx, TFx)}{D_{g_1}^*(F^*(\triangledown_{E_2}^{g_2}(Fx)), F^*(\triangledown_{E_2}^{g_2}(TFx))} \text{ if } & (I-T)Fx \neq 0, \\ \xi_1 & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$L_2(x) = \begin{cases} \frac{D_{g_1}^*(\triangledown_{E_1}^{g_1}(x) - \gamma F^*(\triangledown_{E_2}^{g_2}(Fx) - \triangledown_{E_2}^{g_2}(TFx)), \triangledown_{E_1}^{g_1}(x))}{D_{g_1}^*(F^*(\triangledown_{E_2}^{g_2}(Fx)), F^*(\triangledown_{E_2}^{g_2}(TFx))} \text{ if } & (I-T)Fx \neq 0, \\ \xi_2 & \text{otherwise} \end{cases} \tag{2.3}$$

are well-defined, where γ is any nonnegative real number.
Moreover, for any $(x,p) \in E_1 \times K$, we have

$$D_{g_1}(q,y) \le D_{g_1}(q,x) - (\gamma(1-\phi)L_1(x) - L_2(x)) D_{g_1^*}(F^*(\triangledown_{E_2}^{g_2}(Fx)), F^*(\triangledown_{E_2}^{g_2}(TFx)), \tag{2.4}$$

where

$$y = (\triangledown_{E_1}^{g_1})^{-1}[\triangledown_{E_1}^{g_1}(x) - \gamma F^*(\triangledown_{E_2}^{g_2}(Fx) - \triangledown_{E_2}^{g_2}(TFx))].$$

Lemma. [16] Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma. [31] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,

$$D_g(u, Res^g_{\lambda B} \circ A^g(x)) + D_g(Res^g_{\lambda B}(x), x) \leq D_g(u, x), \text{ for any } u \in (A + B)^{-1}(0^*),$$
$$x \in E \text{ and } \lambda > 0.$$

Lemma. [31] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,
(i) $(A + B)^{-1}(0^*) = Fix(Res^g_{\lambda B} \circ A^g_\lambda)$;
(ii) $Res^g_{\lambda B} \circ A^g_\lambda$ is a BSNE operator with $Fix(Res^g_{\lambda B} \circ A^g_\lambda) = \hat{Fix}(Res^g_{\lambda B} \circ A^g_\lambda)$.

Lemma. [38] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Definition. Let C be a nonempty closed and convex subset of a reflexive Banach space E and $g : E \to (-\infty, +\infty]$ be a strongly coercive Bregman function. A Bregman projection of $x \in int(domg)$ onto $C \subset int(domg)$ is the unique vector $P^g_c(x) \in C$ satisfying

$$D_g(Proj^g_C(x), x) = int\{D_g(y, x) : y \in C\}.$$

Lemma. [36] Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then,
(i) $z = P^g_C(x)$ if and only if $\langle \bigtriangledown^g_E(x) - \bigtriangledown^g_E(z), y - z\rangle \leq 0, \ \forall\ y \in C$.
(ii) $D_g(y, P^g_C(x)) + D_g(P^g_C(x), x) \leq D_g(y, x), \ \forall\ y \in C$.

Lemma. [7, 27] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\gamma_n\}$ be a sequence of real numbers in $(0, 1)$ with conditions $\sum_{n=1}^\infty \gamma_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n d_n, \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfy the condition:

$$\limsup_{k\to\infty}(a_{n_k} - a_{n_k+1}) \leq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3. Main results

We begin this section by establishing the following result needed in the convergence analysis of our main theorem.

Lemma. Let E be a reflexive Banach space, $S : E \to E$ be a Bregman quasi-nonexpansive mapping and $B : E \to 2^{E^*}$ be a maximal monotone operator. Suppose $g : E \to (-\infty, \infty]$ is a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0) \neq \emptyset$. Then, by applying Lemma 2, we have that

$$Fix(S(Res^g_{\lambda B} \circ A^g_\lambda)) = Fix(S) \cap Fix(Res^g_{\lambda B} \circ A^g_\lambda).$$

Proof. Clearly, $Fix(S) \cap Fix(Res^g_{\lambda B} \circ A^g_\lambda) \subseteq Fix(S(Res^g_{\lambda B}))$. We only need to proof that $Fix(S(Res^g_{\lambda B})) \subseteq Fix(S) \cap Fix(Res^g_{\lambda B} \circ A^g_\lambda)$.
Let $x \in Fix(S(Res^g_{\lambda B}))$ and $y \in Fix(S) \cap Fix(Res^g_{\lambda B} \circ A^g_\lambda)$, then

$$\begin{aligned} D_g(y, x) &= D_g(y, S(Res^g_{\lambda B} \circ A^g_\lambda)x) \\ &\leq D_g(y, (Res^g_{\lambda B} \circ A^g_\lambda)x). \end{aligned} \tag{3.1}$$

Now, by applying Lemma 2 and (3.1), we obtain

$$\begin{aligned} D_g(x, (Res^g_{\lambda B} \circ A^g_\lambda)x) &\leq D_g(y, x) - D_g(y, (Res^g_{\lambda B} \circ A^g_\lambda)x) \\ &\leq D_g(y, x) - D_g(y, x) \\ &= 0. \end{aligned}$$

Hence, $x \in Fix(Res^g_{\lambda B} \circ A^g_\lambda)$. Next we show that $x \in Fix(S)$, since

$$x \in Fix(S(Res^g_{\lambda B} \circ A^g_\lambda)) \text{ and } x \in Fix((Res^g_{\lambda B} \circ A^g_\lambda)),$$

we have

$$\begin{aligned} D_g(x, Sx) &= D_g(x, (S(Res^g_{\lambda B} \circ A^g_\lambda)x)) \\ &= D_g(x, x) \\ &= 0. \end{aligned}$$

Hence, $x \in Fix(S)$. This implies that $x \in Fix(S) \cap Fix(Res^g_{\lambda B} \circ A^g_\lambda)$. Therefore, we obtain the desired result.
Throughout this section, we assume that

Assumption

(1) E_1 and E_2 be two reflexive Banach spaces, $g_1 : E_1 \to (-\infty, +\infty]$ and $g_2 : E_2 \to (-\infty, +\infty]$ be strongly coercive Bregman functions which are bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E_1 and E_2 , respectively. Let $\bigtriangledown^{g_1}_{E_1}$ and $\bigtriangledown^{g_2}_{E_2}$ be the gradients of E_1 dependent on g_1 and E_2 dependent on g_2 respectively.

(2) A be BISM mappings of E_1 into E_1^* and B be maximal monotone mappings of E_1 into $2^{E_1^*}$ respectively. Let $Res^{g_1}_{\lambda B}$ be the resolvent of g_1 of B for $\lambda > 0$. Suppose that $F : E_1 \to E_2$ is a bounded linear operator such that $F \neq \emptyset$ and $F^* : E_2^* \to E_1^*$ be the adjoint of F.

(3) $T : E_2 \to E_2$ be Bregman (ϕ_T, σ_T)-demigeneralized mapping such that $\phi_T \in (-\infty, 1)$ and $\sigma_T \in (0, \infty]$, and $S : E_1 \to E_1$ be a Bregman quasi-nonexpansive mapping.

(4) Assume that $\Gamma := \{q \in Fix(S) \cap (A + B)^{-1}(0) : Fq \in Fix(T)\} \neq \emptyset$, let $\gamma > 0$ be a real number and $\alpha_n + \theta_n + \mu_n = 1$ with $0 < a < \theta_n, \mu_n < b < 1$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_n \in (0, 1)$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Theorem. For fixed $u \in E_1$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 \in E_1$ such that

$$\begin{cases} z_n = (\triangledown^{g_1}_{E_1})^{-1}\big[\, \triangledown^{g_1}_{E_1}(x_n) - \gamma F^*(\triangledown^{g_2}_{E_2}(Fx_n) - \triangledown^{g_2}_{E_2}(TFx_n))\big] \\ y_n = (\triangledown^{g_1}_{E_1})^{-1}\big[(1-\beta_n)\, \triangledown^{g_1}_{E_1}(z_n) + \beta_n\, \triangledown^{g_1}_{E_1}(S(Res^{g_1}_{\lambda B} \circ A^{g_1}_{\lambda})z_n)\big] \\ x_{n+1} = (\triangledown^{g_1}_{E_1})^{-1}\big[\alpha_n\, \triangledown^{g_1}_{E_1}(u) + \theta_n\, \triangledown^{g_1}_{E_1}(y_n) + \mu_n\, \triangledown^{g_1}_{E_1}(y_n)\big]. \end{cases} \tag{3.2}$$

Suppose the sequences $\{\xi_{1,n}\}_{n\in\mathbb{N}}$ and $\{\xi_{2,n}\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(iii) there exists a positive real number ρ_1 such that

$$0 < \rho_1 < \liminf_{n\to\infty} \frac{\xi_{2,n}}{(1-\phi_t)\xi_{1,n}} < \gamma,$$

where

$$\xi_{1,n} = \begin{cases} \frac{D_{g_2}(Fx_n, TFx_n)}{D^*_{g_1}(F^*(\triangledown^{g_2}_{E_2}(Fx_n)), F^*(\triangledown^{g_2}_{E_2}(TFx_n))} & \text{if} \quad (I-T)Fx_n \neq 0 \\ \xi_1 & \text{otherwise} \end{cases}$$

and

$$\xi_{2,n} = \begin{cases} \frac{D^*_{g_1}(\triangledown^{g_1}_{E_1}(x_n) - \gamma F^*(\triangledown^{g_2}_{E_2}(Fx_n) - \triangledown^{g_2}_{E_2}(TFx_n)), \triangledown^{g_1}_{E_1}(x_n))}{D^*_{g_1}(F^*(\triangledown^{g_2}_{E_2}(Fx_n)), F^*(\triangledown^{g_2}_{E_2}(TFx_n))} & \text{if} \quad (I-T)Fx_n \neq 0, \\ \xi_2 & \text{otherwise.} \end{cases}$$

Then, the sequence $\{x_n\}$ generated iteratively converges strongly to $z = P^{g_1}_{\Gamma} u$, where $P^{g_1}_{\Gamma}$ is the Bregman projection of E_1 onto Γ.

Proof. Let $x^* \in \Gamma$, then we obtain from Lemma 2 that

$$\begin{aligned} D_{g_1}(x^*, z_n) &= D_{g_1}(x^*, (\triangledown^{g_1}_{E_1})^{-1}\big[\, \triangledown^{g_1}_{E_1}(x_n) - \gamma F^*(\triangledown^{g_2}_{E_2}(Fx_n) - \triangledown^{g_2}_{E_2}(TFx_n))\big]) \\ &\leq D_{g_1}(x^*, x_n) \\ &\quad - (\gamma(1-\phi_T)\xi_{1,n} - \xi_{2,n}) D_{g_1^*}(F^*(\triangledown^{g_2}_{E_2}(Fx_n)), F^*(\triangledown^{g_2}_{E_2}(TFx_n)) \qquad (3.3) \\ &\leq D_{g_1}(x^*, x_n). \qquad (3.4) \end{aligned}$$

Also, from Lemma 2, we get

$$
\begin{aligned}
D_{g_1}(x^*, y_n) &= D_{g_1}(x^*, (\bigtriangledown_{E_1}^{g_1})^{-1}\big[(1-\beta_n)\bigtriangledown_{E_1}^{g_1}(z_n) + \beta_n \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})z_n)\big])\\
&\leq (1-\beta_n)D_{g_1}(x^*, z_n) + \beta_n D_{g_1}(x^*, S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})z_n)\\
&- \beta_n(1-\beta_n)\rho_r(||\bigtriangledown_{E_1}^{g_1}(z_n) - \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g}))z_n||)\\
&\leq (1-\beta_n)D_{g_1}(x^*, z_n) + \beta_n D_{g_1}(x^*, (Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})z_n)\\
&- \beta_n(1-\beta_n)\rho_r(||\bigtriangledown_{E_1}^{g_1}(z_n) - \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g}))z_n||)\\
&\leq (1-\beta_n)D_{g_1}(x^*, z_n) + \beta_n D_{g_1}(x^*, z_n)\\
&- \beta_n(1-\beta_n)\rho_r(||\bigtriangledown_{E_1}^{g_1}(z_n) - \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g}))z_n||)\\
&= D_{g_1}(x^*, z_n)\\
&- \beta_n(1-\beta_n)\rho_r(||\bigtriangledown_{E_1}^{g_1}(z_n) - \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g}))z_n||) \qquad (3.5)\\
&\leq D_{g_1}(x^*, z_n). \qquad (3.6)
\end{aligned}
$$

Using (3.2), (3.4) and (3.5), we get

$$
\begin{aligned}
D_{g_1}(x^*, x_{n+1}) &= D_{g_1}(x^*, (\bigtriangledown_{E_1}^{g_1})^{-1}\big[\alpha_n \bigtriangledown_{E_1}^{g_1}(u) + \theta_n \bigtriangledown_{E_1}^{g_1}(y_n) + \mu_n \bigtriangledown_{E_1}^{g_1}(y_n)\big])\\
&\leq \alpha_n D_{g_1}(x^*, u) + \theta_n D_{g_1}(x^*, y_n) + \mu_n D_{g_1}(x^*, y_n)\\
&\leq \alpha_n D_{g_1}(x^*, u) + (1-\alpha_n)D_{g_1}(x^*, y_n) \qquad (3.7)\\
&\leq \alpha_n D_{g_1}(x^*, u) + (1-\alpha_n)D_{g_1}(x^*, z_n)\\
&\leq \alpha_n D_{g_1}(x^*, u) + (1-\alpha_n)D_{g_1}(x^*, x_n)\\
&\leq \max\{D_{g_1}(x^*, u), D_{g_1}(x^*, x_n)\}\\
&\vdots\\
&\leq \max\{D_{g_1}(x^*, u), D_{g_1}(x^*, x_1)\}.\ \forall\ n \geq 1.
\end{aligned}
$$

Thus, we obtain that the sequence $\{D_{g_1}(x^*, x_n)\}_{n\in\mathbb{N}}$ is bounded. Using Lemma 2, then we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ are bounded.

From (3.3), (3.5) and (3.7), we get

$$
\begin{aligned}
D_{g_1}(x^*, x_{n+1}) &\leq \alpha_n D_{g_1}(x^*, u) + (1-\alpha_n)D_{g_1}(x^*, x_n)\\
&+ (1-\alpha_n)\beta_n(1-\beta_n)\rho_r(||\bigtriangledown_{E_1}^{g_1}(z_n) - \bigtriangledown_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g}))z_n||)\\
&- (1-\alpha_n)(\gamma(1-\phi_T)\xi_{1,n} - \xi_{2,n})D_{g_1^*}(F^*(\bigtriangledown_{E_2}^{g_2}(Fx_n)), F^*(\bigtriangledown_{E_2}^{g_2}(TFx_n)).
\end{aligned}
\qquad (3.8)
$$

Using Lemma 2, (3.4) and (3.6), we obtain

$$
\begin{aligned}
D_{g_1}(z, x_{n+1}) &= D_{g_1}(z, (\nabla_{E_1}^{g_1})^{-1}\big[\alpha_n \nabla_{E_1}^{g_1}(u) + \theta_n \nabla_{E_1}^{g_1}(y_n) + \mu_n \nabla_{E_1}^{g_1}(y_n)\big]) \\
&= V(z, \alpha_n \nabla_{E_1}^{g_1}(u) + (1-\alpha_n)\nabla_{E_1}^{g_1}(y_n)) \\
&\leq V(z, \alpha_n \nabla_{E_1}^{g_1}(u) + (1-\alpha_n)\nabla_{E_1}^{g_1}(y_n) - \alpha_n(\nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z))) \\
&- \langle \nabla_{E_1^*}^{g_1^*}(\alpha_n \nabla_{E_1}^{g_1}(u) + (1-\alpha_n)\nabla_{E_1}^{g_1}(y_n)) - z, -\alpha_n(\nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z))\rangle \\
&= V(z, \alpha_n \nabla_{E_1}^{g_1}(z) + (1-\alpha_n)\nabla_{E_1}^{g_1}(y_n)) \\
&+ \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\
&= D_{g_1}(z, (\nabla_{E_1^*}^{g_1^*})[\alpha_n \nabla_{E_1}^{g_1}(z) + (1-\alpha_n)\nabla_{E_1}^{g_1}(y_n)]) \\
&+ \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\
&\leq \alpha_n D_{g_1}(z,z) + (1-\alpha_n)D_{g_1}(z, y_n) + \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\
&= (1-\alpha_n)D_{g_1}(z, y_n) + \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \\
&\leq (1-\alpha_n)D_{g_1}(z, x_n) + \alpha_n\langle x_{n+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle. \qquad (3.9)
\end{aligned}
$$

In view of Lemma 2, we need to show that $\langle x_{n_k+1} - z, \nabla_{E_1}^{g_1}(u) - \nabla_{E_1}^{g_1}(z)\rangle \leq 0$ for every $\{D_{g_1}(z, x_{n_k})\}$ of $\{D_{g_1}(z, x_n)\}$ satisfying the condition

$$\limsup_{k\to\infty}\{D_{g_1}(z, x_{n_k}) - D_{g_1}(z, x_{n_{n_k}+1})\} \leq 0. \qquad (3.10)$$

From (3.8) and (3.10), we have that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\beta_{n_k}(1-\beta_{n_k})\rho_r(||\nabla_{E_1}^{g_1}(z_{n_k}) - \nabla_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1}\circ A_\lambda^g))z_{n_k}||\Bigg) \\
&\leq \limsup_{k\to\infty}\Bigg(\alpha_{n_k}D_{g_1}(z,u) + (1-\alpha_{n_k})D_{g_1}(z, x_{n_k}) - D_{g_1}(z, x_{n_{k+1}})\Bigg) \\
&= \limsup_{k\to\infty}\Bigg(D_{g_1}(z, x_{n_k}) - D_{g_1}(z, x_{n_{k+1}})\Bigg) \\
&\leq 0. \qquad (3.11)
\end{aligned}
$$

Following the same process as in (3.11), we obtain from (3.8) and (3.10) that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})(\gamma(1-\phi_T)\xi_{1,n_k} - \xi_{2,n_k})D_{g_1^*}(F^*(\nabla_{E_2}^{g_2}(Fx_{n_k})), F^*(\nabla_{E_2}^{g_2}(TFx_{n_k}))\Bigg) \\
&\leq \limsup_{k\to\infty}\Bigg(\alpha_{n_k}D_{g_1}(z,u) + (1-\alpha_{n_k})D_{g_1}(z, x_{n_k}) - D_{g_1}(z, x_{n_{k+1}})\Bigg) \\
&= \limsup_{k\to\infty}\Bigg(D_{g_1}(z, x_{n_k}) - D_{g_1}(z, x_{n_{k+1}})\Bigg) \\
&\leq 0. \qquad (3.12)
\end{aligned}
$$

Therefore, we conclude from (3.11) and (3.12) that

$$\lim_{k\to\infty}\rho_r(||\nabla_{E_1}^{g_1}(z_{n_k}) - \nabla_{E_1}^{g_1}(S(Res_{\lambda B}^{g_1}\circ A_\lambda^g))z_{n_k}|| = 0, \qquad (3.13)$$

and

$$\lim_{k\to\infty} D_{g_1^*}(F^*(\bigtriangledown_{E_2}^{g_2}(Fx_{n_k})), F^*(\bigtriangledown_{E_2}^{g_2}(TFx_{n_k})) = 0. \tag{3.14}$$

So, from Lemma 2 and the properties of $\rho_r, D_{g_1}^*$ and F, we obtain

$$\lim_{k\to\infty} ||Fx_{n_k} - TFx_{n_k}|| = \lim_{k\to\infty} D_{g_2}(Fx_{n_k}, TFx_{n_k}) = 0, \tag{3.15}$$

and

$$\lim_{k\to\infty} ||z_{n_k} - (S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})z_{n_k})|| = 0. \tag{3.16}$$

From (3.2), (3.16) and Lemma 2, we get

$$\lim_{k\to\infty} ||y_{n_k} - z_{n_k}|| = 0. \tag{3.17}$$

Similarly, using (3.2), Lemma 2 and (3.15), we obtain that

$$\lim_{k\to\infty} ||z_{n_k} - x_{n_k}|| = 0. \tag{3.18}$$

Also, from (3.2), (3.17) and Lemma 2, we get

$$\lim_{k\to\infty} ||x_{n_{k+1}} - y_{n_k}|| = 0. \tag{3.19}$$

From (3.17) and (3.19), we get

$$\lim_{k\to\infty} ||x_{n_{k+1}} - z_{n_k}|| = 0. \tag{3.20}$$

We can conclude from (3.18) and (3.20) that

$$\lim_{k\to\infty} ||x_{n_{k+1}} - x_{n_k}|| = 0. \tag{3.21}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to z. Also, from (3.18), we have that the subsequence $\{z_{n_{k_j}}\}$ of $\{z_{n_k}\}$ converges weakly to z. Now, combining Lemma 2 (ii), Lemma 3 and (3.18), we have that $z \in \hat{Fix}(S(Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})) = \hat{Fix}(S) \cap \hat{Fix}(S) \cap \hat{Fix}(Res_{\lambda B}^{g_1} \circ A_\lambda^{g_1})$. Hence, by Lemma 2 (i), we obtain $z \in Fix(S) \cap (A+B)^{-1}(0)$.
Now, from (3.14), we obtain that

$$\lim_{j\to\infty} ||F^*(\bigtriangledown_{E_2}^{g_2}(Fx_{n_{k_j}})) - F^*(\bigtriangledown_{E_2}^{g_2}(TFx_{n_{k_j}}))|| = 0.$$

Since F is a bounded linear operator, so we have that $\lim_{j\to\infty} TFx_{n_{k_j}} = Fz$ and $Fz \in Fix(T)$. Hence, conclude that $z \in \Gamma$.
Next is to show that $\langle x_{n_k+1} - z, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(z)\rangle \leq 0$.

$$\begin{aligned}\limsup_{k\to\infty} \langle x_{n_{k+1}} - x^*, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(x^*)\rangle &= \lim_{j\to\infty} \langle x_{n_{k_j}+1} - x^*, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(x^*)\rangle \\ &\leq \langle z - x^*, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(x^*)\rangle.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\limsup_{k\to\infty} \langle x_{n_k+1} - z, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(z)\rangle &\leq \langle z - x^*, \bigtriangledown_{E_1}^{g_1}(u) - \bigtriangledown_{E_1}^{g_1}(x^*)\rangle \\ &\leq 0.\end{aligned} \tag{3.22}$$

On substituting (3.22) and Lemma 2 into (3.9), we conclude that $\{x_n\}$ converges strongly to z. We obtain the following as a consequence of our main result.
We give the following remarks which also serves as the consequences of our main result.

(1) If $E_1 = H_1, E_2 = H_2$ and $B = 0$ where H_1 and H_2 real Hilbert spaces, then we have the result discussed in [5]. Also E_1 and E_2 are p-uniformly Banach spaces which are also uniformly smooth and $A = 0$, then we have the result discussed in [33].
(2) Also, if $E_1 = H$ where H is a real Hilbert space and E_2 is a uniformly convex and uniformly smooth Banach space with T in (3.2) being equivalent to G, where G is a maximal monotone operator, then we have the result discussed in [24].

It is worth mentioning that the results of [5], [24], [33] and [43] were carried out in Hilbert spaces, and uniformly convex Banach spaces which is also uniformly smooth, whereas the result discussed in our article was carried out in reflexive Banach spaces. This makes their results and many other results discussed in the aforementioned spaces corollaries to our results.

4. Applications

4.1. **Convex Minimization Problem (CMP).** Let C be a nonempty closed and convex subset of a reflexive Banach space E_1 and $h : E_1 \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function which attains its minimum over E_1 and $g_1 : E_1 \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subset, and uniformly convex and uniformly smooth on bounded subset of E_1. Then, the CMP is to find $x \in E_1$ such that

$$h(x) = \min_{y \in E} h(y). \tag{4.1}$$

It is generally known that (4.1) can be formulated as follows: find $x \in E_1$ such that

$$0^* \in \partial h(x), \tag{4.2}$$

where $\partial h = \{\xi \in E^* : \langle \xi, y - x\rangle \leq h(y) - h(x) \ \forall \ x \in E_1\}$. It is known that ∂h is a maximal monotone operator whenever h is a proper, convex and lower semi-continuous function. Hence, by taking $\partial h = B$ and $A = 0$ in (3.2), we obtain a strong convergence result for approximation solutions of SCFPP and CMP (4.1).

4.2. **Variational Inequality Problem.** Let C be a nonempty closed and convex subset of a reflexive Banach space E_1 with E_1^* its dual. Let $A : E_1 \to E_1^*$ be a BISM operator. Then, the classical Variational Inequality Problem (VIP) is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \ \forall \ y \in C. \tag{4.3}$$

VIP is one of the most important problems in optimization as it is used in studying differential equations, minimax problems, and has certain applications to mechanics and economic theory. We denote by $VI(C,A)$, the set of solutions of VIP (4.3). Suppose that $g_1 : E_1 \to \mathbb{R}$ is a Legendre and totally convex function which satisfies the range condition $ran(\bigtriangledown g_1 - A) \subset ran(\bigtriangledown g_1)$, (see [21], Proposition 12), then

$$VI(C,A) = Fix(Proj_C^{g_1} \circ A_\lambda^{g_1}). \tag{4.4}$$

In addition, if g_1 is uniformly Frechet differentiable and bounded on a bounded subset of E_1, then the anti-resolvent is single-valued and a BSNE operator which satisfies $Fix(A^{g_1}) = \hat{Fix}(A^{g_1})$, (see [37, 42]). Therefore, $Proj_C^{g_1} \circ A^{g_1}$ is also BSNE operator satisfying $Fix(Proj_C^{g_1} \circ A^{g_1}) = \hat{Fix}(Proj_C^{g_1} \circ A^{g_1})$ (see [37], Remark 3). Then (3.2) becomes

$$\begin{cases} z_n = (\bigtriangledown_{E_1}^{g_1})^{-1}\big[\bigtriangledown_{E_1}^{g_1}(x_n) - \gamma F^*(\bigtriangledown_{E_2}^{g_2}(Fx_n) - \bigtriangledown_{E_2}^{g_2}(TFx_n))\big] \\ y_n = (\bigtriangledown_{E_1}^{g_1})^{-1}\big[(1-\beta_n)\bigtriangledown_{E_1}^{g_1}(z_n) + \beta_n \bigtriangledown_{E_1}^{g_1}(S(Proj_C^{g_1} \circ A^{g_1}))z_n)\big] \\ x_{n+1} = (\bigtriangledown_{E_1}^{g_1})^{-1}\big[\alpha_n \bigtriangledown_{E_1}^{g_1}(u) + \theta_n \bigtriangledown_{E_1}^{g_1}(y_n) + \mu_n \bigtriangledown_{E_1}^{g_1}(y_n)\big]. \end{cases} \tag{4.5}$$

Acknowledgement. The first and second authors acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Post-Doctoral Fellowship. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

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Received: September 6, 2020; Accepted: July 14, 2021.