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POSITIVE SOLUTIONS OF NONLINEAR THIRD-ORDER BOUNDARY VALUE PROBLEMS INVOLVING STIELTJES INTEGRAL CONDITIONS

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Abstract. In this paper, by using the Guo-Krasnoselskii theorem, we investigate the existence and nonexistence of positive solutions of a class of boundary value problem of third-order nonlinear differential equation involving Stieltjes integral conditions. Under some growth conditions imposed on the nonlinear term, we obtain explicit ranges of values of parameters with which the problem has a positive solution and has no positive solution respectively. An example is given to illustrate the main results of the paper.
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1. Introduction

Third order differential equations arise in a variety of different areas of applied mathematics and physics, such as dynamics, Newtonian fluid mechanics, the theory of boundary layer, the theory of heat transfer and so on(see in [7]). In recent years, the existence and multiplicity of positive solutions for third-order differential equations subject to various boundary conditions of local or nonlocal type have been studied by several authors. For examples, Anderson [1] established the existence of at least three positive solutions to problem

$$\begin{cases} x'''(t) = f(x(t)),\ t \in (0,1) \\ x(0) = x'(t_2) = x''(1) = 0, \quad t_2 \in (0,1) \end{cases}$$

where $f : R \to [0, +\infty)$ is continuous and $1/2 \leq t_2 < 1$. By using Guo-Krasnoselskii fixed point theorem [8], Palamides and Smyrlis [17] proved that there exists at least one positive solution for third-order three-point boundary value problem

$$\begin{cases} x'''(t) = a(t)f(t,\ x(t)),\ t \in (0,1) \\ x''(\eta) = 0,\ x(0) = x(1) = 0,\ \eta \in (0,1). \end{cases}$$

Guo, Sun and Zhao [9] established the positive solutions for following third-order three-point problem

$$\begin{cases} x'''(t) = a(t)f(x(t)), \ t \in (0,1) \\ x(0) = x'(0) = 0, \ x'(1) = x'(\eta), \ \eta \in (0,1) \end{cases}$$

By using Leray-Schauder continuation principle, Hopkins and Kosmatov [10] obtained the sign-changing solution for problem

$$\begin{cases} x'''(t) = a(t)f(x(t)), \ t \in (0,1) \\ x(0) = x'(0) = x''(1) = 0 \text{ or } x(0) = x'(1) = x''(1) = 0, \end{cases}$$

Anderson [2] considered the third-order three-point boundary value problem

$$\begin{cases} x'''(t) = f(t, \ x(t)), \ t_1 \le t \le t_3 \\ x(t_1) = x'(t_2) = 0, \ \gamma x(t_3) + \delta x''(t_3) = 0 \end{cases}$$

By using the Leggett-Williams [13] fixed point theorem, the author established the existence of at least three positive solutions. For more existence results for third-order boundary value problems, one can see [3], [5], [15], [16], [14], [20], [6], [12], [4] and references therein.

In this paper, we investigate positive solution for third-order boundary value problem $(P, \ \lambda)$

$$(P, \ \lambda) \begin{cases} -u'''(t) = \lambda f(t, \ u(t)), \ t \in [0,1] \\ \\ u''(0) = 0, \ u'(0) = \alpha[u], \ u'(1) + \beta[u] = 0, \end{cases}$$

where $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and α, β are linear functionals on $C[0, \ 1]$ that are given by

$$\alpha[u] = \int_0^1 u(s)dA(s), \beta[u] = \int_0^1 u(s)dB(s),$$

involving Riemann-Stieltjes with $A, \ B$ functions of bounded variation, that is $dA, \ dB$ can be sign changing measures. An advantage is that these boundary conditions includes the local case when $\alpha, \ \beta$ are indentically 0 and also includes the multipoint and integral conditions in a single framework. For examples, $\alpha[u] = \int_0^1 u(t)dA(t)$ appearing in the problem $(P, \ \lambda)$ covers various boundary conditions including the following:

$$\alpha[u] = \alpha_0 u(\eta), \eta \in (0,1), \ \alpha_0 \in R,$$

$$\alpha[u] = \sum_{i=1}^{n-2} \alpha_i u(\eta_i), \eta_i \in (0,1), \ \alpha_i \in R, \ i = 1,2,\cdots, \ n-2,$$

$$\alpha[u] = \int_0^1 u(t)g(t)dt, \ g \in C([0,1], \ R).$$

Motivated by Webb and Infante [18], [11] and Webb and Lan [19], which established new existence results of positive solutions of a Hammerstein integral equation by an unified way, under some growth condition imposed on the nonlinear term, we obtain

explicit ranges of values of λ with which the problem (P, λ) has a positive solution and has no positive solution respectively. An example is given to show how our results may be applied to obtain eigenvalues yielding existence of positive solutions.
The main tool used is the following fixed point theorem by Guo and Krasnoselskii [8].

Lemma 1.1. [8] *Let E be a Banach space and $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\Omega_2 \setminus \overline{\Omega}_1) \to K$$

be a completely continuous operator such that

$$\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_1, and\ \|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_2$$

or

$$\|Tu\| \geq \|u\|, u \in K \cap \partial\Omega_1, and\ \|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2,$$

then T has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

2. Preliminaries

In relation to problem (P, λ), we introduce the following linear problem

$$-u'''(t) = y(t), t \in [0, 1] \tag{2.1}$$

$$u''(0) = 0,\ u'(0) = \alpha[u],\ u'(1) + \beta[u] = 0. \tag{2.2}$$

We will use the notations $\hat{1},\ \hat{t}$ to denote the functions with values 1 and t respectively and we let H denote the heavside function,

$$H(x) := \begin{cases} 1, & \text{if} \quad x \geq 0 \\ 0, & \text{if} \quad x < 0 \end{cases}$$

For $y \in C[0,\ 1]$, let

$$Jy(t) := \int_0^t \frac{1}{2}(t-s)^2 y(s)ds,$$

$$J_A(s) := \int_s^1 \frac{1}{2}(t-s)^2 dA(t),$$

$$J_B(s) := \int_s^1 \frac{1}{2}(t-s)^2 dB(t).$$

Lemma 2.1. *Suppose that*

$$\Lambda := (1 + \beta[\hat{t}])\alpha(\hat{1}) + (1 - \alpha[\hat{t}])\beta[\hat{1}] \neq 0.$$

Let $y(t) \in C[0,1]$. A unique solution of the boundary value problem (2.1)-(2.2) is given by

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = -\frac{1}{2}(t-s)^2 H(t-s) + \frac{1}{\Lambda}\big(1 + \beta[\hat{t}] - t\beta[\hat{1}]\big)J_A(s) + \frac{1}{\Lambda}\big(1 - \alpha[\hat{t}] + t\alpha[\hat{1}]\big)(1 - s + J_B(s)).$$

Proof. By integration Eq 2.1, we have

$$u''(t) = u''(0) - \int_0^t y(s)ds.$$

$$u'(t) = u'(0) + u''(0)t - \int_0^t (t-s)y(s)ds.$$

$$u(t) = u(0) + u'(0)t + \frac{1}{2}u''(0)t^2 - \int_0^t \frac{1}{2}(t-s)^2 y(s)ds.$$

Considering the boundary conditions $u''(0) = 0,\ u'(0) = \alpha[u],\ u'(1) + \beta[u] = 0$, we have

$$u(t) = u(0) + \alpha[u]t - \int_0^t \frac{1}{2}(t-s)^2 y(s)ds \tag{2.3}$$

and

$$\alpha[u] + \beta[u] = \int_0^1 (1-s)y(s)ds.$$

Applying the functionals $\alpha,\ \beta$ to equation (2.3) and considering

$$\alpha[u] + \beta[u] = \int_0^1 (1-s)y(s)ds,$$

gives

$$\alpha[\hat{1}]u(0) - (1 - \alpha[\hat{t}])\alpha[u] = \alpha[Jy]$$

$$\beta[\hat{1}]u(0) + (1 + \beta[\hat{t}])\alpha[u] = \int_0^1 (1-s)y(s)ds + \beta[Jy]$$

On solving this system for $u(0)$ and $\alpha[u]$, we see that

$$u(0) = \frac{1}{\Lambda}\left((1+\beta[\hat{t}])\alpha[Jy] + (1-\alpha[\hat{t}])\left(\int_0^1 (1-s)y(s)ds + \beta[Jy]\right)\right),$$

$$\alpha[u] = \frac{1}{\Lambda}\left(\alpha[\hat{1}]\int_0^1 (1-s)y(s)ds + \alpha[\hat{1}]\beta[Jy] - \beta[\hat{1}]\alpha(Jy)\right).$$

By changing the order of integration, we find that

$$\alpha[Jy] = \int_0^1 J_A(s)y(s)ds,\ \beta[Jy] = \int_0^1 J_B(s)y(s)ds.$$

On substituting into (2.3), we see that

$$\begin{aligned} u(t) &= u(0) + \alpha[u]t - \int_0^t \frac{1}{2}(t-s)^2 y(s)ds \\ &= -\int_0^t \frac{1}{2}(t-s)^2 y(s)ds + \frac{1}{\Lambda}\left((1+\beta[\hat{t}])\alpha[Jy] + (1-\alpha[\hat{t}])\left(\int_0^1 (1-s)y(s)ds + \beta[Jy]\right)\right) \\ &\quad + \frac{t}{\Lambda}\left(\alpha[\hat{1}]\int_0^1 (1-s)y(s)ds + \alpha[\hat{1}]\beta[Jy] - \beta[\hat{1}]\alpha(Jy)\right) \\ &= \int_0^1 G(t,\ s)y(s)ds. \end{aligned}$$

Now we suppose that the following conditions holds:

$(H1)$ $1+\beta[\hat{t}] \geq \beta[\hat{1}] > 0, 1-s+J_B(s) \geq (1+\beta[\hat{t}])(1-s)^2, B(s) \geq 0$ and $J_B(s) \leq \beta[\hat{t}]$,
$(H2)$ $A(s) \geq 0, J_A(s) \geq 0$ for all $s \in [0,\ 1]$,
$(H3)$ $0 \leq \alpha[\hat{1}]\big(1-s+J_B(s)\big) - \beta[\hat{1}]J_A(s) \leq \dfrac{(1-s)^2}{2}$.

Lemma 2.2. *Suppose that $\beta,\ \alpha$ are linear functions and satisfy $(H1)-(H3)$ respectively. Then if $\Lambda > 0$ it follows that*

$$0 \leq G(t,\ s) \leq G(s,\ s) for\ all\ t,\ s \in [0,1]. \tag{2.4}$$

Proof.

$$G(t,s) = -\frac{1}{2}(t-s)^2H(t-s)+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-\alpha[\hat{t}]+t\alpha[\hat{1}])(1-s+J_B(s))$$

$$\geq -\frac{1}{2}(t-s)^2H(t-s)+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-\alpha[\hat{t}]+t\alpha[\hat{1}])(1+\beta[\hat{t}])(1-s)^2$$

$$\geq -\frac{1}{2}(t-s)^2H(t-s)+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}((1-\alpha[\hat{t}])\beta[\hat{1}]+t\alpha[\hat{1}](1+\beta[\hat{t}]))(1-s)^2$$

$$= -\frac{1}{2}(t-s)^2H(t-s)+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(t\Lambda+(1-t)(1-\alpha[\hat{t}])\beta[\hat{1}])(1-s)^2$$

$$= -\frac{1}{2}(t-s)^2H(t-s)+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-t)(1-\alpha[\hat{t}])\beta[\hat{1}](1-s)^2+t(1-s)^2.$$

Thus for $t \leq s$,

$$G(t,\ s) \geq \frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-t)(1-\alpha[\hat{t}])\beta[\hat{1}](1-s)^2+t(1-s)^2 \geq 0.$$

For $t \geq s$,

$$G(t,\ s) \geq -\frac{1}{2}(t-s)^2+\frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-t)(1-\alpha[\hat{t}])\beta[\hat{1}](1-s)^2+t(1-s)^2$$

$$= \frac{1}{\Lambda}(1+\beta[\hat{t}]-t\beta[\hat{1}])J_A(s)+\frac{1}{\Lambda}(1-t)(1-\alpha[\hat{t}])\beta[\hat{1}](1-s)^2+t+s^2t-\frac{1}{2}(t+s)^2 \geq 0.$$

This ensures that the leftside of inequality (2.4) holds. Considering that $t \mapsto G(t,\ s)$ is a piecewise affine map, so, for a fixed s, the maximum and minimum values occur when $t = 0$, or $t = s$ or $t = 1$. We will see that the maximum of $t \mapsto G(t,\ s)$ occurs when $t = s$. A direct calculation together with condition (H3)show that

$$G(s,\ s) - G(0,\ s) = \frac{s}{\Lambda}\bigg(\alpha[\hat{1}](1-s+J_B(s)) - \beta[\hat{1}]J_A(s)\bigg) \geq 0,$$

$$G(s,\ s) - G(1,\ s) = \frac{(1-s)}{\Lambda}\big(\beta[\hat{1}]J_A(s) - \alpha[\hat{1}](1-s+J_B(s))\big) + \frac{1}{2}(1-s)^2 \geq 0$$

Hence $G(t,\ s) \leq G(s,\ s),\ t,\ s \in [0,1]$. This completes the proof of Lemma 2.2.
To give results on positive solutions one needs to establish the key property as following:

Lemma 2.3. *Suppose that $\beta,\ \alpha$ are linear functions and satisfy $(H1)-(H3)$ respectively. if $\Lambda > 0$ and $\alpha[\hat{1}] > 0$, it follows that*

$$\phi(t)G(s,\ s) \leq G(t,\ s) \leq G(s,\ s)\ for\ all\ t,\ s \in [0,1], \tag{2.5}$$

where

$$\phi(t)=\min\left\{1-\frac{(1-t)\alpha[\hat{1}]}{1-\alpha[\hat{t}]+\alpha[\hat{1}]},\ \frac{\Lambda t(1-t)}{\alpha[\hat{1}](1+\beta[\hat{t}])},\ 1-\frac{\beta[\hat{1}]t}{1+\beta[\hat{t}]}\right\}.$$

Proof. For $t\le s$, we see that

$$G(t,\ s)-\phi(t)G(s,\ s)=\frac{1}{\Lambda}\Big((1-\phi(t))(1+\beta[\hat{t}])-(t-\phi(t)s)\beta[\hat{1}]\Big)J_A(s)$$
$$+\frac{1}{\Lambda}\Big((1-\alpha(\hat{t})+t\alpha[\hat{1}])-\phi(t)(1-\alpha[\hat{t}]+s\alpha[\hat{1}])\Big)(1-s+J_B(s))$$

Hence it suffices to have, for all $t<s\le 1$,

$$(1-\phi(t))(1+\beta[\hat{t}])-(t-\phi(t)s)\beta[\hat{1}]\ge 0$$

and

$$1-\alpha[t]+t\alpha[\hat{1}]-\phi(t)(1-\alpha[\hat{t}]+s\alpha[\hat{1}])\ge 0.$$

The first inequality holds for $\phi(t)<1$ since $1-\phi(t)\ge t-s\phi(t)$ for $s>t$ and $1+\beta[\hat{t}]\ge\beta[\hat{1}]$. The second inequality holds if

$$\phi(t)\le\frac{1-\alpha[\hat{t}]+t\alpha[\hat{1}]}{1-\alpha[\hat{t}]+\alpha[\hat{1}]}=1-\frac{(1-t)\alpha[1]}{1-\alpha[\hat{t}]+\alpha[\hat{1}]}.$$

For the case $s\le t$, we have

$$G(t,\ s)-\phi(t)G(s,\ s)=-\frac{1}{2}(t-s)^2+\frac{1}{\Lambda}\Big((1-\phi(t))(1+\beta[\hat{t}])-(t-\phi(t)s)\beta[\hat{1}]\Big)J_A(s)$$
$$+\frac{1}{\Lambda}\Big((1-\alpha(\hat{t})+t\alpha[\hat{1}])-\phi(t)(1-\alpha[\hat{t}]+s\alpha[\hat{1}])\Big)(1-s+J_B(s))$$

We suppose that $\phi(t)\le 1-\dfrac{\beta[\hat{1}]}{1+\beta[\hat{t}]}t$, which ensures that the term

$$(1-\phi(t))(1+\beta[\hat{t}])-(t-\phi(t)s)\beta[\hat{1}]$$

is nonnegative for all $s\le t$. For the left part of the other term, we have

$$-\frac{1}{2}(t-s)^2+\frac{1}{\Lambda}\Big((1-\alpha(\hat{t})+t\alpha[\hat{1}])-\phi(t)(1-\alpha[\hat{t}]+s\alpha[\hat{1}])\Big)(1-s+J_B(s)$$
$$\ge-\frac{1}{2}(t-s)^2+\Big((1-\alpha[\hat{t}])\frac{\beta[\hat{1}]}{1+\beta[\hat{t}]}t+(t-s\phi(t))\alpha[\hat{1}]\Big)(1+\beta[\hat{t}])(1-s)^2$$
$$=-\frac{1}{2}(t-s)^2+\Big(t\Lambda-s\phi(t)\alpha[\hat{1}](1+\beta[\hat{t}])\Big)(1-s)^2$$

and it suffices to have

$$\Big(t\Lambda-s\phi(t)\alpha[\hat{1}](1+\beta[\hat{t}])\Big)(1-s)^2\ge\Lambda\frac{1}{2}(t-s)^2,\ \text{for all } s\le t$$

or

$$\Big(t(1-s)^2-\frac{1}{2}(t-s)^2\Big)\Lambda\ge s(1-s)^2\alpha[\hat{1}](1+\beta[\hat{t}])\phi(t),\ \text{for all } s\le t.$$

Hence we want

$$\phi(t) \le \frac{t(1-t)^2\Lambda}{\alpha[\hat{1}](1+\beta[\hat{t}])}$$

To summarize, if $\alpha[\hat{1}] > 0$, we may choose

$$\phi(t) = \min\left\{1 - \frac{(1-t)\alpha[\hat{1}]}{1-\alpha[\hat{t}]+\alpha[\hat{1}]},\ \frac{\Lambda t(1-t)^2}{\alpha[\hat{1}](1+\beta[\hat{t}])},\ 1 - \frac{\beta[\hat{1}]t}{1+\beta[\hat{t}]}\right\}.$$

This completes the proof of Lemma 2.3.

Remark 2.1. The inequality (2.5) suffices the condition $\alpha[\hat{1}] > 0$. When the case $\alpha[u] = 0$ is considered, a similar result also holds where

$$\phi(t) = 1 - \frac{\beta[\hat{1}]t}{1+\beta[\hat{t}]}.$$

3. Existence results of positive solution

Here we introduce the following extreme limits:

$$f_0^s = \lim_{u\to 0^+}\sup \max_{t\in[0,\ 1]} \frac{f(t,\ u)}{u},\ f_0^i = \lim_{u\to 0^+}\inf \min_{t\in[0,1]} \frac{f(t,\ u)}{u},$$

$$f_\infty^s = \lim_{u\to \infty}\sup \max_{t\in[0,\ 1]} \frac{f(t,\ u)}{u},\ f_\infty^i = \lim_{u\to \infty}\inf \min_{t\in[0,\ 1]} \frac{f(t,\ u)}{u}.$$

For the convenience, we denote

$$\gamma = \sup_{t\in[a,\ b]} \phi(t)ds,$$

where $[a,\ b]$ is a arbitrary subinterval on [0,1].

We define the cone $P \subset X = C[0,1]$ by

$$P = \{u \in X \mid u(t) \ge 0,\ \inf_{a\le t\le b} u(t)\ \ge \gamma\|u\|\}.$$

Define the operator $T : X \to X$ by

$$T(u(t)) := \lambda \int_0^1 G(t,\ s)f(s,\ u(s))ds.$$

Lemma 3.1. *$T : P \to P$ is completely continuous.*

Proof. By using the properties of function $G(t,\ s)$ and the Arzela-Ascoli theorem, the proof of Lemma 3.1 is standard and it is omitted here.

Define the positive constants

$$K_1 = \frac{1}{\gamma\tilde{A}f_\infty^i},\ K_2 = \frac{1}{Af_0^s}$$

where

$$A = \int_0^1 G(s,\ s)ds,\ \tilde{A} = \gamma\int_a^b G(s,\ s)ds.$$

Theorem 3.1. *Assume that $(H1)-(H3)$ hold. $f_0^s,\ f_\infty^i \in (0,\ \infty)$, $K_1 < K_2$, then for $\lambda \in (K_1,\ K_2)$, the problem (P) has at least one positive solutions $u(t),\ t \in [0,\ 1]$.*

Proof. Let $\lambda \in (K_1,\ K_2),\ \varepsilon$ is a positive number such that $f_\infty^i > \varepsilon$ and

$$\frac{1}{\gamma \tilde{A}(f_\infty^i - \varepsilon)} < \lambda < \frac{1}{A(f_0^s + \varepsilon)}.$$

There exists $R_1 > 0$ such that for $t \in [0,\ 1],\ u(t) \geq 0$ and $u(t) \leq R_1$,

$$f(t,\ u(t)) \leq (f_0^s + \varepsilon)u(t).$$

We define the set

$$\Omega_1 = \{u(t) \in X,\ \|u\| < R_1\}.$$

Let $u \in P \cap \partial\Omega_1$,

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\leq \lambda \int_0^1 G(t,\ s)(f_0^s + \varepsilon) u(s) ds \\ &\leq \lambda (f_0^s + \varepsilon) \int_0^1 G(t,\ s) \|u\| ds \\ &\leq \lambda A (f_0^s + \varepsilon) \|u\| \\ &\leq \|u\|. \end{aligned}$$

On the other side, by condition $(H1)$ and the definition of f_∞^i there exists $\overline{R}_2 > 0$ such that for $t \in [a,\ b],\ u(t) \geq 0,\ u(t) \geq \overline{R}_2$,

$$f(t,\ u(t)) \geq (f_\infty^i - \varepsilon)u(t).$$

We consider $R_2 = \max\{2R_1,\ \overline{R}_2/\gamma\}$ and we define the set

$$\Omega_2 = \{u(t) \in X,\ \|u\| < R_2\}.$$

Let $u \in P \cap \Omega_2$, then for $u \in P$ with $\|u\| = R_2$, we have

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s)(f_\infty^i - \varepsilon) u(s) ds \\ &\geq \lambda \gamma \int_a^b G(t,\ s)(f_\infty^i - \varepsilon) \|u\| ds \\ &\geq \lambda \gamma \tilde{A} (f_\infty^i - \varepsilon) \|u\| \\ &\geq \|u\|. \end{aligned}$$

By using Lemma 1.1, T has a fixed point $u \in P \cap (\Omega_2 \setminus \overline{\Omega}_1)$.
By similar analysis, we can consider the case that above limits achieve 0 or ∞.
We give the main results here and the omit the proofs.

Theorem 3.2. *Assume that $(H1) - (H3)$ hold. If $f_0^s = 0,\ f_\infty^i \in (0,\ \infty)$, then for $\lambda \in (K_1,\ \infty)$, problem (P, λ) has at least one positive solutions $u(t),\ t \in [0,\ 1]$.*

Theorem 3.3. *Assume that* $(H1)-(H3)$ *hold. If* $f_\infty^i = \infty,\ f_0^s \in (0,\ \infty)$, *then for* $\lambda \in (0,\ K_2)$, *problem* (P, λ) *has at least one positive solutions* $u(t),\ t \in [0,\ 1]$.

Denote positive constants

$$L_1 = \frac{1}{\gamma \tilde{A} f_0^i},\ L_2 = \frac{1}{A f_\infty^s}.$$

Theorem 3.4. *Assume that* $(H1)-(H3)$ *hold,* $f_0^i,\ f_\infty^s \in (0,\ \infty), L_1 < L_2$. *Then for* $\lambda \in (L_1,\ L_2)$, *the problem* (P, λ) *has at least one positive solution* $u(t),\ t \in [0,\ 1]$.

Proof. Let$\lambda \in (L_1,\ L_2)$, ε is a positive number such that $f_0^i > \varepsilon$ and

$$\frac{1}{\gamma \tilde{A}(f_0^i - \varepsilon)} < \lambda < \frac{1}{A(f_\infty^s + \varepsilon)}.$$

There exists $R_3 > 0$ such that for $t \in [a,\ b],\ u(t) \geq 0$ and $u(t) \leq R_3$,

$$f(t,\ u(t)) \geq (f_0^i - \varepsilon)u(t).$$

We define the set

$$\Omega_3 = \{u(t) \in X,\ \|u\| < R_3\}.$$

Let $u \in P \cap \partial\Omega_3$,

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s)(f_0^i - \varepsilon) u(s) ds \\ &\geq \lambda\gamma \int_a^b G(t,\ s)(f_0^i - \varepsilon)\|u\| ds \\ &\geq \lambda\gamma\tilde{A}(f_0^i - \varepsilon)\|u\| \\ &\geq \|u\|. \end{aligned}$$

On the other side, we define the functions $f^* :\ [0,\ 1] \times R^+ \longrightarrow R^+$,

$$f^*(t,\ x) = \max_{0 \leq u \leq x} f(t,\ u),\ t \in [0,\ 1],\ x \geq 0.$$

Then

$$f(t,\ u) \leq f^*(t,\ x), t \in [0,\ 1],\ u \geq 0,\ u \leq x.$$

The functions $f^*(t,\ \cdot)$ are nondecreasing for each $t \in [0,\ 1]$ and satisfy the conditions

$$\limsup_{x \to \infty} \max_{t \in [0,\ 1]} \frac{f^*(t,\ x)}{x} \leq f_\infty^s.$$

Thus, for $\varepsilon > 0$, there exist $\overline{R}_4 > 0$ such that for all $x \geq \overline{R}_4,\ t \in [0,\ 1]$,

$$\frac{f^*(t,\ x)}{x} \leq \limsup_{x \to \infty} \max_{t \in [0,\ 1]} \frac{f^*(t,\ x)}{x} + \varepsilon \leq f_\infty^s + \varepsilon.$$

Then

$$f^*(t,\ x) \leq (f_\infty^s + \varepsilon)x.$$

Let $R_4 = \max\{2R_3,\ \overline{R}_4\}$ and $\Omega_4 = \{u \in X,\ \|u\| < R_4\}$. Let $u \in P \cap \partial\Omega_4$, then

$$f(t,\ u(t)) \le f^*(t,\ \|u\|)\ t \in [0,\ 1].$$

Thus

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\le \lambda \int_0^1 G(t,\ s) f^*(t,\ \|u\|) ds \\ &\le \lambda \int_0^1 G(t,\ s)(f_\infty^s + \varepsilon)\|u\| ds \\ &\le \lambda A (f_\infty^s + \varepsilon)\|u\| \\ &\le \|u\|. \end{aligned}$$

By using Lemma 1.1, T has a fixed point $u \in P \cap (\Omega_4 \setminus \overline{\Omega}_3)$.

We can also consider the case that above limits achieve 0 or ∞.

Theorem 3.5. *Assume that $(H1)-(H3)$ hold. If $f_\infty^s = 0,\ f_0^i \in (0,\ \infty)$, then for $\lambda \in (L_1,\ \infty)$, problem (P,λ) has at least one positive solution $u(t),\ t \in [0,\ 1]$.*

Theorem 3.6. *Assume that $(H1)-(H3)$ hold. If $f_0^i = \infty,\ f_\infty^s \in (0,\ \infty)$, then for $\lambda \in (0,\ L_2)$, problem (P,λ) has at least one positive solution $u(t),\ t \in [0,\ 1]$.*

The proofs are similar to the proof of Theorem 3.4 and we omit it here.

4. Nonexistence results of positive solution

In this section we shall consider sufficient conditions on λ and f such that problem (P,λ) has no positive solution.

Theorem 4.1. *Assume that $(H1)-(H3)$ hold. If $f_0^s,\ f_\infty^s < \infty$, then there exist positive constant λ_0 such that for every $\lambda \in (0,\ \lambda_0)$, problem (P,λ) has no positive solution.*

Proof. From the condition $f_0^s,\ f_\infty^s < \infty$, there exist $M_1 > 0$ such that

$$f(t,\ u) \le M_1 u,\ t \in [0,\ 1],\ u \ge 0.$$

Define positive constants

$$\lambda_0 = \frac{a}{AM_1}.$$

Let $\lambda \in (0,\ \lambda_0)$, suppose that problem (P,λ) has a positive solution $u(t)$, $t \in [0,\ 1]$. Thus,

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\le \lambda \int_0^1 G(t,\ s) M_1 u(s) ds \\ &\le \lambda M_1 \int_0^1 G(t,\ s)\|u\| ds \\ &< \|u\|. \end{aligned}$$

This induces that $\|u\| < \|u\|$, which is a contradiction. So the boundary value problem (P, λ) has no positive solution.

Theorem 4.2. *Assume that $(H1) - (H3)$ hold. If $f_0^i,\ f_\infty^i > 0$, then there exists a positive constant $\tilde{\lambda}_0$ such that for every $\lambda \in (\tilde{\lambda}_0,\ \infty)$, the boundary value problem (P, λ) has no positive solution.*

Proof. From the definitions of $f_0^i,\ f_\infty^i$ and the condition $f_0^i,\ f_\infty^i > 0$, there exist positive numbers m_1 such that

$$f(t,\ u) \geq m_1 u,\ t \in [a,\ b],\ u \geq 0.$$

Define positive constants

$$\tilde{\lambda}_0 = \frac{1}{\gamma \tilde{A} m_1}.$$

Let $\lambda \in (\tilde{\lambda}_0,\ \infty)$, we suppose that problem (P, λ) has a positive solution $u(t),\ t \in [0,\ 1]$. Then for $t \in [0,\ 1]$, we have

$$\begin{aligned} T(u)(t) &= \lambda \int_0^1 G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s) f(s,\ u(s)) ds \\ &\geq \lambda \int_a^b G(t,\ s) m_1 u(s) ds \\ &\geq \lambda \gamma \int_a^b G(t,\ s) m_1 \|u\| ds \\ &\geq \lambda \gamma \tilde{A} m_1 \|u\| \\ &> \|u\|, \end{aligned}$$

which induces a contradiction. Thus the boundary value problem (P, λ) has no positive solution.

5. Example

Consider the third-order four-point boundary value problem

$$-u^{'''}(t) = \lambda f(t,\ u),\ 0 \leq t \leq 1, \tag{5.1}$$

$$u''(0) = 0,\ u'(0) = 0,\ u'(1) + \beta_1 u(\eta_1) + \beta_2 u(\eta_2) = 0. \tag{5.2}$$

where $\beta_1 = \dfrac{3}{2},\ \beta_2 = -\dfrac{2}{3},\ \eta_1 = \dfrac{1}{3},\ \eta_2 = \dfrac{1}{2}$,

$$f(t,\ u) = \frac{\sqrt{t+3}(100u+1)(3+sin(u))u}{u+1}.$$

We notice that $\beta_1,\ \beta_2$ are not the same sign. By a simple calculation we have

$$\beta[\hat{1}] = \beta_1 + \beta_2,\ \beta[\hat{t}] = \beta_1 \eta_1 + \beta_2 \eta_2$$

and

$$J_B(s) = \begin{cases} 0, \ \eta_2 \leq s \leq 1 \\ \beta_2(\eta_2 - s), \ \eta_1 \leq s \leq \eta_2 \\ \beta_1(\eta_1 - s) + \beta_2(\eta_2 - s), \ 0 \leq s \leq \eta_1 \end{cases}$$

Thus the condition $1 + \beta[\hat{t}] \geq \beta[\hat{1}] > 0$ imposes a restriction

$$1 + \beta_1\eta_1 + \beta_2\eta_2 \geq \beta_1 + \beta_2 > 0. \tag{r1}$$

The condition $J_B(s) \leq \beta[\hat{t}] > 0$ imposes three restrictions

$$\beta_1\eta_1 + \beta_2\eta_2 \geq 0, \ \eta_2 \leq s \leq 1, \tag{5.3}$$

$$\beta_1\eta_1 + \beta_2 s \geq 0, \ \eta_1 \leq s \leq \eta_2, \tag{5.4}$$

$$(\beta_1 + \beta_2)s \geq 0, \ 0 \leq s \leq \eta_1. \tag{5.5}$$

The condition $r1$ implies that the Eq.(5.5) holds and (5.3-5.5) can simplify to conditions

$$\beta_1\eta_1 + \beta_2\eta_2 > 0, \ and \ \beta_1\eta_1 + \beta_2 s \geq 0, \eta_1 \leq s \leq \eta_2. \tag{r2}$$

The condition $1 - s + J_B(s) \geq (1 + \beta[\hat{t}])(1 - s)^2$ imposes three restrictions

$$1 - s \leq (1 + \beta_1 + \beta_2\eta_2)(1 - s)^2, \ \eta_2 \leq s \leq 1, \tag{5.6}$$

$$1 - s + \beta_2(\eta_2 - s) \leq (1 + \beta_1 + \beta_2\eta_2)(1 - s)^2, \ \eta_1 \leq s \leq \eta_2, \tag{5.7}$$

$$1 - s + \beta_1(\eta_1 - s) + \beta_2(\eta_2 - s) \leq (1 + \beta_1 + \beta_2\eta_2)(1 - s)^2, \ 0 \leq s \leq \eta_1, \tag{5.8}$$

This simplify to the conditions

$$(1 + \beta_1\eta_1 + \beta_2\eta_2)(1 - \eta_2) \leq 1 \tag{r3}$$

$$(1 + \beta_1\eta_1 + \beta_2\eta_2)s^2 + (\beta_2 - 1 - 2\beta_1\eta_1 - 2\beta_2\eta_2)s + \beta_1\eta_1 \leq 0, \ \eta_1 \leq s \leq \eta_2, \tag{r4}$$

$$(1 + \beta_1\eta_1 + \beta_2\eta_2)s + (\beta_2 + \beta_1 - 1 - 2\beta_1\eta_1 - 2\beta_2\eta_2)s \leq 0, \ 0 \leq s \leq \eta_1, \tag{r5}$$

By substituting the value $\beta_1, \ \beta_2, \ \eta_1, \ \eta_2$, we can check that the restrictions of $(r1)$ – $(r5)$ holds. We notice that

$$\phi(t) = 1 - \frac{\beta[\hat{1}]t}{1 + \beta[\hat{t}]}$$

and we can choose the subinterval $[a, \ b] = [0, \ 1]$. A direct calculation shows that

$$\begin{aligned} A &= \int_0^1 G(s, \ s)ds = \int_0^1 \frac{1}{\beta[\hat{1}]}(1 - s + J_B(s))ds \\ &= \frac{1}{\beta[\hat{1}]}\int_0^{\eta_1}(1 - s + \beta_1(\eta_1 - s) + \beta_2(\eta_2 - s))ds \\ &+ \int_{\eta_1}^{\eta_2}(1 - s + \beta_2(\eta_2 - s))ds + \int_{\eta_2}^1 (1 - s)ds = \frac{1}{2}(1 + \beta_1\eta_1^2 + \beta_2\eta_2^2). \end{aligned}$$

and

$$\gamma = \sup_{0 \leq t \leq 1} c(t) = 1 - \frac{\beta[\hat{1}]}{1 + \beta[\hat{t}]} = \frac{1 + \beta_1\eta_1 + \beta_2\eta_2 - \beta_1 - \beta_2}{1 + \beta_1\eta_1 + \beta_2\eta_2}$$

$$\tilde{A} = \gamma\int_0^1 G(s, \ s)ds = \frac{1}{2}\frac{1 + \beta_1\eta_1 + \beta_2\eta_2 - \beta_1 - \beta_2}{1 + \beta_1\eta_1 + \beta_2\eta_2}(1 + \beta_1\eta_1^2 + \beta_2\eta_2^2)$$

That is,

$$A = \frac{1}{2}(1 + \beta_1\eta_1^2 + \beta_2\eta_2^2) = \frac{1}{2},$$
$$\gamma = \frac{1 + \beta_1\eta_1 + \beta_2\eta_2 - \beta_1 - \beta_2}{1 + \beta_1\eta_1 + \beta_2\eta_2} = \frac{2}{7},$$
$$\tilde{A} = \frac{1}{2}\frac{1 + \beta_1\eta_1 + \beta_2\eta_2 - \beta_1 - \beta_2}{1 + \beta_1\eta_1 + \beta_2\eta_2}(1 + \beta_1\eta_1^2 + \beta_2\eta_2^2) = \frac{1}{7}$$

We have

$$f_0^s = 6,\ f_0^i = 3\sqrt{3},\ f_\infty^s = 800,\ f_\infty^i = 200\sqrt{3}, M = 800,\ m = 3\sqrt{3}$$

and

$$K_1 = \frac{1}{\gamma\tilde{A}f_\infty^i} \approx 0.0707,\ K_2 = \frac{1}{Af_0^s} \approx 0.3333,$$
$$\lambda_0 = \frac{1}{AM} = 0.0025,\ \tilde{\lambda}_0 = \frac{1}{\gamma\tilde{A}m} \approx 4.7150$$

Then

(1) from Theorem 3.1, for $\lambda \in (K_1, K_2)$, the problem (5.1) – (5.2) has a positive solution.

(2) from Theorem 4.1, for $\lambda \in (0,\ \lambda_0)$, the problem (5.1) – (5.2) has no positive solution.

(3) from Theorem 4.2, for $\lambda \in (\tilde{\lambda}_0,\ \infty)$, the problem (5.1) – (5.2) has no positive solution.

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